where

$$\Psi(t) = \frac{f(t)}{W'(t)} = \frac{4\pi G\rho - 2\omega^2}{g_P(t)} \tag{4–142}$$

depends only on the parameter t labeling the equisurfaces and contains the *physics* of the problem: the density ρ, the rotational velocity ω, the potential W and gravity g: we recall that

$$g_P(t) = -W'(t) = -\frac{dW(t)}{dt} \tag{4–143}$$

represents gravity along the rotation axis ($\Theta = 0$).

On the right-hand side of (4–141) we have quantities characterizing the *geometry* of the stratification:

$$X = [N(t, \Theta)]^2 \quad , \tag{4–144}$$

where

$$N = \frac{dn}{dt} \tag{4–145}$$

is a measure of the distance between neighboring equisurfaces, and

$$Y = Y(t, \Theta) = 2JN - \partial \ln N/\partial t \quad , \tag{4–146}$$

J denoting the mean curvature of the equisurfaces.

4.3.1 General Formulas for X and Y

We shall first derive formulas for the quantity N, the mean curvature J, and hence of X and Y, for a general surface of revolution. We use spherical coordinates r, θ, λ. Because of rotational symmetry, there is no dependence on longitude λ, and please distinguish the spherical distance θ from the parameter Θ labeling the plumb lines (sec. 3.2.1).

Let the meridian section (λ = const.) of the surface of revolution have the equation

$$r = r(\theta) \quad . \tag{4–147}$$

By a standard formula which can be found in any text on elementary calculus, the radius of curvature of the meridian in plane polar coordinates r, θ is given by

$$\frac{1}{R_1} = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}} \quad , \tag{4–148}$$

where

$$r_\theta = \frac{\partial r}{\partial \theta}, \qquad r_{\theta\theta} = \frac{\partial^2 r}{\partial \theta^2} \tag{4–149}$$

as usual. This is already one principal radius of curvature for our surface.

The other principal radius is the normal radius of curvature R_2 well-known from ellipsoidal geometry. It is the length of the surface normal from a surface point to its intersection with the rotation axis which for the time being, we take as x-axis (in

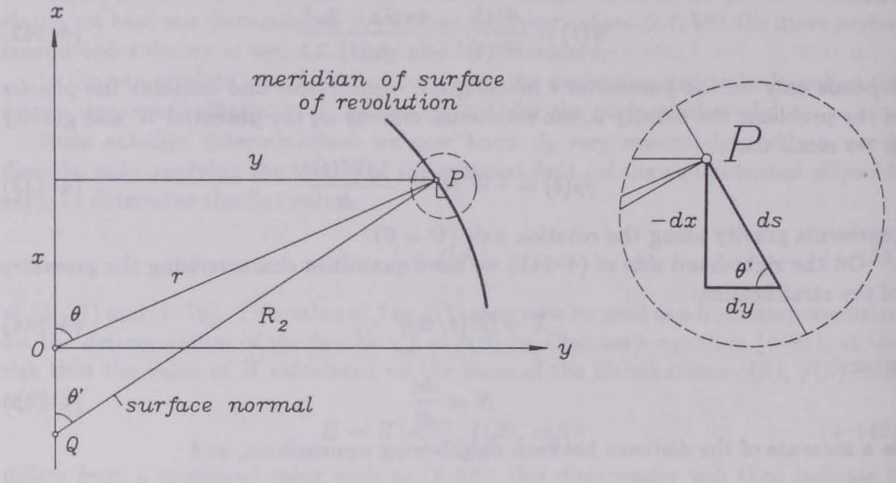

FIGURE 4.7: The normal radius of curvature

order to have $x = r\cos\theta$, $y = r\sin\theta$ as usual for plane polar coordinates). This holds not only for the ellipsoid, but also for an arbitrary surface of revolution; cf. sec. 1.4.

From Fig. 4.7 we read

$$y = r\sin\theta = R_2\sin\theta' \quad ,$$

whence

$$R_2 = r\frac{\sin\theta}{\sin\theta'} \quad . \tag{4–150}$$

The elementary triangle at P, shown in a magnified manner next to the main diagram (Fig. 4.7), gives

$$\sin\theta' = -\frac{dx}{ds} \quad . \tag{4–151}$$

Differentiating $x = r\cos\theta$ we have

$$dx = dr\cos\theta - r\sin\theta d\theta \quad . \tag{4–152}$$

Furthermore,

$$ds^2 = dr^2 + r^2 d\theta^2 \quad . \tag{4–153}$$

In both formulas we put

$$dr = r_\theta d\theta \tag{4–154}$$

by (4–149); in fact, by (4–147), r depends on θ only, so that here

$$r_\theta = \frac{\partial r}{\partial\theta} = \frac{dr}{d\theta} \tag{4–155}$$

(for the sake of generality, we keep the notation $\partial r/\partial\theta$ because later on r will depend on t as well).

