A final word on the determination of the flattening may be in order. For conceptual clarity we base our discussion on the first-order theory of sec. 2.7, but the more precise second-order theory of sec. 4.2.3 may also be considered.

In the pre-satellite era, J_2 was unknown, so the derivation had to be based on the known dynamical ellipticity H, solving (2–154) for the surface value of f.

From satellite determinations we now know J_2 very accurately and can use it directly, only applying the theory of the external field (of the equipotential ellipsoid, say), to determine the flattening

$$f = f(J_2) \quad ; \tag{4–139}$$

cf. (1–77) and (1–79). This value of $f = f(1)$ may now be used as a boundary condition for the determination of the function $f = f(\beta)$ by Clairaut's equation (4–91), at the risk that the value of H calculated on the basis of the distributions $\rho(\beta)$, $f(\beta)$, and $\kappa(\beta)$:

$$H = H\left[\rho(\beta),\, f(\beta),\, \kappa(\beta)\right] \tag{4–140}$$

differs from a measured value such as (1–85); this discrepancy will then indicate a deviation of the earth from hydrostatic equilibrium. There is an enormous literature on this subject; as examples we mention (Caputo, 1965), (Khan, 1968, 1969), and (Nakiboglu, 1979), with references to earlier work.

Since the surface f is precisely known if J_2 is given, it would, in the author's opinion, be inappropriate not to take it into account. Thus, deliberately ignoring this value and using (2–153), with J_2 and H given (knowing that they may be incompatible in the case of hydrostatic equilibrium!) to calculate a "hydrostatic flattening" f_H (on the order of 1/299 or 1/300), seems to be somewhat artificial.

Recent computations show that the results significantly depend on the choice of density distribution, decreasing the discrepancy between "real" and "hydrostatic" flattening. For a detailed discussion we refer again to (Denis, 1989); from the preprint (Denis, 1985) we quote the final statement: "All in all, it may be worthwhile to study the possibility of deriving a model with a physically plausible density distribution which satisfies the supplementary astrogeodetic constraint that its hydrostatic surface flattening is about 1/298.25, thus agreeing with one of the recommendations issued by the Standard Earth Committee (see Lapwood and Usami, 1981, p. 213)."

4.3 Derivation from Wavre's Theory

The basic differential equations of Clairaut, to a second-order approximation, and of Darwin can also be derived, in an elegant and instructive way, from Wavre's geometric theory described in sec. 3.2.

We start from eq. (3–45) with (3–47):

$$\Psi(t) = \frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta} \quad , \tag{4–141}$$

where

$$\Psi(t) = \frac{f(t)}{W'(t)} = \frac{4\pi G\rho - 2\omega^2}{g_P(t)} \tag{4–142}$$

depends only on the parameter t labeling the equisurfaces and contains the *physics* of the problem: the density ρ, the rotational velocity ω, the potential W and gravity g: we recall that

$$g_P(t) = -W'(t) = -\frac{dW(t)}{dt} \tag{4–143}$$

represents gravity along the rotation axis ($\Theta = 0$).

On the right-hand side of (4–141) we have quantities characterizing the *geometry* of the stratification:

$$X = [N(t,\,\Theta)]^2 \quad , \tag{4–144}$$

where

$$N = \frac{dn}{dt} \tag{4–145}$$

is a measure of the distance between neighboring equisurfaces, and

$$Y = Y(t,\,\Theta) = 2JN - \partial \ln N/\partial t \quad , \tag{4–146}$$

J denoting the mean curvature of the equisurfaces.

4.3.1 General Formulas for X and Y

We shall first derive formulas for the quantity N, the mean curvature J, and hence of X and Y, for a general surface of revolution. We use spherical coordinates r, θ, λ. Because of rotational symmetry, there is no dependence on longitude λ, and please distinguish the spherical distance θ from the parameter Θ labeling the plumb lines (sec. 3.2.1).

Let the meridian section (λ = const.) of the surface of revolution have the equation

$$r = r(\theta) \quad . \tag{4–147}$$

By a standard formula which can be found in any text on elementary calculus, the radius of curvature of the meridian in plane polar coordinates r, θ is given by

$$\frac{1}{R_1} = \frac{r^2 + 2r_\theta^2 - r r_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}} \quad , \tag{4–148}$$

where

$$r_\theta = \frac{\partial r}{\partial \theta}, \qquad r_{\theta\theta} = \frac{\partial^2 r}{\partial \theta^2} \tag{4–149}$$

as usual. This is already one principal radius of curvature for our surface.

The other principal radius is the normal radius of curvature R_2 well-known from ellipsoidal geometry. It is the length of the surface normal from a surface point to its intersection with the rotation axis which for the time being, we take as x-axis (in

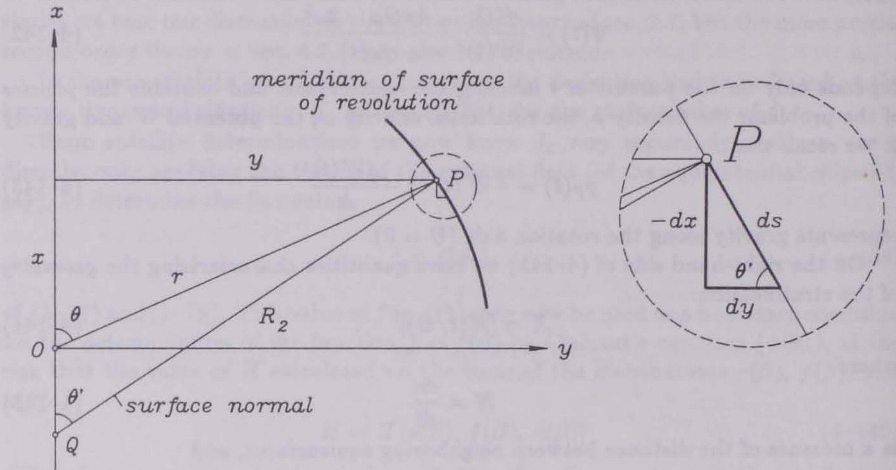

FIGURE 4.7: The normal radius of curvature

order to have $x = r\cos\theta$, $y = r\sin\theta$ as usual for plane polar coordinates). This holds not only for the ellipsoid, but also for an arbitrary surface of revolution; cf. sec. 1.4.

From Fig. 4.7 we read

$$y = r\sin\theta = R_2 \sin\theta' \quad ,$$

whence

$$R_2 = r\frac{\sin\theta}{\sin\theta'} \quad . \tag{4–150}$$

The elementary triangle at P, shown in a magnified manner next to the main diagram (Fig. 4.7), gives

$$\sin\theta' = -\frac{dx}{ds} \quad . \tag{4–151}$$

Differentiating $x = r\cos\theta$ we have

$$dx = dr\cos\theta - r\sin\theta d\theta \quad . \tag{4–152}$$

Furthermore,

$$ds^2 = dr^2 + r^2 d\theta^2 \quad . \tag{4–153}$$

In both formulas we put

$$dr = r_\theta d\theta \tag{4–154}$$

by (4–149); in fact, by (4–147), r depends on θ only, so that here

$$r_\theta = \frac{\partial r}{\partial\theta} = \frac{dr}{d\theta} \tag{4–155}$$

(for the sake of generality, we keep the notation $\partial r/\partial\theta$ because later on r will depend on t as well).

In view of (4–154) we may write (4–152) and (4–153) as

$$\begin{aligned} -dx &= r\sin\theta\left(1-\frac{r_\theta}{r}\cot\theta\right)d\theta \quad , \\ ds &= \sqrt{r^2+r_\theta^2}\,d\theta \quad , \end{aligned} \tag{4–156}$$

and substitute into (4–151) and then into (4–150). The result is

$$\frac{1}{R_2}=\frac{1}{\sqrt{r^2+r_\theta^2}}\left(1-\frac{r_\theta}{r}\cot\theta\right) \quad . \tag{4–157}$$

Combining (4–148) and (4–157) we thus have for the mean curvature (1–20)

$$J=\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{1}{2\sqrt{r^2+r_\theta^2}}\left(2-\frac{r_\theta}{r}\cot\theta+\frac{r_\theta^2-rr_{\theta\theta}}{r^2+r_\theta^2}\right) \quad . \tag{4–158}$$

Consider now Wavre's function (4–145),

$$N=\frac{dn}{dt} \quad ,$$

using Fig. 4.8. Along the straight line OP' we obviously have $\theta=$ const., so that

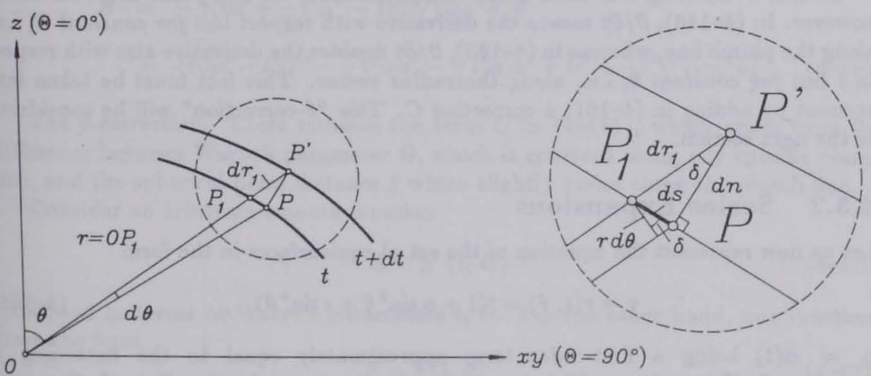

FIGURE 4.8: The distance between two neighboring equisurfaces

$$dr_1=r_t dt=\frac{\partial r}{\partial t}\,dt \quad ,$$

which is the change of r because of t only. From the enlarged part of Fig. 4.8 we read

$$dn=dr_1\cos\delta=r_t dt\cos\delta \quad ,$$

whence

$$N = \frac{dn}{dt} = r_t \cos\delta \quad .$$

On the other hand the enlarged figure shows that

$$\cos\delta = \frac{r\,d\theta}{ds} = \frac{r}{\sqrt{r^2 + r_\theta^2}}$$

by (4–156). Thus we find

$$N = \frac{r r_t}{\sqrt{r^2 + r_\theta^2}} \quad . \tag{4–159}$$

Eqs. (4–158) and (4–159) are basic. Their substitution into (4–144) and (4–146) finally gives

$$X = N^2 = \frac{r^2 r_t^2}{r^2 + r_\theta^2} \quad , \tag{4–160}$$

$$Y = A - B - C \quad , \tag{4–161}$$

$$A = 2JN = \frac{r r_t}{r^2 + r_\theta^2}\left(2 - \frac{r_\theta}{r}\cot\theta + \frac{r_\theta^2 - r r_{\theta\theta}}{r^2 + r_\theta^2}\right) \quad , \tag{4–162}$$

$$B = \frac{\partial \ln N}{\partial t} = \frac{r_t}{r} + \frac{r_{tt}}{r_t} - \frac{r r_t + r_\theta r_{\theta t}}{r^2 + r_\theta^2} \quad . \tag{4–163}$$

So far, everything has been quite straightforward. A fine point must be made, however. In (4–146), $\partial/\partial t$ means the derivative with respect to t *for constant* Θ, i.e., along the plumb line, whereas in (4–163), $\partial/\partial t$ denotes the derivative also with respect to t but *for constant* θ, i.e., along the radius vector. This fact must be taken into account by adding in (4–161) a correction C. This "θ-correction" will be considered in the next section.

4.3.2 Series Expansions

Let us now represent the equation of the set of equisurfaces in the form

$$r = r(t, \theta) = t(1 + \alpha \sin^2\theta + \epsilon \sin^4\theta) \quad , \tag{4–164}$$

$\alpha = \alpha(t)$ being a first-order term approximately equal to the flattening f ($\alpha = f + O(f^2)$), and $\epsilon = \epsilon(t)$ being a second-order term of order $f^2 \doteq \alpha^2$. Terms of order higher than two will consistently be neglected. If $t =$ const., then we get the equation of an equisurface, which plainly is of form (4–147).

The above representation is equivalent to a spherical-harmonic expansion to $n = 4$, containing P_2 and P_4 such as (4–11), but it is easier to manipulate for our present purpose. For later reference we form the partial derivatives:

$$\begin{aligned}
r_t &= 1+(\alpha+t\alpha')\sin^2\theta+(\epsilon+t\epsilon')\sin^4\theta \quad , \\
r_\theta &= t\cos\theta\sin\theta(2\alpha+4\epsilon\sin^2\theta) \quad , \\
r_{tt} &= (2\alpha'+t\alpha'')\sin^2\theta+(2\epsilon'+t\epsilon'')\sin^4\theta \quad , \\
r_{t\theta} &= 2(\alpha+t\alpha')\cos\theta\sin\theta+4(\epsilon+t\epsilon')\cos\theta\sin^3\theta \quad , \\
r_{\theta\theta} &= 2t\alpha+(-4t\alpha+12t\epsilon)\sin^2\theta-16t\epsilon\sin^4\theta \quad .
\end{aligned} \tag{4–165}$$

The prime denotes differentiation with respect to t:

$$\alpha' = \frac{d\alpha}{dt} , \quad \alpha'' = \frac{d^2\alpha}{dt^2} , \quad \text{etc.} \tag{4–166}$$

Now it is straightforward though somewhat laborious to substitute the series (4–165) into (4–160), (4–162), and (4–163), consistently neglecting terms of order higher than two. The result is

$$\begin{aligned}
X &= 1+(2\alpha+2t\alpha'-4\alpha^2)\sin^2\theta+ \\
&+ (5\alpha^2+2t\alpha\alpha'+t^2\alpha'^2+2\epsilon+2t\epsilon')\sin^4\theta \quad ,
\end{aligned} \tag{4–167}$$

$$\begin{aligned}
A &= \frac{2}{t}\left[1-2\alpha+(3\alpha+t\alpha'-2t\alpha\alpha'-8\epsilon)\sin^2\theta+\right. \\
&+ \left.(-\alpha^2+2t\alpha\alpha'+10\epsilon+t\epsilon')\sin^4\theta\right] \quad ,
\end{aligned} \tag{4–168}$$

$$\begin{aligned}
B &= \frac{2}{t}\left[(t\alpha'+\frac{1}{2}t^2\alpha''-2t\alpha\alpha')\sin^2\theta+\right. \\
&+ \left.(t\alpha\alpha'-t^2\alpha'^2-\frac{1}{2}t^2\alpha\alpha''-\frac{1}{2}t^3\alpha'\alpha''+t\epsilon'+\frac{1}{2}t^2\epsilon'')\sin^4\theta\right] \quad .
\end{aligned} \tag{4–169}$$

The θ-correction. There remains the term C in (4–161), which arises from the difference between Wavre's parameter Θ, which is constant along any specific plumb line, and the spherical polar distance θ which slightly varies along the plumb line.

Consider an arbitrary smooth function

$$F = F^*(t, \Theta) \tag{4–170}$$

expressed in terms of Wavre's parameters t, Θ. On the other hand, our functions have the form

$$F = F(t, \theta) \quad , \tag{4–171}$$

expressed in terms of the polar distance (note that the parameters t, Θ form an orthogonal system whereas t, θ don't). Regarding the system (t, θ) as functions of (t, Θ):

$$\begin{aligned}
t &= t \quad , \\
\theta &= \theta(\Theta, t) \quad ,
\end{aligned} \tag{4–172}$$

we have

$$F = F(t, \theta) = F(t, \theta(\Theta, t)) = F^*(t, \Theta) \quad , \tag{4–173}$$

and hence

$$\frac{\partial F^*}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t} \quad , \tag{4–174}$$

$$\left.\frac{\partial F}{\partial t}\right)_{\Theta=\text{const.}} = \left.\frac{\partial F}{\partial t}\right)_{\theta=\text{const.}} + F_\theta \frac{\partial \theta}{\partial t} \quad , \tag{4–175}$$

in an obvious notation. Thus, in order to get $\partial F/\partial t$ in Wavre's sense, we have to add to $\partial F/\partial t$ in our present sense a "θ-correction".

The factor $\partial\theta/\partial t$ is the change of θ along the normal to the equisurface passing through the point $(t,\,\theta)$ under consideration. It is easily found as follows (Fig. 4.9). The infinitesimal distance PF can be expressed in two ways:

$$-\,r d\theta = \delta dr \tag{4–176}$$

(we have put the minus sign since in Fig. 4.8 we had taken $r = OP_1$, whereas now

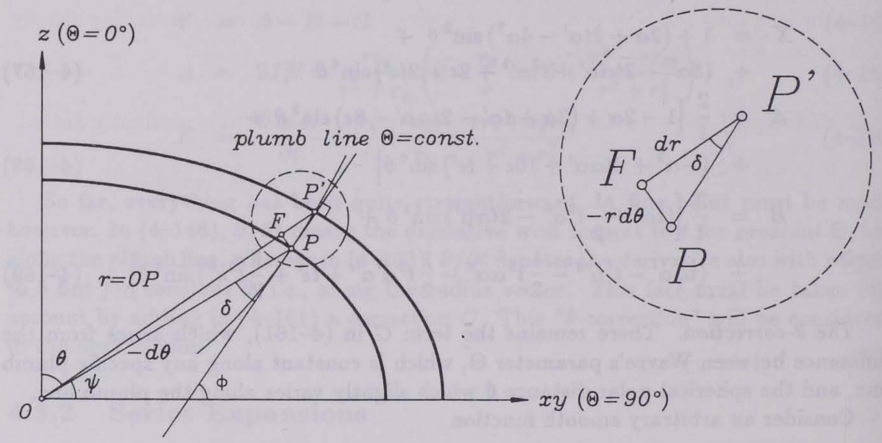

FIGURE 4.9: The θ-correction

$r = OP$; so to speak, in Fig. 4.8 we went from P' to P, whereas in Fig. 4.9 we go from P to P'). Thus

$$\frac{\partial \theta}{\partial r} = -\frac{\delta}{r} \quad , \tag{4–177}$$

where the very small angle δ is nothing else than the difference between the geographic latitude ϕ and the geocentric latitude ψ (Fig. 4.9), which is given by (1–76):

$$\delta = \phi - \psi = 2f\cos\theta\sin\theta \quad , \tag{4–178}$$

neglecting higher-order terms. (This is a standard formula from ellipsoidal geometry: to this accuracy, the level surfaces can be considered ellipsoids of revolution.) To the same accuracy, we may in (4–178) replace r by t, obtaining

$$\frac{\partial\theta}{\partial t} = -2t^{-1} f \cos\theta \sin\theta + O(f^2) \quad . \tag{4–179}$$

Comparing (4–175) with (4–163), we see that in our case

$$F = \ln N \quad , \tag{4–180}$$

so that C represents the θ-correction for B; cf. (4–161) and (4–163). Thus

$$C = \frac{\partial \ln N}{\partial\theta}\frac{\partial\theta}{\partial t} = \frac{1}{2}\frac{\partial \ln N^2}{\partial\theta}\frac{\partial\theta}{\partial t} = \frac{1}{2N^2}\frac{\partial N^2}{\partial\theta}\frac{\partial\theta}{\partial t} \tag{4–181}$$

and finally, by (4–144),

$$C = \frac{1}{2X}\frac{\partial X}{\partial\theta}\frac{\partial\theta}{\partial t} \quad . \tag{4–182}$$

