Since κ is small of second order, we may again replace the polar radius t by the mean radius β without loss of accuracy:

$$\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa =$$
$$= 3\left(1 - \frac{\delta}{D}\right)f^2 + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta f\dot{f} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{f}^2 . \qquad (4\text{–}204)$$

This is *Darwin's equation* which we already know (eq. (4–123)), but which appears in a new light by the present derivation; clearly f can be replaced by e in the second-order terms on the right-hand side. To repeat: the differential equations (4–204) and (4–201) are equivalent, but (4–204) is practically more useful, whereas (4–201) is theoretically particularly interesting.

4.3.5 Clairaut's Equation

The derivation of Clairaut's equation accurate to $O(f^2)$ starts from (4–193). Using (4–167) and (4–184), taking into account (4–187) and (4–188), we thus can write

$$t\Psi(t) = \frac{3\alpha - \frac{1}{2}t^2\alpha'' + 2\alpha^2 + 2t\alpha\alpha' - 8\epsilon}{\alpha + t\alpha' - 2\alpha^2} . \qquad (4\text{–}205)$$

From (2–104) we take, to first order,

$$\frac{W}{4\pi G} = \frac{1}{\beta}\int_0^\beta \delta \cdot \beta^2 d\beta + \int_\beta^1 \delta \cdot \beta d\beta + \frac{\omega^2\beta^2}{12\pi G} , \qquad (4\text{–}206)$$

where, as usual,

$$\delta = \frac{\rho}{\rho_m} \qquad (4\text{–}207)$$

denotes the dimensionless "normalized density" and β the (normalized) mean radius of the equisurface passing through the point P at which W is considered (the fact that it is also used as an integration variable in our customary way will cause no confusion).

Differentiation gives

$$\frac{1}{4\pi G}\frac{dW}{d\beta} = -\frac{1}{\beta^2}\int_0^\beta \delta \cdot \beta^2 d\beta + \beta\delta - \beta\delta + \frac{\omega^2\beta}{6\pi G} \qquad (4\text{–}208)$$

or, by (4–56),

$$\frac{dW}{d\beta} = -\frac{4\pi G}{3}D\beta + \frac{2}{3}\omega^2\beta . \qquad (4\text{–}209)$$

By definition,

$$\beta = \sqrt[3]{t(1+f) \cdot t(1+f) \cdot t} \qquad (4\text{–}210)$$

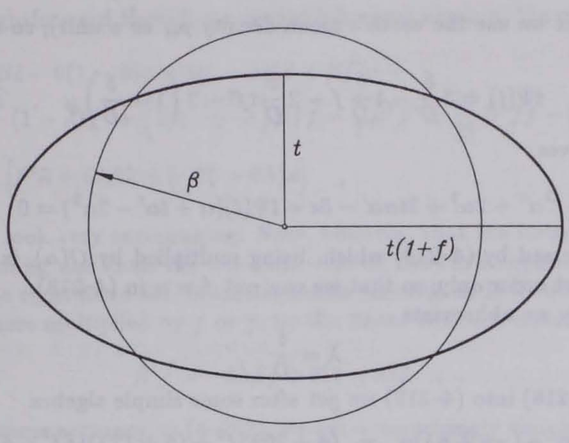

FIGURE 4.10: Polar radius t and mean radius β

is the geometric mean of all three axes (Fig. 4.10). (In a more familiar notation this is $R = \sqrt[3]{a^2b}$, the sphere being defined as having the same volume as the ellipsoid.) In view of the smallness of f, (4–210) reduces in the linear approximation to

$$\beta = t\left(1 + \frac{2}{3}f\right) \quad , \tag{4–211}$$

$$t = \beta\left(1 - \frac{2}{3}f\right) \quad . \tag{4–212}$$

Hence,

$$W'(t) = \frac{dW}{dt} = \frac{dW}{d\beta}\frac{d\beta}{dt} = \frac{dW}{d\beta}\left(1 + \frac{2}{3}f + \frac{2}{3}tf'\right) \quad . \tag{4–213}$$

Using (4–209) with (4–211), this gives

$$W'(t) = -\frac{4\pi G}{3}Dt\left(1 + \frac{2}{3}f\right)\left(1 + \frac{2}{3}f + \frac{2}{3}tf'\right) + \frac{2}{3}\omega^2 t \tag{4–214}$$

(since $\omega^2 = O(f)$, we have been able simply to replace β by t in the last term).

Introducing the dimensionless quantity (4–66), in the present units

$$\mu = \frac{3}{4\pi G}\frac{\omega^2}{D} \quad , \tag{4–215}$$

which is $O(f)$, we thus have to $O(f)$

$$g_P = -W'(t) = \frac{4\pi G}{3}Dt\left(1 + \frac{4}{3}f + \frac{2}{3}tf' - \frac{2}{3}\mu\right) \quad . \tag{4–216}$$

Now

$$\frac{4\pi G\rho - 2\omega^2}{\frac{4\pi}{3}GtD} = \frac{3}{t}\frac{\delta}{D} - \frac{2}{t}\mu \tag{4–217}$$

(we have $\rho = \delta$ if we use the earth's mean density ρ_m as a unit), so that by (4–142) and (4–216)

$$t\Psi(t) = 3\frac{\delta}{D} - 4\frac{\delta}{D}f - 2\frac{\delta}{D}tf' - 2\left(1 - \frac{\delta}{D}\right)\mu \quad . \tag{4–218}$$

Then (4–205) gives

$$3\alpha - \frac{1}{2}t^2\alpha'' + 2\alpha^2 + 2t\alpha\alpha' - 8\epsilon - t\Psi(t)(\alpha + t\alpha' - 2\alpha^2) = 0 \tag{4–219}$$

with $t\Psi(t)$ expressed by (4–218) which, being multiplied by $O(\alpha)$, is indeed seen to be needed to first order only, so that we can put $f = \alpha$ in (4–218).

