$$\begin{aligned}\frac{\partial X}{\partial \theta} &= 2\sin\theta\cos\theta(X_1 + 2X_2\sin^2\theta) \quad , \\ \frac{\partial Y}{\partial \theta} &= \frac{2}{t}\,2\sin\theta\cos\theta(Y_1 + 2Y_2\sin^2\theta) \quad ,\end{aligned} \tag{4–189}$$

and (4–186) becomes

$$\frac{1}{2}\,t\Psi(t) = \frac{Y_1 + 2Y_2\sin^2\theta}{X_1 + 2X_2\sin^2\theta} \quad . \tag{4–190}$$

Since X_2, $Y_2 \ll X_1$, Y_1, we may again expand:

$$\begin{aligned}\frac{1}{2}\,t\Psi(t) &= \frac{Y_1}{X_1}\left(1 + 2\frac{Y_2}{Y_1}\sin^2\theta\right)\left(1 + 2\frac{X_2}{X_1}\sin^2\theta\right)^{-1} = \\ &= \frac{Y_1}{X_1}\left[1 + 2\left(\frac{Y_2}{Y_1} - \frac{X_2}{X_1}\right)\sin^2\theta + (\ldots)\sin^4\theta + \cdots\right] \quad .\end{aligned} \tag{4–191}$$

Now comes the essential reasoning: since this equation is an identity in θ and since the left-hand side is independent of θ, the right-hand side must also be independent of θ. This requires

$$\frac{Y_2}{Y_1} - \frac{X_2}{X_1} = 0 \tag{4–192}$$

and consequently

$$\frac{1}{2}\,t\Psi(t) = \frac{Y_1}{X_1} \quad . \tag{4–193}$$

These are the basic equations for our problem: (4–192) will lead to Darwin's equation, whereas (4–193) will give Clairaut's equation accurate to second order in f. We immediately note that (4–192) corresponds to the condition (3–46) which is "weaker" than (3–45) as we have remarked at the end of sec. 3.2.1. Thus (3–46) is sufficient to derive Darwin's but not Clairaut's equation.

4.3.4 Darwin's Equation

Eq. (4–192) is equivalent to

$$X_1Y_2 - X_2Y_1 = 0 \quad . \tag{4–194}$$

X_1 and X_2 are the terms (truncated series) on the right-hand side of (4–167) multiplied by $\sin^2\theta$ and $\sin^4\theta$, respectively, and similarly for Y_1 and Y_2 with (4–184); cf. (4–187) and (4–188).

We substitute these series into (4–194), keeping terms of order α^3 but neglecting $O(\alpha^4)$. The result is

$$\begin{aligned}&(t^2\alpha + t^3\alpha')\epsilon'' + (6t\alpha - t^3\alpha'')\epsilon' - (14\alpha + 20t\alpha' + t^2\alpha'')\epsilon = \\ &= -21\alpha^3 - 14t\alpha^2\alpha' - 3t^2\alpha\alpha'^2 + 2t^3\alpha'^3 + \\ &+ \frac{7}{2}\,t^2\alpha^2\alpha'' + 3t^3\alpha\alpha'\alpha'' + \frac{3}{2}\,t^4\alpha'^2\alpha'' \quad .\end{aligned} \tag{4–195}$$

Now we transform to the standard parameters f (flattening) and κ (deviation), contained in eq. (4–3):

$$r = a\left[1 - f\cos^2\theta - \left(\frac{3}{8}f^2 + \kappa\right)\sin^2 2\theta\right] \quad ; \tag{4–196}$$

remember that κ is a second-order quantity which is zero for an exact ellipsoid.

Since $t = b$, the semiminor axis of the equisurface under consideration, and since $b = a(1-f)$, it is easy to transform (4–196) into the form

$$r = t\left[1 + \left(f - \frac{1}{2}f^2 - 4\kappa\right)\sin^2\theta + \left(\frac{3}{2}f^2 + 4\kappa\right)\sin^4\theta\right] \quad , \tag{4–197}$$

which by comparison with (4–164) shows that

$$\alpha = f - \frac{1}{2}f^2 - 4\kappa \quad , \tag{4–198}$$

$$\epsilon = \frac{3}{2}f^2 + 4\kappa \quad , \tag{4–199}$$

confirming the first-order equality

$$\alpha \doteq f \quad . \tag{4–200}$$

This is now used to transform (4–195). It is readily recognized that for ϵ we need (4–199), whereas for α, which is always multiplied by second-order terms, (4–200) is sufficient. The result is

$$\begin{aligned}(t^2 f + t^3 f')\kappa'' &+ (6tf - t^3 f'')\kappa' - (14f + 20tf' + t^2 f'')\kappa = \\ &= -\frac{1}{2}tf^2 f' - \frac{3}{2}t^2 f f'^2 - \frac{1}{4}t^3 f'^3 + \frac{1}{2}t^2 f^2 f'' + \frac{3}{4}t^3 f f' f'' + \frac{3}{8}t^4 f'^2 f''\end{aligned} \tag{4–201}$$

This is a second-order linear ordinary differential equation for the deviation $\kappa = \kappa(t)$, which has extraordinary theoretical interest: It shows that, given the flattening $f = f(t)$ (which implies knowing the derivatives f' and f''), the quantity κ is fully determined (apart from the usual boundary conditions). The density distribution does not enter here!

This is fully in the spirit of Wavre's theory which aims at separating the geometry from the physics to the largest possible extent.

Practically it may be preferable to eliminate f'' by Clairaut's equation (2–114) or (4–124):

$$t^2 f'' = -6\frac{\delta}{D}tf' + 6\left(1 - \frac{\delta}{D}\right)f \quad . \tag{4–202}$$

In the linear approximation we have $t \doteq \beta$, $e \doteq f$; this linear approximation is, of course, sufficient since f'' is multiplied by terms of $O(f^2)$. After some straightforward calculations we thus obtain, the factor $f + tf'$ canceling "miraculously",

$$\begin{aligned}t^2\kappa'' &+ 6\frac{\delta}{D}t\kappa' + \left(-20 + 6\frac{\delta}{D}\right)\kappa = \\ &= 3\left(1 - \frac{\delta}{D}\right)f^2 + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)tff' - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)t^2 f'^2 \quad .\end{aligned} \tag{4–203}$$

Since κ is small of second order, we may again replace the polar radius t by the mean radius β without loss of accuracy:

$$\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa =$$
$$= 3\left(1 - \frac{\delta}{D}\right)f^2 + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta f\dot{f} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{f}^2 \, . \qquad (4\text{–}204)$$

This is *Darwin's equation* which we already know (eq. (4–123)), but which appears in a new light by the present derivation; clearly f can be replaced by e in the second-order terms on the right-hand side. To repeat: the differential equations (4–204) and (4–201) are equivalent, but (4–204) is practically more useful, whereas (4–201) is theoretically particularly interesting.

4.3.5 Clairaut's Equation

The derivation of Clairaut's equation accurate to $O(f^2)$ starts from (4–193). Using (4–167) and (4–184), taking into account (4–187) and (4–188), we thus can write

$$t\Psi(t) = \frac{3\alpha - \frac{1}{2}t^2\alpha'' + 2\alpha^2 + 2t\alpha\alpha' - 8\epsilon}{\alpha + t\alpha' - 2\alpha^2} \, . \qquad (4\text{–}205)$$

From (2–104) we take, to first order,

$$\frac{W}{4\pi G} = \frac{1}{\beta}\int_0^\beta \delta \cdot \beta^2 d\beta + \int_\beta^1 \delta \cdot \beta d\beta + \frac{\omega^2\beta^2}{12\pi G} \, , \qquad (4\text{–}206)$$

where, as usual,

$$\delta = \frac{\rho}{\rho_m} \qquad (4\text{–}207)$$

denotes the dimensionless "normalized density" and β the (normalized) mean radius of the equisurface passing through the point P at which W is considered (the fact that it is also used as an integration variable in our customary way will cause no confusion).

Differentiation gives

$$\frac{1}{4\pi G}\frac{dW}{d\beta} = -\frac{1}{\beta^2}\int_0^\beta \delta \cdot \beta^2 d\beta + \beta\delta - \beta\delta + \frac{\omega^2\beta}{6\pi G} \qquad (4\text{–}208)$$

or, by (4–56),

$$\frac{dW}{d\beta} = -\frac{4\pi G}{3}D\beta + \frac{2}{3}\omega^2\beta \, . \qquad (4\text{–}209)$$

By definition,

$$\beta = \sqrt[3]{t(1+f)\cdot t(1+f)\cdot t} \qquad (4\text{–}210)$$