In view of (4–154) we may write (4–152) and (4–153) as

$$\begin{aligned} -dx &= r\sin\theta\left(1-\frac{r_\theta}{r}\cot\theta\right)d\theta \quad , \\ ds &= \sqrt{r^2+r_\theta^2}\,d\theta \quad , \end{aligned} \tag{4–156}$$

and substitute into (4–151) and then into (4–150). The result is

$$\frac{1}{R_2}=\frac{1}{\sqrt{r^2+r_\theta^2}}\left(1-\frac{r_\theta}{r}\cot\theta\right) \quad . \tag{4–157}$$

Combining (4–148) and (4–157) we thus have for the mean curvature (1–20)

$$J=\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{2\sqrt{r^2+r_\theta^2}}\left(2-\frac{r_\theta}{r}\cot\theta+\frac{r_\theta^2-rr_{\theta\theta}}{r^2+r_\theta^2}\right) \quad . \tag{4–158}$$

Consider now Wavre's function (4–145),

$$N=\frac{dn}{dt} \quad ,$$

using Fig. 4.8. Along the straight line OP' we obviously have $\theta=$ const., so that

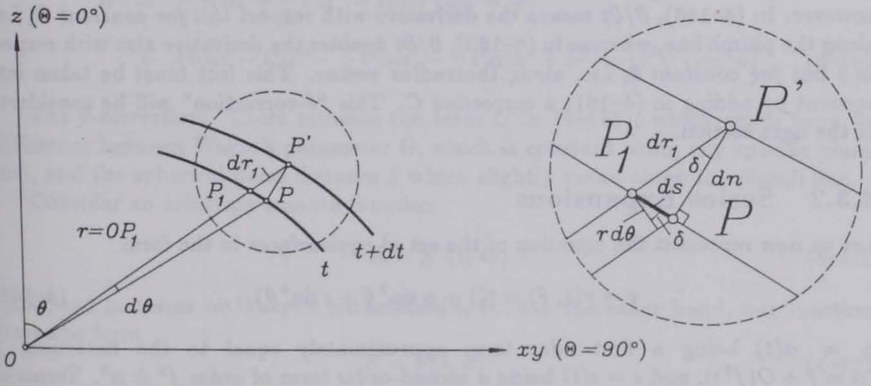

FIGURE 4.8: The distance between two neighboring equisurfaces

$$dr_1=r_t dt=\frac{\partial r}{\partial t}\,dt \quad ,$$

which is the change of r because of t only. From the enlarged part of Fig. 4.8 we read

$$dn=dr_1\cos\delta=r_t dt\cos\delta \quad ,$$

whence

$$N = \frac{dn}{dt} = r_t \cos\delta \quad .$$

On the other hand the enlarged figure shows that

$$\cos\delta = \frac{r\,d\theta}{ds} = \frac{r}{\sqrt{r^2 + r_\theta^2}}$$

by (4–156). Thus we find

$$N = \frac{r r_t}{\sqrt{r^2 + r_\theta^2}} \quad . \tag{4–159}$$

Eqs. (4–158) and (4–159) are basic. Their substitution into (4–144) and (4–146) finally gives

$$X = N^2 = \frac{r^2 r_t^2}{r^2 + r_\theta^2} \quad , \tag{4–160}$$

$$Y = A - B - C \quad , \tag{4–161}$$

$$A = 2JN = \frac{r r_t}{r^2 + r_\theta^2}\left(2 - \frac{r_\theta}{r}\cot\theta + \frac{r_\theta^2 - r r_{\theta\theta}}{r^2 + r_\theta^2}\right) \quad , \tag{4–162}$$

$$B = \frac{\partial \ln N}{\partial t} = \frac{r_t}{r} + \frac{r_{tt}}{r_t} - \frac{r r_t + r_\theta r_{\theta t}}{r^2 + r_\theta^2} \quad . \tag{4–163}$$

So far, everything has been quite straightforward. A fine point must be made, however. In (4–146), $\partial/\partial t$ means the derivative with respect to t *for constant* Θ, i.e., along the plumb line, whereas in (4–163), $\partial/\partial t$ denotes the derivative also with respect to t but *for constant* θ, i.e., along the radius vector. This fact must be taken into account by adding in (4–161) a correction C. This "θ-correction" will be considered in the next section.

4.3.2 Series Expansions

Let us now represent the equation of the set of equisurfaces in the form

$$r = r(t, \theta) = t(1 + \alpha\sin^2\theta + \epsilon\sin^4\theta) \quad , \tag{4–164}$$

$\alpha = \alpha(t)$ being a first-order term approximately equal to the flattening f ($\alpha = f + O(f^2)$), and $\epsilon = \epsilon(t)$ being a second-order term of order $f^2 \doteq \alpha^2$. Terms of order higher than two will consistently be neglected. If t = const., then we get the equation of an equisurface, which plainly is of form (4–147).

The above representation is equivalent to a spherical-harmonic expansion to $n = 4$, containing P_2 and P_4 such as (4–11), but it is easier to manipulate for our present purpose. For later reference we form the partial derivatives: