*Level ellipsoid.* If the bounding surface of the equilibrium figure is an ellipsoid of revolution, then

$$\kappa(1) = 0 \quad .$$

Adding this as a boundary condition would result in three boundary conditions: $\kappa(0)$, $\kappa(1)$ and $\dot{\kappa}(1)$, which in general are incompatible for a second-order differential equation. This gives the

*Theorem of Ledersteger*

A level ellipsoid cannot in general be an equilibrium figure.

An exception is the Maclaurin ellipsoid (sec. 5.4) which, however, is homogeneous and in no way similar to the real earth.

This theorem was shown in second-order approximation only, but it will hold *a forteriori* for a rigorous ellipsoid.

The argument is very simple and intuitively convincing, especially in the light of later developments (Chapter 5 and secs. 6.2 and 6.4), which show that the earth is certainly not another exceptional case. A direct proof, going beyond the second-order approximation, would be desirable but seems to be very difficult.

Note that, as a first-order approximation (Clairaut's theory), heterogeneous ellipsoidal earth–like equilibrium figures do exist, but deviations start already in the second order.

### 4.2.5 Practical Comments and Results

The most important and recent pre-satellite determination of the flattening $f$, related to the ellipticity $e$ by (4–48):

$$f = e + \frac{5}{42} e^2 - \frac{4}{7} \kappa \quad , \tag{4–134}$$

and of the deviation $\kappa$ by solving Clairaut's and Darwin's equations was made by Bullard (1948), with modifications by Jones (1954).

Bullard gets the value (4–1), and Jones the closely similar value

$$f^{-1} = 297.300 \pm 0.065 \quad . \tag{4–135}$$

Bullard finds for de Sitter's numerical constants $\lambda_1$ and $\eta_S$ the values

$$\lambda_1 = 0.00016 \pm 0.00018 \quad (!) \quad , \tag{4–136}$$

$$\eta_S = 0.565 \quad , \tag{4–137}$$

and for the surface value of $\kappa$, $\kappa_1 = \kappa(1)$ (not to be confused with (4–132)):

$$\kappa_1 = 68 \times 10^{-8} \quad , \tag{4–138}$$

corresponding to a deviation of the spheroid from the ellipsoid of 4.3 meters at latitude 45° (see Fig. 4.1).

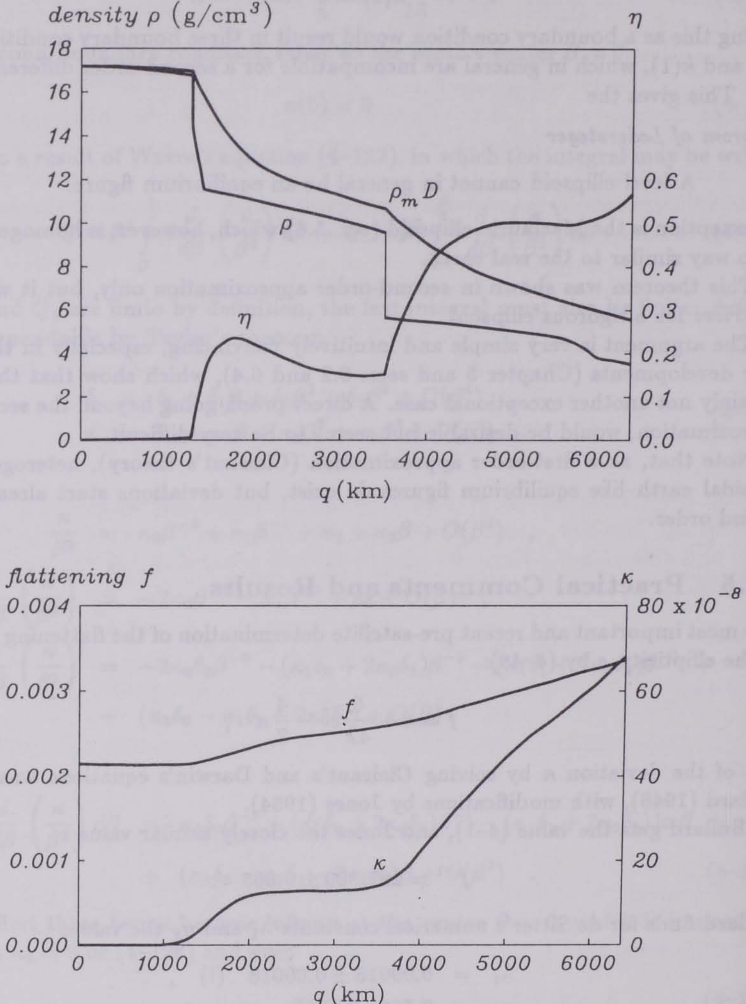


**FIGURE 4.5:** Density $\rho$, mean density $\rho_m D$, and $\eta$ (above) and flattening $f$ and deviation $\kappa$ (below) as a function of the average radius $q = R\beta$ (in kilometers)

Fig. 4.5 shows the distribution of density $\rho$, mean density $\rho_m D$, Radau's parameter $\eta$, flattening $f$, and deviation $\kappa$ in the earth's interior, following Bullard (1948) and Jones (1954). The density model is now obsolete, cf. Fig. 1.7, as well as the surface value for $f$, but the diagrams are nevertheless extremely instructive.

Recent determinations are extensively and carefully discussed in (Denis, 1989). As we have already remarked, instead of solving Clairaut's and Darwin's differential equations, we may also solve corresponding integro-differential equations such as (4–79) and (4–122) by iterative procedures described in (Zharkov and Trubitsyn, 1978, secs. 36 and 37) and in (Denis, 1989); the latter work is an excellent complement of the present book, especially as regards numerical aspects and results; it also contains extensive additional references. A modern counterpart of Fig. 4.5 is Fig. 4.6, following the preprint (Denis, 1985) which was available when the present book was written. The dependence of $f$ on the underlying density model is remarkable.

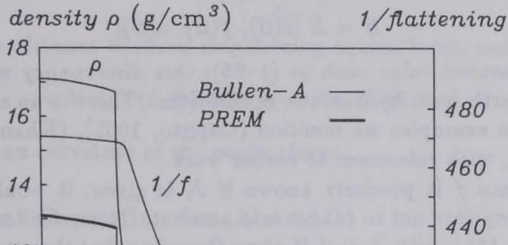


**FIGURE 4.6**: Inverse flattening $f^{-1}$ for two different models of density $\rho$

Modern determinations of $\kappa_1$, comparable to (4–138), lie between 64 and $78 \times 10^{-8}$. So it may be expected that the plot of $\kappa$ in Fig. 4.5 is still reasonably representative.

A final word on the determination of the flattening may be in order. For conceptual clarity we base our discussion on the first-order theory of sec. 2.7, but the more precise second-order theory of sec. 4.2.3 may also be considered.

In the pre-satellite era, $J_2$ was unknown, so the derivation had to be based on the known dynamical ellipticity $H$, solving (2–154) for the surface value of $f$.

From satellite determinations we now know $J_2$ very accurately and can use it directly, only applying the theory of the external field (of the equipotential ellipsoid, say), to determine the flattening

$$f = f(J_2) \quad ; \tag{4–139}$$

cf. (1–77) and (1–79). This value of $f = f(1)$ may now be used as a boundary condition for the determination of the function $f = f(\beta)$ by Clairaut's equation (4–91), at the risk that the value of $H$ calculated on the basis of the distributions $\rho(\beta)$, $f(\beta)$, and $\kappa(\beta)$:

$$H = H\left[\rho(\beta),\, f(\beta),\, \kappa(\beta)\right] \tag{4–140}$$

differs from a measured value such as (1–85); this discrepancy will then indicate a deviation of the earth from hydrostatic equilibrium. There is an enormous literature on this subject; as examples we mention (Caputo, 1965), (Khan, 1968, 1969), and (Nakiboglu, 1979), with references to earlier work.

Since the surface $f$ is precisely known if $J_2$ is given, it would, in the author's opinion, be inappropriate not to take it into account. Thus, deliberately ignoring this value and using (2–153), with $J_2$ and $H$ given (knowing that they may be incompatible in the case of hydrostatic equilibrium!) to calculate a "hydrostatic flattening" $f_H$ (on the order of 1/299 or 1/300), seems to be somewhat artificial.

Recent computations show that the results significantly depend on the choice of density distribution, decreasing the discrepancy between "real" and "hydrostatic" flattening. For a detailed discussion we refer again to (Denis, 1989); from the preprint (Denis, 1985) we quote the final statement: "All in all, it may be worthwhile to study the possibility of deriving a model with a physically plausible density distribution which satisfies the supplementary astrogeodetic constraint that its hydrostatic surface flattening is about 1/298.25, thus agreeing with one of the recommendations issued by the Standard Earth Committee (see Lapwood and Usami, 1981, p. 213)."

## 4.3 Derivation from Wavre's Theory

The basic differential equations of Clairaut, to a second-order approximation, and of Darwin can also be derived, in an elegant and instructive way, from Wavre's geometric theory described in sec. 3.2.

We start from eq. (3–45) with (3–47):

$$\Psi(t) = \frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta} \quad , \tag{4–141}$$