becomes, using (4–107),

$$\frac{J_2}{H} = \frac{2}{3}\left(1 - \frac{2}{3}e\right) - \frac{4}{3}\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta + \frac{2}{3}J_2 + O(e^2) \quad , \tag{4–111}$$

noting that in our units,

$$M = \frac{4}{3}\pi R^3 \rho_m = \frac{4\pi}{3} \tag{4–112}$$

and

$$\frac{C-A}{M} = \frac{C-A}{MR^2} \doteq \frac{C-A}{Ma^2} = J_2 \quad .$$

To the same order we have, by (2–151)

$$J_2 = \frac{2}{3}e - \frac{1}{3}m \tag{4–113}$$

since $e = f + O(f^2)$. Thus (4–111) becomes

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - 2\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta\right] \quad , \tag{4–114}$$

from which we eliminate the integral by (4–102).

Hence

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - \frac{2}{5}\left(1 - \frac{2}{3}e\right)\frac{\sqrt{1+\eta_S}}{1+\lambda_1}\right] \quad . \tag{4–115}$$

For η_S we have by (4–95) and (4–96) with $\beta = 1$,

$$1 + \eta_S = \frac{5}{2}\frac{m}{e} - 1 + \frac{4}{7}e - \frac{6}{7}m + \frac{10}{21}\frac{m^2}{e} \quad . \tag{4–116}$$

Eqs. (4–115) and (4–116) provide the extension of (2–153) to second order (Jones, 1954).

4.2.4 Darwin's Equation

It is now not difficult to derive a differential equation for the deviation $\kappa = \kappa(\beta)$. We start from the equilibrium condition (4–70) with (4–68). This gives the identity

$$(3e^2 - 8\kappa)D - 6eS + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–117}$$

We eliminate S by means of (4–88):

$$S = De - \frac{1}{3}D\beta\dot{e} + O(e^2) \quad , \tag{4–118}$$

obtaining

$$(-3e^2 + 2\beta e\dot{e} - 8\kappa)D + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–119}$$

D, P and Q are given by (4–56). To eliminate P, multiply by β^7 and differentiate. The result, using (4–81), is

$$\begin{aligned}&(2\beta e\ddot{e} - 4e\dot{e} + 2\beta\dot{e}^2 - 8\dot{\kappa})\beta^7 D + \\ &\quad +(2\beta e\dot{e} - 3e^2 - 8\kappa)(4\beta^6 D + 3\beta^6\delta) + \\ &\quad +3\delta\left[7\beta^6\left(e^2 + \frac{8}{9}\kappa\right) + \beta^7\left(2e\dot{e} + \frac{8}{9}\dot{\kappa}\right)\right] + \\ &\quad +\frac{8}{3}\beta^9\delta(2\beta^{-3}\kappa - \beta^{-2}\dot{\kappa}) + 24\beta^6 Q = 0 \quad . \end{aligned} \tag{4–120}$$

Again we eliminate $\ddot{e}$ by Clairaut's equation (4–83). The rest is elementary but cumbersome algebra, leading to the surprisingly simple result

$$2\beta e\dot{e} + \beta^2\dot{e}^2 - 16\kappa - 4\beta\dot{\kappa} + 12D^{-1}Q = 0 \quad , \tag{4–121}$$

which in view of (4–56) gives the beautiful integro-differential *equation of Wavre* (1932, eq. (177)):

$$4(4\kappa + \beta\dot{\kappa}) = \beta\dot{e}(2e + \beta\dot{e}) + 12\frac{\beta^2}{D}\int_\beta^1 \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right)d\beta \quad . \tag{4–122}$$

This equation is extensively studied in Wavre (1932, pp. 109–113).

We shall, however, eliminate also Q. For this purpose we multiply (4–121) by $\beta^{-2}D$ and differentiate. Again we take (4-81) into account and eliminate $\ddot{e}$ by (4–83). The result is *Darwin's equation*

$$\begin{aligned}\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa = 3\left(1 - \frac{\delta}{D}\right)e^2 + \\ +\left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta e\dot{e} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{e}^2 \quad .\end{aligned} \tag{4–123}$$

This equation is not unlike the simple Clairaut equation

$$\beta^2\ddot{e} + 6\frac{\delta}{D}\beta\dot{e} + \left(-6 + 6\frac{\delta}{D}\right)e = 0 \quad , \tag{4–124}$$

but in contrast to (4–124), the right-hand side of (4–123) is not zero: Darwin's equation is *inhomogeneous*. Using Radau's parameter (4–96), the right-hand side of (4–123) takes the slightly simpler form

$$e^2\left[3\left(1 - \frac{\delta}{D}\right) + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\eta - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\eta^2\right] \tag{4–125}$$

(Bullard, 1948; Jones, 1954, p. 12).

