same accuracy, we may in (4–178) replace r by t, obtaining

$$\frac{\partial \theta}{\partial t} = -2t^{-1} f \cos\theta \sin\theta + O(f^2) \quad . \tag{4–179}$$

Comparing (4–175) with (4–163), we see that in our case

$$F = \ln N \quad , \tag{4–180}$$

so that C represents the θ-correction for B; cf. (4–161) and (4–163). Thus

$$C = \frac{\partial \ln N}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{1}{2}\frac{\partial \ln N^2}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{1}{2N^2}\frac{\partial N^2}{\partial \theta}\frac{\partial \theta}{\partial t} \tag{4–181}$$

and finally, by (4–144),

$$C = \frac{1}{2X}\frac{\partial X}{\partial \theta}\frac{\partial \theta}{\partial t} \quad . \tag{4–182}$$

By (4–167), $\partial X/\partial\theta$ will be of order $\alpha \doteq f$, and so is (4–179). So, C will be of order f^2, so that we may put $f = \alpha$ and $X = 1$ without loss of accuracy, obtaining simply

$$C = -(4t^{-1}\alpha^2 + 4\alpha\alpha')(\sin^2\theta - \sin^4\theta) \quad . \tag{4–183}$$

Combining (4–168), (4–169) and (4–183) according to (4–161), we finally get

$$\begin{aligned} Y &= \frac{2}{t}\Big[1 - 2\alpha + \Big(3\alpha + 2\alpha^2 + 2t\alpha\alpha' - \frac{1}{2}t^2\alpha'' - 8\epsilon\Big)\sin^2\theta + \\ &+ \Big(-3\alpha^2 - t\alpha\alpha' + t^2\alpha'^2 + \frac{1}{2}t^2\alpha\alpha'' + \frac{1}{2}t^3\alpha'\alpha'' + \\ &+ \ 10\epsilon - \frac{1}{2}t^2\epsilon''\Big)\sin^4\theta\Big] \quad . \end{aligned} \tag{4–184}$$

4.3.3 Basic Equations

From (4–173) we find

$$\frac{\partial F}{\partial \Theta} = \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial \Theta} \quad . \tag{4–185}$$

For t = const., the factor $\partial\theta/\partial\Theta$ cancels in the numerator and the denominator on the right-hand side of (4–141), so that we also have

$$\Psi(t) = \frac{\partial Y/\partial\theta}{\partial X/\partial\theta} \quad . \tag{4–186}$$

The functions X and Y are given by (4–167) and (4–184), which we write in the form

$$X = 1 + X_1\sin^2\theta + X_2\sin^4\theta \quad , \tag{4–187}$$

$$Y = \frac{2}{t}(Y_0 + Y_1\sin^2\theta + Y_2\sin^4\theta) \quad , \tag{4–188}$$

where the functions X_i and Y_i are series depending on t only. Thus

$$\begin{aligned}\frac{\partial X}{\partial \theta} &= 2\sin\theta\cos\theta(X_1 + 2X_2\sin^2\theta) \quad , \\ \frac{\partial Y}{\partial \theta} &= \frac{2}{t}\, 2\sin\theta\cos\theta(Y_1 + 2Y_2\sin^2\theta) \quad ,\end{aligned} \tag{4–189}$$

and (4–186) becomes

$$\frac{1}{2}\, t\Psi(t) = \frac{Y_1 + 2Y_2\sin^2\theta}{X_1 + 2X_2\sin^2\theta} \quad . \tag{4–190}$$

Since X_2, $Y_2 \ll X_1$, Y_1, we may again expand:

$$\begin{aligned}\frac{1}{2}\, t\Psi(t) &= \frac{Y_1}{X_1}\left(1 + 2\frac{Y_2}{Y_1}\sin^2\theta\right)\left(1 + 2\frac{X_2}{X_1}\sin^2\theta\right)^{-1} = \\ &= \frac{Y_1}{X_1}\left[1 + 2\left(\frac{Y_2}{Y_1} - \frac{X_2}{X_1}\right)\sin^2\theta + (\ldots)\sin^4\theta + \cdots\right] \quad .\end{aligned} \tag{4–191}$$

Now comes the essential reasoning: since this equation is an identity in θ and since the left-hand side is independent of θ, the right-hand side must also be independent of θ. This requires

$$\frac{Y_2}{Y_1} - \frac{X_2}{X_1} = 0 \tag{4–192}$$

and consequently

$$\frac{1}{2}\, t\Psi(t) = \frac{Y_1}{X_1} \quad . \tag{4–193}$$

These are the basic equations for our problem: (4–192) will lead to Darwin's equation, whereas (4–193) will give Clairaut's equation accurate to second order in f. We immediately note that (4–192) corresponds to the condition (3–46) which is "weaker" than (3–45) as we have remarked at the end of sec. 3.2.1. Thus (3–46) is sufficient to derive Darwin's but not Clairaut's equation.

4.3.4 Darwin's Equation

Eq. (4–192) is equivalent to

$$X_1 Y_2 - X_2 Y_1 = 0 \quad . \tag{4–194}$$

X_1 and X_2 are the terms (truncated series) on the right-hand side of (4–167) multiplied by $\sin^2\theta$ and $\sin^4\theta$, respectively, and similarly for Y_1 and Y_2 with (4–184); cf. (4–187) and (4–188).

We substitute these series into (4–194), keeping terms of order α^3 but neglecting $O(\alpha^4)$. The result is

$$\begin{aligned}&(t^2\alpha + t^3\alpha')\epsilon'' + (6t\alpha - t^3\alpha'')\epsilon' - (14\alpha + 20t\alpha' + t^2\alpha'')\epsilon = \\ &= -21\alpha^3 - 14t\alpha^2\alpha' - 3t^2\alpha\alpha'^2 + 2t^3\alpha'^3 + \\ &+ \frac{7}{2}\, t^2\alpha^2\alpha'' + 3t^3\alpha\alpha'\alpha'' + \frac{3}{2}\, t^4\alpha'^2\alpha'' \quad .\end{aligned} \tag{4–195}$$