For simplicity we abbreviate

$$\lambda = \frac{\delta}{D} \quad . \tag{4–220}$$

Substituting (4–218) into (4–219) we get after some simple algebra

$$\begin{aligned} t^2\alpha'' + 6\lambda t\alpha' + (-6 + 6\lambda)\alpha &= (4 + 20\lambda)f^2 + (4 + 12\lambda)tff' + 4\lambda t^2 f'^2 - \\ &- 16\epsilon + 4(1 - \lambda)(f + tf')\mu \end{aligned} \tag{4–221}$$

where, on the right-hand side, we have put $f = \alpha$ because it contains quadratic terms only.

The left-hand side represents the linear Clairaut equation for α, and the right-hand side, rather than being zero, is now $O(f^2)$. Thus (4–221) may already be regarded as some second-order generalization of Clairaut's equation, but it is better to change from α, t to the flattening f and the mean radius β by means of (4–198), (4–199), and (4–212).

The final result becomes still simpler if we use, instead of the flattening f, the "ellipticity"

$$e = f - \frac{5}{42}f^2 + \frac{4}{7}\kappa \tag{4–222}$$

(with $e^2 \doteq f^2$), already introduced in eq. (4–48).

By (4–198), (4–199), and (4–222) we have

$$\alpha = e - \frac{8}{21}e^2 - \frac{32}{7}\kappa \quad , \tag{4–223}$$

$$\epsilon = \frac{3}{2}e^2 + 4\kappa \quad . \tag{4–224}$$

This is inserted into (4–221). Furthermore we substitute, from (4–212),

$$t = \beta\left(1 - \frac{2}{3}e\right), \qquad t^2 = \beta^2\left(1 - \frac{4}{3}e\right). \tag{4–225}$$

Finally we replace all derivatives with respect to t by derivatives with respect to β, denoted by a dot as before, cf. eq. (4–78):

$$f' = \frac{df}{dt} = \frac{df}{d\beta}\frac{d\beta}{dt} = \dot{f}\left(1 + \frac{2}{3}f + \frac{2}{3}\beta\dot{f}\right) \quad , \tag{4–226}$$

$$f'' = \ddot{f} + \frac{4}{3}f\ddot{f} + 2\beta\dot{f}\ddot{f} + \frac{4}{3}\dot{f}^2 \quad . \tag{4–227}$$

This is straightforward though somewhat laborious algebra; the result is

$$\beta^2\ddot{e} + 6\lambda\beta\dot{e} - 6(1-\lambda)e = 4(1-\lambda)(f+\beta\dot{f})\mu -$$
$$-\frac{156}{7}(1-\lambda)f^2 + \left(4+\frac{116}{7}\lambda\right)\beta f\dot{f} - \frac{4}{7}\beta^2\dot{f}^2 + \frac{16}{21}\beta^2 f\ddot{f} - 2\beta^3\dot{f}\ddot{f} +$$
$$+\frac{32}{7}\left[\beta^2\ddot{\kappa} + 6\lambda\beta\dot{\kappa} + (-20+6\lambda)\kappa\right] \quad , \qquad (4\text{–}228)$$

which does not look very encouraging. Note, however, that the term between parentheses [ ] is nothing else than the left-hand side of Darwin's equation (4–204). Replacing it by the right-hand side of this equation removes κ. If we do this and finally eliminate $\ddot{f}$, where multiplied by f or $\dot{f}$, by the linear Clairaut equation:

$$\beta^2\ddot{f} = -6\lambda\beta\dot{f} + 6(1-\lambda)f \quad , \qquad (4\text{–}229)$$

which has the same accuracy as (4–202), we get a surprisingly simple result:

$$\begin{aligned}\beta^2\ddot{e} + 6\lambda\beta\dot{e} - 6(1-\lambda)e \;&=\; -\frac{4}{7}(1-\lambda)(7f^2 + 6\beta f\dot{f} + 3\beta^2\dot{f}^2) + \\ &+\; 4(1-\lambda)(f+\beta\dot{f})\mu \quad , \qquad (4\text{–}230)\end{aligned}$$

which is nothing else than our old friend, the second-order Clairaut equation (4–91) with (4–92) or (4–90); note that $e = f$ in second-order terms as usual.

We thus have derived this equation and also Darwin's equation in an alternative geometric way. This method, proceeding from Wavre's theory, is simple and transparent in principle, though the detailed calculations may be laborious. In principle, it is nothing else than an extension of the method of sec. 3.2.5 to second order. It is completely different and independent of the method of sec. 4.2; in particular, it does not use spherical harmonic series with a somewhat difficult convergence behavior.

Generally, the present method may be considered more elementary and direct, avoiding tricky manipulations with spherical harmonics and equally tricky differentiation of integrals. On the other hand it should be noted that we only get the *differential equations* for f and κ, *but not the boundary conditions*. For those people who do not appreciate the esthetic appeal of this Wavre-type approach, it will at least serve as a very useful independent check.