whence

$$N = \frac{dn}{dt} = r_t \cos\delta \quad .$$

On the other hand the enlarged figure shows that

$$\cos\delta = \frac{r\,d\theta}{ds} = \frac{r}{\sqrt{r^2 + r_\theta^2}}$$

by (4–156). Thus we find

$$N = \frac{r r_t}{\sqrt{r^2 + r_\theta^2}} \quad . \tag{4–159}$$

Eqs. (4–158) and (4–159) are basic. Their substitution into (4–144) and (4–146) finally gives

$$X = N^2 = \frac{r^2 r_t^2}{r^2 + r_\theta^2} \quad , \tag{4–160}$$

$$Y = A - B - C \quad , \tag{4–161}$$

$$A = 2JN = \frac{r r_t}{r^2 + r_\theta^2}\left(2 - \frac{r_\theta}{r}\cot\theta + \frac{r_\theta^2 - r r_{\theta\theta}}{r^2 + r_\theta^2}\right) \quad , \tag{4–162}$$

$$B = \frac{\partial \ln N}{\partial t} = \frac{r_t}{r} + \frac{r_{tt}}{r_t} - \frac{r r_t + r_\theta r_{\theta t}}{r^2 + r_\theta^2} \quad . \tag{4–163}$$

So far, everything has been quite straightforward. A fine point must be made, however. In (4–146), $\partial/\partial t$ means the derivative with respect to t *for constant* Θ, i.e., along the plumb line, whereas in (4–163), $\partial/\partial t$ denotes the derivative also with respect to t but *for constant* θ, i.e., along the radius vector. This fact must be taken into account by adding in (4–161) a correction C. This "θ-correction" will be considered in the next section.

4.3.2 Series Expansions

Let us now represent the equation of the set of equisurfaces in the form

$$r = r(t, \theta) = t(1 + \alpha \sin^2\theta + \epsilon \sin^4\theta) \quad , \tag{4–164}$$

$\alpha = \alpha(t)$ being a first-order term approximately equal to the flattening f ($\alpha = f + O(f^2)$), and $\epsilon = \epsilon(t)$ being a second-order term of order $f^2 \doteq \alpha^2$. Terms of order higher than two will consistently be neglected. If $t =$ const., then we get the equation of an equisurface, which plainly is of form (4–147).

The above representation is equivalent to a spherical-harmonic expansion to $n = 4$, containing P_2 and P_4 such as (4–11), but it is easier to manipulate for our present purpose. For later reference we form the partial derivatives:

$$\begin{aligned}
r_t &= 1 + (\alpha + t\alpha')\sin^2\theta + (\epsilon + t\epsilon')\sin^4\theta \quad , \\
r_\theta &= t\cos\theta\sin\theta(2\alpha + 4\epsilon\sin^2\theta) \quad , \\
r_{tt} &= (2\alpha' + t\alpha'')\sin^2\theta + (2\epsilon' + t\epsilon'')\sin^4\theta \quad , \\
r_{t\theta} &= 2(\alpha + t\alpha')\cos\theta\sin\theta + 4(\epsilon + t\epsilon')\cos\theta\sin^3\theta \quad , \\
r_{\theta\theta} &= 2t\alpha + (-4t\alpha + 12t\epsilon)\sin^2\theta - 16t\epsilon\sin^4\theta \quad .
\end{aligned} \tag{4–165}$$

The prime denotes differentiation with respect to t:

$$\alpha' = \frac{d\alpha}{dt} \, , \quad \alpha'' = \frac{d^2\alpha}{dt^2} \, , \quad \text{etc.} \tag{4–166}$$