By (4–167), $\partial X/\partial\theta$ will be of order $\alpha \doteq f$, and so is (4–179). So, C will be of order f^2, so that we may put $f = \alpha$ and $X = 1$ without loss of accuracy, obtaining simply

$$C = -(4t^{-1}\alpha^2 + 4\alpha\alpha')(\sin^2\theta - \sin^4\theta) \quad . \tag{4–183}$$

Combining (4–168), (4–169) and (4–183) according to (4–161), we finally get

$$\begin{aligned} Y &= \frac{2}{t}\Big[1 - 2\alpha + \Big(3\alpha + 2\alpha^2 + 2t\alpha\alpha' - \frac{1}{2}t^2\alpha'' - 8\epsilon\Big)\sin^2\theta + \\ &+ \Big(-3\alpha^2 - t\alpha\alpha' + t^2\alpha'^2 + \frac{1}{2}t^2\alpha\alpha'' + \frac{1}{2}t^3\alpha'\alpha'' + \\ &+ \quad 10\epsilon - \frac{1}{2}t^2\epsilon''\Big)\sin^4\theta\Big] \quad . \end{aligned} \tag{4–184}$$

4.3.3 Basic Equations

From (4–173) we find

$$\frac{\partial F}{\partial\Theta} = \frac{\partial F}{\partial\theta}\frac{\partial\theta}{\partial\Theta} \quad . \tag{4–185}$$

For t = const., the factor $\partial\theta/\partial\Theta$ cancels in the numerator and the denominator on the right-hand side of (4–141), so that we also have

$$\Psi(t) = \frac{\partial Y/\partial\theta}{\partial X/\partial\theta} \quad . \tag{4–186}$$

The functions X and Y are given by (4–167) and (4–184), which we write in the form

$$\begin{aligned} X &= 1 + X_1\sin^2\theta + X_2\sin^4\theta \quad , &(4\text{–}187)\\ Y &= \frac{2}{t}(Y_0 + Y_1\sin^2\theta + Y_2\sin^4\theta) \quad , &(4\text{–}188) \end{aligned}$$

where the functions X_i and Y_i are series depending on t only. Thus

$$\begin{aligned}\frac{\partial X}{\partial\theta} &= 2\sin\theta\cos\theta(X_1+2X_2\sin^2\theta) \quad , \\ \frac{\partial Y}{\partial\theta} &= \frac{2}{t}\,2\sin\theta\cos\theta(Y_1+2Y_2\sin^2\theta) \quad ,\end{aligned} \tag{4–189}$$

and (4–186) becomes

$$\frac{1}{2}\,t\Psi(t) = \frac{Y_1+2Y_2\sin^2\theta}{X_1+2X_2\sin^2\theta} \quad . \tag{4–190}$$

Since X_2, $Y_2 \ll X_1$, Y_1, we may again expand:

$$\begin{aligned}\frac{1}{2}\,t\Psi(t) &= \frac{Y_1}{X_1}\left(1+2\frac{Y_2}{Y_1}\sin^2\theta\right)\left(1+2\frac{X_2}{X_1}\sin^2\theta\right)^{-1} = \\ &= \frac{Y_1}{X_1}\left[1+2\left(\frac{Y_2}{Y_1}-\frac{X_2}{X_1}\right)\sin^2\theta+(\ldots)\sin^4\theta+\cdots\right] \quad .\end{aligned} \tag{4–191}$$

Now comes the essential reasoning: since this equation is an identity in θ and since the left-hand side is independent of θ, the right-hand side must also be independent of θ. This requires

$$\frac{Y_2}{Y_1}-\frac{X_2}{X_1}=0 \tag{4–192}$$

and consequently

$$\frac{1}{2}\,t\Psi(t) = \frac{Y_1}{X_1} \quad . \tag{4–193}$$

These are the basic equations for our problem: (4–192) will lead to Darwin's equation, whereas (4–193) will give Clairaut's equation accurate to second order in f. We immediately note that (4–192) corresponds to the condition (3–46) which is "weaker" than (3–45) as we have remarked at the end of sec. 3.2.1. Thus (3–46) is sufficient to derive Darwin's but not Clairaut's equation.

4.3.4 Darwin's Equation

Eq. (4–192) is equivalent to

$$X_1Y_2 - X_2Y_1 = 0 \quad . \tag{4–194}$$

X_1 and X_2 are the terms (truncated series) on the right-hand side of (4–167) multiplied by $\sin^2\theta$ and $\sin^4\theta$, respectively, and similarly for Y_1 and Y_2 with (4–184); cf. (4–187) and (4–188).

We substitute these series into (4–194), keeping terms of order α^3 but neglecting $O(\alpha^4)$. The result is

$$\begin{aligned}&(t^2\alpha+t^3\alpha')\epsilon''+(6t\alpha-t^3\alpha'')\epsilon'-(14\alpha+20t\alpha'+t^2\alpha'')\epsilon = \\ &= -21\alpha^3-14t\alpha^2\alpha'-3t^2\alpha\alpha'^2+2t^3\alpha'^3+ \\ &+\frac{7}{2}\,t^2\alpha^2\alpha''+3t^3\alpha\alpha'\alpha''+\frac{3}{2}\,t^4\alpha'^2\alpha'' \quad .\end{aligned} \tag{4–195}$$

Now we transform to the standard parameters f (flattening) and κ (deviation), contained in eq. (4–3):

$$r = a\left[1 - f\cos^2\theta - \left(\frac{3}{8}f^2 + \kappa\right)\sin^2 2\theta\right] \quad ; \tag{4–196}$$

remember that κ is a second-order quantity which is zero for an exact ellipsoid.

Since $t = b$, the semiminor axis of the equisurface under consideration, and since $b = a(1-f)$, it is easy to transform (4–196) into the form

$$r = t\left[1 + \left(f - \frac{1}{2}f^2 - 4\kappa\right)\sin^2\theta + \left(\frac{3}{2}f^2 + 4\kappa\right)\sin^4\theta\right] \quad , \tag{4–197}$$

which by comparison with (4–164) shows that

$$\alpha = f - \frac{1}{2}f^2 - 4\kappa \quad , \tag{4–198}$$

$$\epsilon = \frac{3}{2}f^2 + 4\kappa \quad , \tag{4–199}$$

confirming the first-order equality

$$\alpha \doteq f \quad . \tag{4–200}$$

This is now used to transform (4–195). It is readily recognized that for ϵ we need (4–199), whereas for α, which is always multiplied by second-order terms, (4–200) is sufficient. The result is

$$\begin{aligned}(t^2 f + t^3 f')\kappa'' &+ (6tf - t^3 f'')\kappa' - (14f + 20tf' + t^2 f'')\kappa = \\ &= -\frac{1}{2}tf^2 f' - \frac{3}{2}t^2 f f'^2 - \frac{1}{4}t^3 f'^3 + \frac{1}{2}t^2 f^2 f'' + \frac{3}{4}t^3 f f' f'' + \frac{3}{8}t^4 f'^2 f''\end{aligned} \tag{4–201}$$

This is a second-order linear ordinary differential equation for the deviation $\kappa = \kappa(t)$, which has extraordinary theoretical interest: It shows that, given the flattening $f = f(t)$ (which implies knowing the derivatives f' and f''), the quantity κ is fully determined (apart from the usual boundary conditions). The density distribution does not enter here!

This is fully in the spirit of Wavre's theory which aims at separating the geometry from the physics to the largest possible extent.

Practically it may be preferable to eliminate f'' by Clairaut's equation (2–114) or (4–124):

$$t^2 f'' = -6\frac{\delta}{D}tf' + 6\left(1 - \frac{\delta}{D}\right)f \quad . \tag{4–202}$$

In the linear approximation we have $t \doteq \beta$, $e \doteq f$; this linear approximation is, of course, sufficient since f'' is multiplied by terms of $O(f^2)$. After some straightforward calculations we thus obtain, the factor $f + tf'$ canceling "miraculously",

$$\begin{aligned}t^2\kappa'' &+ 6\frac{\delta}{D}t\kappa' + \left(-20 + 6\frac{\delta}{D}\right)\kappa = \\ &= 3\left(1 - \frac{\delta}{D}\right)f^2 + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)tff' - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)t^2 f'^2 \quad .\end{aligned} \tag{4–203}$$

Since κ is small of second order, we may again replace the polar radius t by the mean radius β without loss of accuracy:

$$\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa =$$
$$= 3\left(1 - \frac{\delta}{D}\right)f^2 + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta f\dot{f} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{f}^2 . \qquad (4\text{–}204)$$

This is *Darwin's equation* which we already know (eq. (4–123)), but which appears in a new light by the present derivation; clearly f can be replaced by e in the second-order terms on the right-hand side. To repeat: the differential equations (4–204) and (4–201) are equivalent, but (4–204) is practically more useful, whereas (4–201) is theoretically particularly interesting.

4.3.5 Clairaut's Equation

The derivation of Clairaut's equation accurate to $O(f^2)$ starts from (4–193). Using (4–167) and (4–184), taking into account (4–187) and (4–188), we thus can write

$$t\Psi(t) = \frac{3\alpha - \frac{1}{2}t^2\alpha'' + 2\alpha^2 + 2t\alpha\alpha' - 8\epsilon}{\alpha + t\alpha' - 2\alpha^2} . \qquad (4\text{–}205)$$

From (2–104) we take, to first order,

$$\frac{W}{4\pi G} = \frac{1}{\beta}\int_0^\beta \delta \cdot \beta^2 d\beta + \int_\beta^1 \delta \cdot \beta d\beta + \frac{\omega^2\beta^2}{12\pi G} , \qquad (4\text{–}206)$$

where, as usual,

$$\delta = \frac{\rho}{\rho_m} \qquad (4\text{–}207)$$

denotes the dimensionless "normalized density" and β the (normalized) mean radius of the equisurface passing through the point P at which W is considered (the fact that it is also used as an integration variable in our customary way will cause no confusion).

Differentiation gives

$$\frac{1}{4\pi G}\frac{dW}{d\beta} = -\frac{1}{\beta^2}\int_0^\beta \delta \cdot \beta^2 d\beta + \beta\delta - \beta\delta + \frac{\omega^2\beta}{6\pi G} \qquad (4\text{–}208)$$

or, by (4–56),

$$\frac{dW}{d\beta} = -\frac{4\pi G}{3}D\beta + \frac{2}{3}\omega^2\beta . \qquad (4\text{–}209)$$

By definition,

$$\beta = \sqrt[3]{t(1+f) \cdot t(1+f) \cdot t} \qquad (4\text{–}210)$$

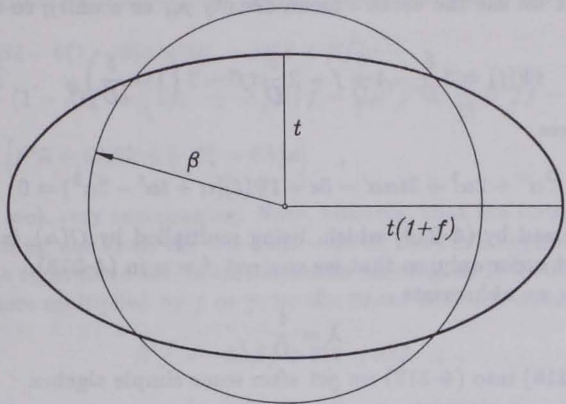

FIGURE 4.10: Polar radius t and mean radius β

is the geometric mean of all three axes (Fig. 4.10). (In a more familiar notation this is $R = \sqrt[3]{a^2 b}$, the sphere being defined as having the same volume as the ellipsoid.) In view of the smallness of f, (4–210) reduces in the linear approximation to

$$\beta = t\left(1 + \frac{2}{3} f\right) \quad , \tag{4–211}$$

$$t = \beta\left(1 - \frac{2}{3} f\right) \quad . \tag{4–212}$$

Hence,

$$W'(t) = \frac{dW}{dt} = \frac{dW}{d\beta}\frac{d\beta}{dt} = \frac{dW}{d\beta}\left(1 + \frac{2}{3} f + \frac{2}{3} t f'\right) \quad . \tag{4–213}$$

Using (4–209) with (4–211), this gives

$$W'(t) = -\frac{4\pi G}{3} Dt\left(1 + \frac{2}{3} f\right)\left(1 + \frac{2}{3} f + \frac{2}{3} t f'\right) + \frac{2}{3}\omega^2 t \tag{4–214}$$

(since $\omega^2 = O(f)$, we have been able simply to replace β by t in the last term).

Introducing the dimensionless quantity (4–66), in the present units

$$\mu = \frac{3}{4\pi G}\frac{\omega^2}{D} \quad , \tag{4–215}$$

which is $O(f)$, we thus have to $O(f)$

$$g_P = -W'(t) = \frac{4\pi G}{3} Dt\left(1 + \frac{4}{3} f + \frac{2}{3} t f' - \frac{2}{3}\mu\right) \quad . \tag{4–216}$$

Now

$$\frac{4\pi G\rho - 2\omega^2}{\frac{4\pi}{3} GtD} = \frac{3}{t}\frac{\delta}{D} - \frac{2}{t}\mu \tag{4–217}$$

(we have $\rho = \delta$ if we use the earth's mean density ρ_m as a unit), so that by (4–142) and (4–216)

$$t\Psi(t) = 3\frac{\delta}{D} - 4\frac{\delta}{D} f - 2\frac{\delta}{D} t f' - 2\left(1 - \frac{\delta}{D}\right)\mu \quad . \tag{4–218}$$

Then (4–205) gives

$$3\alpha - \frac{1}{2} t^2 \alpha'' + 2\alpha^2 + 2t\alpha\alpha' - 8\epsilon - t\Psi(t)(\alpha + t\alpha' - 2\alpha^2) = 0 \tag{4–219}$$

with $t\Psi(t)$ expressed by (4–218) which, being multiplied by $O(\alpha)$, is indeed seen to be needed to first order only, so that we can put $f = \alpha$ in (4–218).

For simplicity we abbreviate

$$\lambda = \frac{\delta}{D} \quad . \tag{4–220}$$

Substituting (4–218) into (4–219) we get after some simple algebra

$$\begin{aligned} t^2\alpha'' + 6\lambda t\alpha' + (-6 + 6\lambda)\alpha &= (4 + 20\lambda) f^2 + (4 + 12\lambda) t f f' + 4\lambda t^2 f'^2 - \\ &- 16\epsilon + 4(1 - \lambda)(f + t f')\mu \end{aligned} \tag{4–221}$$

where, on the right-hand side, we have put $f = \alpha$ because it contains quadratic terms only.

The left-hand side represents the linear Clairaut equation for α, and the right-hand side, rather than being zero, is now $O(f^2)$. Thus (4–221) may already be regarded as some second-order generalization of Clairaut's equation, but it is better to change from α, t to the flattening f and the mean radius β by means of (4–198), (4–199), and (4–212).

The final result becomes still simpler if we use, instead of the flattening f, the "ellipticity"

$$e = f - \frac{5}{42} f^2 + \frac{4}{7}\kappa \tag{4–222}$$

(with $e^2 \doteq f^2$), already introduced in eq. (4–48).

By (4–198), (4–199), and (4–222) we have

$$\alpha = e - \frac{8}{21} e^2 - \frac{32}{7}\kappa \ , \tag{4–223}$$

$$\epsilon = \frac{3}{2} e^2 + 4\kappa \quad . \tag{4–224}$$

This is inserted into (4–221). Furthermore we substitute, from (4–212),

$$t = \beta\left(1 - \frac{2}{3} e\right), \qquad t^2 = \beta^2\left(1 - \frac{4}{3} e\right). \tag{4–225}$$

Finally we replace all derivatives with respect to t by derivatives with respect to β, denoted by a dot as before, cf. eq. (4–78):

$$f' = \frac{df}{dt} = \frac{df}{d\beta}\frac{d\beta}{dt} = \dot{f}\left(1 + \frac{2}{3} f + \frac{2}{3}\beta\dot{f}\right) \quad , \tag{4–226}$$

$$f'' = \ddot{f} + \frac{4}{3} f\ddot{f} + 2\beta\dot{f}\ddot{f} + \frac{4}{3}\dot{f}^2 \quad . \tag{4–227}$$

This is straightforward though somewhat laborious algebra; the result is

$$\begin{aligned}\beta^2\ddot{e}+6\lambda\beta\dot{e}-6(1-\lambda)e&=4(1-\lambda)(f+\beta\dot{f})\mu-\\&-\frac{156}{7}(1-\lambda)f^2+\left(4+\frac{116}{7}\lambda\right)\beta f\dot{f}-\frac{4}{7}\beta^2\dot{f}^2+\frac{16}{21}\beta^2 f\ddot{f}-2\beta^3\dot{f}\ddot{f}+\\&+\frac{32}{7}\left[\beta^2\ddot{\kappa}+6\lambda\beta\dot{\kappa}+(-20+6\lambda)\kappa\right]\quad,\end{aligned}\tag{4–228}$$

which does not look very encouraging. Note, however, that the term between parentheses [ ] is nothing else than the left-hand side of Darwin's equation (4–204). Replacing it by the right-hand side of this equation removes κ. If we do this and finally eliminate $\ddot{f}$, where multiplied by f or $\dot{f}$, by the linear Clairaut equation:

$$\beta^2\ddot{f}=-6\lambda\beta\dot{f}+6(1-\lambda)f\quad,\tag{4–229}$$

which has the same accuracy as (4–202), we get a surprisingly simple result:

$$\begin{aligned}\beta^2\ddot{e}+6\lambda\beta\dot{e}-6(1-\lambda)e\;&=\;-\frac{4}{7}(1-\lambda)(7f^2+6\beta f\dot{f}+3\beta^2\dot{f}^2)+\\&+\;4(1-\lambda)(f+\beta\dot{f})\mu\quad,\end{aligned}\tag{4–230}$$

which is nothing else than our old friend, the second-order Clairaut equation (4–91) with (4–92) or (4–90); note that $e=f$ in second-order terms as usual.

We thus have derived this equation and also Darwin's equation in an alternative geometric way. This method, proceeding from Wavre's theory, is simple and transparent in principle, though the detailed calculations may be laborious. In principle, it is nothing else than an extension of the method of sec. 3.2.5 to second order. It is completely different and independent of the method of sec. 4.2; in particular, it does not use spherical harmonic series with a somewhat difficult convergence behavior.

Generally, the present method may be considered more elementary and direct, avoiding tricky manipulations with spherical harmonics and equally tricky differentiation of integrals. On the other hand it should be noted that we only get the *differential equations* for f and κ, *but not the boundary conditions*. For those people who do not appreciate the esthetic appeal of this Wavre-type approach, it will at least serve as a very useful independent check.