5.2 The Level Ellipsoid and Its External Field

We shall assume that the normal figure of the earth is a *level ellipsoid*, that is, an ellipsoid of revolution which is an equipotential surface of a *normal gravity field*. This assumption is natural because the ellipsoid is to be the normal form of the geoid, which is an equipotential surface of the actual gravity field. Denoting the potential of the normal gravity field by

$$U = U(x, y, z) \quad ,$$

we see that the level ellipsoid, being a surface $U =$ const., exactly corresponds to the geoid, defined as a surface $W =$ const.; W is the actual gravity potential.

The basic point here is that by postulating that the given ellipsoid be an equipotential surface of the normal gravity field, and by prescribing the total mass M, we completely and uniquely determine the normal potential U. The detailed density distribution inside the ellipsoid, which produces the potential U, need not be known at all.

What follows is a rigorous version of the theory that has been treated as a first-order approximation already in sec. 2.1, where we have not distinguished between U and W.

As we have already noticed in that section, the normal gravity potential U is completely determined by

1. the shape of the ellipsoid of revolution, that is, its semiaxes a and b,
2. the total mass M, and
3. the angular velocity ω

("Stokes constants"; cf. sec. 3.2.2).

The calculation will now be carried out in detail. The given ellipsoid S_0,

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad , \tag{5-34}$$

is by definition an equipotential surface

$$U(x, y, z) = U_0 \quad . \tag{5-35}$$

It is now convenient to introduce ellipsoidal coordinates u, $\bar{\theta}$, λ. The ellipsoid S_0 is taken as the reference ellipsoid $u = b$. In addition we shall use, instead of $\bar{\theta}$, the reduced latitude

$$\beta = 90° - \bar{\theta} \quad . \tag{5-36}$$

For its definition recall Fig. 5.1 and sec. 1.4, but note that β refers to the *coordinate ellipsoid* $u =$ const. which in general is different from the *reference ellipsoid* $u = b$; see Fig. 5.2.

Since the gravitational part, V, of the normal potential U will be harmonic outside the ellipsoid S_0, we use the series (5–26). The field V has rotational symmetry and hence does not depend on the longitude λ. Therefore, all non-zonal terms, which depend on λ, must be zero, and there remains

$$V(u,\beta)=\sum_{n=0}^{\infty}\frac{Q_n\left(i\frac{u}{E}\right)}{Q_n\left(i\frac{b}{E}\right)}a_nP_n(\sin\beta)\quad, \tag{5-37}$$

where

$$E=\sqrt{a^2-b^2} \tag{5-38}$$

is the linear excentricity. The centrifugal potential Φ is given by

$$\Phi=\frac{1}{2}\omega^2(x^2+y^2)=\frac{1}{2}\omega^2(u^2+E^2)\cos^2\beta\quad. \tag{5-39}$$

Hence the total normal gravity potential may be written

$$U(u,\beta)=\sum_{n=0}^{\infty}\frac{Q_n\left(i\frac{u}{E}\right)}{Q_n\left(i\frac{b}{E}\right)}a_nP_n(\sin\beta)+\frac{1}{2}\omega^2(u^2+E^2)\cos^2\beta\quad. \tag{5-40}$$

On the ellipsoid S_0 we have $u=b$ and $U=U_0$. Hence

$$U(b,\beta)=\sum_{n=0}^{\infty}a_nP_n(\sin\beta)+\frac{1}{2}\omega^2(u^2+E^2)\cos^2\beta=U_0\quad. \tag{5-41}$$

This equation must hold for all points of S_0, that is, for all values of β. Since

$$b^2+E^2=a^2 \tag{5-42}$$

and by (1–33),

$$\cos^2\beta=\frac{2}{3}\left[1-P_2(\sin\beta)\right]\quad, \tag{5-43}$$

we have

$$\sum_{n=0}^{\infty}a_nP_n(\sin\beta)+\frac{1}{3}\omega^2a^2-\frac{1}{3}\omega^2a^2P_2(\sin\beta)-U_0=0$$

or

$$\begin{aligned}&\left(a_0+\frac{1}{3}\omega^2a^2-U_0\right)P_0(\sin\beta)+a_1P_1(\sin\beta)+\\&\quad+\left(a_2-\frac{1}{3}\omega^2a^2\right)P_2(\sin\beta)+\sum_{n=3}^{\infty}a_nP_n(\sin\beta)=0\quad.\end{aligned} \tag{5-44}$$

This equation will hold for all values of β only if the coefficient of every $P_n(\sin\beta)$ is zero. Thus we get

$$\begin{aligned}a_0&=U_0-\frac{1}{3}\omega^2a^2\,, & a_1&=0\,,\\ a_2&=\frac{1}{3}\omega^2a^2\,, & a_3&=a_4=\ldots=0\end{aligned} \tag{5-45}$$

(don't confuse the coefficients a_n with the semimajor axis a!).

Inserting these into (5–37) gives

$$V(u,\beta) = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{Q_0\left(i\frac{u}{E}\right)}{Q_0\left(i\frac{b}{E}\right)} + \frac{1}{3}\omega^2 a^2 \frac{Q_2\left(i\frac{u}{E}\right)}{Q_2\left(i\frac{b}{E}\right)} P_2(\sin\beta) \quad . \tag{5–46}$$

This formula is basically the solution of Dirichlet's problem for the level ellipsoid, but we can give it more convenient forms.

We express Q_0 and Q_2 by (5–20) and (5–21), introducing the real quantities

$$q = \frac{1}{2}\left[\left(1 + 3\frac{u^2}{E^2}\right)\arctan\frac{E}{u} - 3\frac{u}{E}\right] , \tag{5–47}$$

$$q_0 = \frac{1}{2}\left[\left(1 + 3\frac{b^2}{E^2}\right)\arctan\frac{E}{b} - 3\frac{b}{E}\right] . \tag{5–48}$$

Thus (5–46) reduces to

$$V(u,\beta) = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{\arctan\frac{E}{u}}{\arctan\frac{E}{b}} + \frac{1}{3}\omega^2 a^2 \frac{q}{q_0} P_2(\sin\beta) \quad . \tag{5–49}$$

Now we can express U_0 in terms of the mass M. For large values of u we have

$$\arctan\frac{E}{u} = \frac{E}{u} + O\left(\frac{1}{u^3}\right) . \tag{5–50}$$

From the expressions (1–26) for spherical coordinates and from equations (5–1) for ellipsoidal coordinates we find

$$x^2 + y^2 + z^2 = r^2 \doteq u^2 + E^2\cos^2\beta \quad , \tag{5–51}$$

so that for large values of r we have

$$\frac{1}{u} = \frac{1}{r} + O\left(\frac{1}{r^3}\right) \tag{5–52}$$

and

$$\arctan\frac{E}{u} = \frac{E}{r} + O\left(\frac{1}{r^3}\right) . \tag{5–53}$$

For very large distances r, the first term in (5–49) is dominant, so that asymptotically

$$V = \left(U_0 - \frac{1}{3}\omega^2 a^2\right) \frac{E}{\arctan(E/b)} \frac{1}{r} + O\left(\frac{1}{r^3}\right) . \tag{5–54}$$

We know from sec. 2.1 that

$$V = \frac{GM}{r} + O\left(\frac{1}{r^3}\right) . \tag{5–55}$$

We multiply both equations by r, whence

$$GM = \lim_{r\to\infty}(rV) = \left(U_0 - \frac{1}{3}\omega^2 a^2\right)\frac{E}{\arctan(E/b)} \quad . \tag{5–56}$$

Now it is appropriate to introduce the second excentricity of the reference ellipsoid by (1–57):

$$e' = \frac{E}{b} \quad . \tag{5–57}$$

Note that the prime in e' is *not* a sign of differentiation, but only serves to distinguish e' from the first excentricity

$$e = \frac{E}{a} \quad . \tag{5–58}$$

(A confusion with (4–222) and with $\dot{e}$, etc., is not to be expected.)

Thus we get

$$\begin{aligned} GM &= \left(U_0 - \frac{1}{3}\omega^2 a^2\right)\frac{E}{\arctan e'} \quad , & (5\text{–}59)\\ U_0 &= \frac{GM}{E}\arctan e' + \frac{1}{3}\omega^2 a^2 & (5\text{–}60) \end{aligned}$$

as equivalent relations between total mass M and "sea-level potential" U_0.

We can substitute these relations into the expression for V, given by (5–49), and express P_2 as

$$P_2(\sin\beta) = \frac{3}{2}\sin^2\beta - \frac{1}{2} \quad . \tag{5–61}$$

Finally, if we add the centrifugal potential Φ (5–39), we get the normal gravity potential U as

$$U(u,\beta) = \frac{GM}{E}\arctan\frac{E}{u} + \frac{1}{2}\omega^2 a^2 \frac{q}{q_0}\left(\sin^2\beta - \frac{1}{3}\right) + \frac{1}{2}\omega^2(u^2+E^2)\cos^2\beta \quad . \tag{5–62}$$

The only constants that occur in this formula are a, b, GM, and ω. This expresses the well-known fact that four independent constants are necessary and sufficient to fully determine the level ellipsoid together with its external gravity field.

Normal gravity. Normal gravity γ on the reference ellipsoid is defined by

$$\gamma = |\mathrm{grad}U|_{u=b} \tag{5–63}$$

and can be computed by

$$\gamma = -\frac{\partial U}{\partial n} = -\frac{\partial U}{\partial u}\Big/\frac{dn}{du} \quad , \tag{5–64}$$

$\partial/\partial n$ denoting the derivative along the ellipsoidal normal. Obviously

$$dn = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}\, du \quad . \tag{5–65}$$

We differentiate (5–62) with respect to u to get $\partial U/\partial u$, and calculate dn/du by (5–65) with (5–1). The result, on replacing $\beta = 90° - \bar{\theta}$ by the geographic latitude ϕ by (1–66) and putting $u = b$, is *Somigliana's* (1930) *formula*

$$\gamma = \frac{a\gamma_e \cos^2\phi + b\gamma_p \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \quad . \tag{5–66}$$

Computationally more convenient is the equivalent form

$$\gamma = \gamma_e \frac{1 + k\sin^2\phi}{\sqrt{1 - e^2\sin^2\phi}} \quad , \tag{5–67}$$

where e is (5–58) and

$$k = \frac{b\gamma_p}{a\gamma_e} - 1 \quad . \tag{5–68}$$

Gravity at the equator, γ_e, and at the poles, γ_p, are related to the semiaxes a and b by the remarkable formula

$$\frac{a-b}{a} + \frac{\gamma_p - \gamma_e}{\gamma_e} = \frac{\omega^2 b}{\gamma_e}\left(1 + \frac{e' q_0'}{2q_0}\right) \quad , \tag{5–69}$$

where e' is (5–57), q_0 is given by (5–48), and

$$q_0' = 3\left(1 + \frac{1}{e'^2}\right)\left(1 - \frac{\arctan e'}{e'}\right) - 1 \quad . \tag{5–70}$$

Eq. (5–69) is *Clairaut's theorem* (2–26) in a closed form for the level ellipsoid.

Spherical-harmonic coefficients. We shall need in (1–39)

$$J_2 = \frac{C - A}{Ma^2} = \frac{e^2}{3}\left(1 - \frac{2}{15}\frac{me'}{q_0}\right) \tag{5–71}$$

with

$$m = \frac{\omega^2 a^2 b}{GM} \tag{5–72}$$

as usual (eq. (1–83)) and, by (1–40),

$$J_4 = \frac{3}{35} e^4 \left(1 - 10\frac{J_2}{e^2}\right) \quad . \tag{5–73}$$

For derivations and more details cf. (Heiskanen and Moritz, 1967, secs. 2–8 and 2–9). Computational formulas and numerical results are found in (Moritz, 1984) for the Geodetic Reference System 1980; see also sec. 1.5. For the transition from the present closed expressions to the approximate formulas of sec. 2.1 cf. (Heiskanen and Moritz, 1967, sec. 2–10). An interesting alternative presentation is (Hotine, 1969, Chapter 23).