Now it is straightforward though somewhat laborious to substitute the series (4–165) into (4–160), (4–162), and (4–163), consistently neglecting terms of order higher than two. The result is

$$\begin{aligned}
X &= 1 + (2\alpha + 2t\alpha' - 4\alpha^2)\sin^2\theta + \\
&+ (5\alpha^2 + 2t\alpha\alpha' + t^2\alpha'^2 + 2\epsilon + 2t\epsilon')\sin^4\theta \quad ,
\end{aligned} \tag{4–167}$$

$$\begin{aligned}
A &= \frac{2}{t}\Big[1 - 2\alpha + (3\alpha + t\alpha' - 2t\alpha\alpha' - 8\epsilon)\sin^2\theta + \\
&+ (-\alpha^2 + 2t\alpha\alpha' + 10\epsilon + t\epsilon')\sin^4\theta\Big] \quad ,
\end{aligned} \tag{4–168}$$

$$\begin{aligned}
B &= \frac{2}{t}\Big[(t\alpha' + \frac{1}{2}t^2\alpha'' - 2t\alpha\alpha')\sin^2\theta + \\
&+ (t\alpha\alpha' - t^2\alpha'^2 - \frac{1}{2}t^2\alpha\alpha'' - \frac{1}{2}t^3\alpha'\alpha'' + t\epsilon' + \frac{1}{2}t^2\epsilon'')\sin^4\theta\Big] \quad .
\end{aligned} \tag{4–169}$$

The θ-correction. There remains the term C in (4–161), which arises from the difference between Wavre's parameter Θ, which is constant along any specific plumb line, and the spherical polar distance θ which slightly varies along the plumb line.

Consider an arbitrary smooth function

$$F = F^*(t, \Theta) \tag{4–170}$$

expressed in terms of Wavre's parameters t, Θ. On the other hand, our functions have the form

$$F = F(t, \theta) \quad , \tag{4–171}$$

expressed in terms of the polar distance (note that the parameters t, Θ form an orthogonal system whereas t, θ don't). Regarding the system (t, θ) as functions of (t, Θ):

$$\begin{aligned}
t &= t \quad , \\
\theta &= \theta(\Theta, t) \quad ,
\end{aligned} \tag{4–172}$$

we have

$$F = F(t, \theta) = F(t, \theta(\Theta, t)) = F^*(t, \Theta) \quad , \tag{4–173}$$

and hence

$$\frac{\partial F^*}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t} \quad , \tag{4–174}$$

$$\left.\frac{\partial F}{\partial t}\right)_{\Theta=\text{const.}} = \left.\frac{\partial F}{\partial t}\right)_{\theta=\text{const.}} + F_\theta \frac{\partial \theta}{\partial t} \quad , \tag{4–175}$$

in an obvious notation. Thus, in order to get $\partial F/\partial t$ in Wavre's sense, we have to add to $\partial F/\partial t$ in our present sense a "θ-correction".

The factor $\partial\theta/\partial t$ is the change of θ along the normal to the equisurface passing through the point (t, θ) under consideration. It is easily found as follows (Fig. 4.9). The infinitesimal distance PF can be expressed in two ways:

$$-r d\theta = \delta dr \tag{4–176}$$

(we have put the minus sign since in Fig. 4.8 we had taken $r = OP_1$, whereas now

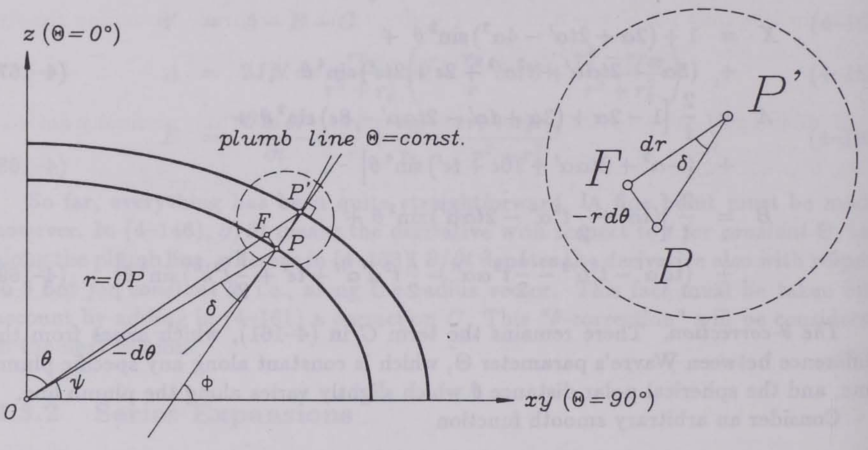

FIGURE 4.9: The θ-correction

$r = OP$; so to speak, in Fig. 4.8 we went from P' to P, whereas in Fig. 4.9 we go from P to P'). Thus

$$\frac{\partial \theta}{\partial r} = -\frac{\delta}{r} \quad , \tag{4–177}$$

where the very small angle δ is nothing else than the difference between the geographic latitude ϕ and the geocentric latitude ψ (Fig. 4.9), which is given by (1–76):

$$\delta = \phi - \psi = 2f\cos\theta\sin\theta \quad , \tag{4–178}$$

neglecting higher-order terms. (This is a standard formula from ellipsoidal geometry: to this accuracy, the level surfaces can be considered ellipsoids of revolution.) To the same accuracy, we may in (4–178) replace r by t, obtaining

$$\frac{\partial\theta}{\partial t} = -2t^{-1} f \cos\theta \sin\theta + O(f^2) \quad . \tag{4–179}$$

Comparing (4–175) with (4–163), we see that in our case

$$F = \ln N \quad , \tag{4–180}$$

so that C represents the θ-correction for B; cf. (4–161) and (4–163). Thus

$$C = \frac{\partial \ln N}{\partial\theta}\frac{\partial\theta}{\partial t} = \frac{1}{2}\frac{\partial \ln N^2}{\partial\theta}\frac{\partial\theta}{\partial t} = \frac{1}{2N^2}\frac{\partial N^2}{\partial\theta}\frac{\partial\theta}{\partial t} \tag{4–181}$$

and finally, by (4–144),

$$C = \frac{1}{2X}\frac{\partial X}{\partial\theta}\frac{\partial\theta}{\partial t} \quad . \tag{4–182}$$

By (4–167), $\partial X/\partial\theta$ will be of order $\alpha \doteq f$, and so is (4–179). So, C will be of order f^2, so that we may put $f = \alpha$ and $X = 1$ without loss of accuracy, obtaining simply

$$C = -(4t^{-1}\alpha^2 + 4\alpha\alpha')(\sin^2\theta - \sin^4\theta) \quad . \tag{4–183}$$

Combining (4–168), (4–169) and (4–183) according to (4–161), we finally get

$$\begin{aligned} Y &= \frac{2}{t}\left[1 - 2\alpha + \left(3\alpha + 2\alpha^2 + 2t\alpha\alpha' - \frac{1}{2}t^2\alpha'' - 8\epsilon\right)\sin^2\theta + \right. \\ &+ \left(-3\alpha^2 - t\alpha\alpha' + t^2\alpha'^2 + \frac{1}{2}t^2\alpha\alpha'' + \frac{1}{2}t^3\alpha'\alpha'' + \right. \\ &+ \left.\left. 10\epsilon - \frac{1}{2}t^2\epsilon''\right)\sin^4\theta\right] \quad . \end{aligned} \tag{4–184}$$

4.3.3 Basic Equations

From (4–173) we find

$$\frac{\partial F}{\partial\Theta} = \frac{\partial F}{\partial\theta}\frac{\partial\theta}{\partial\Theta} \quad . \tag{4–185}$$

For t = const., the factor $\partial\theta/\partial\Theta$ cancels in the numerator and the denominator on the right-hand side of (4–141), so that we also have

$$\Psi(t) = \frac{\partial Y/\partial\theta}{\partial X/\partial\theta} \quad . \tag{4–186}$$

The functions X and Y are given by (4–167) and (4–184), which we write in the form

$$\begin{aligned} X &= 1 + X_1\sin^2\theta + X_2\sin^4\theta \quad , \\ Y &= \frac{2}{t}(Y_0 + Y_1\sin^2\theta + Y_2\sin^4\theta) \quad , \end{aligned} \tag{4–187, 4–188}$$

where the functions X_i and Y_i are series depending on t only. Thus