Boundary conditions. One boundary condition we get from Wavre's equation (4–122) with $\beta = 1$:

$$\dot{\kappa} = -4\kappa + \frac{1}{2}e\dot{e} + \frac{1}{4}\dot{e}^2 \quad , \tag{4–126}$$

whence by (2–118) with $R = 1$ and $f = e$ on the surface:

$$\dot{\kappa} = -4\kappa - \frac{5}{4}\,e\,m + \frac{25}{16}\,m^2 \quad . \tag{4–127}$$

The second boundary condition refers to the earth's center $\beta = 0$:

$$\kappa(0) = 0 \quad . \tag{4–128}$$

This is also a result of Wavre's equation (4–122), in which the integral may be written

$$Q = \beta^2 \int\limits_{\beta}^{1} \delta \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta = Q_1 - \beta^2 \int\limits_{0}^{\beta} \delta \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta \quad . \tag{4–129}$$

Since Q and Q_1 are finite by definition, the last integral must also be finite. Assume δ and κ expandable by Taylor's theorem

$$\begin{aligned} \delta &= \delta_0 + \delta_1\beta + \delta_2\beta^2 + \delta_3\beta^3 + O(\beta^4) \quad , \\ \kappa &= \kappa_0 + \kappa_1\beta + \kappa_2\beta^2 + \kappa_3\beta^3 + O(\beta^4) \quad . \end{aligned} \tag{4–130}$$

Then

$$\begin{aligned} \frac{\kappa}{\beta^2} &= \kappa_0\beta^{-2} + \kappa_1\beta^{-1} + \kappa_2 + \kappa_3\beta + O(\beta^2) \quad , \\ \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) &= -2\kappa_0\beta^{-3} - \kappa_1\beta^{-2} + \kappa_3 + O(\beta) \quad , \\ \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) &= -2\kappa_0\delta_0\beta^{-3} - (\kappa_1\delta_0 + 2\kappa_0\delta_1)\beta^{-2} - (\kappa_1\delta_1 + 2\kappa_0\delta_2)\beta^{-1} + \\ &+ (\kappa_3\delta_0 - \kappa_1\delta_2 - 2\kappa_0\delta_3) + O(\beta) \end{aligned}$$

and

$$\begin{aligned} \int \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta &= \kappa_0\delta_0\beta^{-2} + (\kappa_1\delta_0 + 2\kappa_0\delta_1)\beta^{-1} - (\kappa_1\delta_1 + 2\kappa_0\delta_2)\ln\beta + \\ &+ (\kappa_3\delta_0 - \kappa_1\delta_2 - 2\kappa_0\delta_3)\beta + O(\beta^2) \quad . \end{aligned} \tag{4–131}$$

Now the first three terms become infinite at the center $\beta = 0$, which is impossible. This gives $\kappa_0 = 0$ or (4–128) and even

$$\kappa_1 = 0 \quad , \tag{4–132}$$

so that the expansion (4–130) must begin with $\kappa_2\beta^2$:

$$\kappa = \kappa_2\beta^2 + O(\beta^3) \quad . \tag{4–133}$$

Note that the boundary conditions for Darwin's equation: $\kappa(0)$ and $\dot{\kappa}(1)$, have a character different from those for Clairaut's equation: $f(1)$ and $\dot{f}(1)$.

Level ellipsoid. If the bounding surface of the equilibrium figure is an ellipsoid of revolution, then

$$\kappa(1) = 0 \quad .$$

Adding this as a boundary condition would result in three boundary conditions: $\kappa(0)$, $\kappa(1)$ and $\dot{\kappa}(1)$, which in general are incompatible for a second-order differential equation. This gives the

Theorem of Ledersteger

A level ellipsoid cannot in general be an equilibrium figure.

An exception is the Maclaurin ellipsoid (sec. 5.4) which, however, is homogeneous and in no way similar to the real earth.

This theorem was shown in second-order approximation only, but it will hold *a forteriori* for a rigorous ellipsoid.

The argument is very simple and intuitively convincing, especially in the light of later developments (Chapter 5 and secs. 6.2 and 6.4), which show that the earth is certainly not another exceptional case. A direct proof, going beyond the second-order approximation, would be desirable but seems to be very difficult.

Note that, as a first-order approximation (Clairaut's theory), heterogeneous ellipsoidal earth–like equilibrium figures do exist, but deviations start already in the second order.

4.2.5 Practical Comments and Results

The most important and recent pre-satellite determination of the flattening f, related to the ellipticity e by (4–48):

$$f = e + \frac{5}{42}e^2 - \frac{4}{7}\kappa \quad , \tag{4–134}$$

and of the deviation κ by solving Clairaut's and Darwin's equations was made by Bullard (1948), with modifications by Jones (1954).

Bullard gets the value (4–1), and Jones the closely similar value

$$f^{-1} = 297.300 \pm 0.065 \quad . \tag{4–135}$$

Bullard finds for de Sitter's numerical constants λ_1 and η_S the values

$$\lambda_1 = 0.00016 \pm 0.00018 \quad (!) \quad , \tag{4–136}$$

$$\eta_S = 0.565 \quad , \tag{4–137}$$

and for the surface value of κ, $\kappa_1 = \kappa(1)$ (not to be confused with (4–132)):

$$\kappa_1 = 68 \times 10^{-8} \quad , \tag{4–138}$$

corresponding to a deviation of the spheroid from the ellipsoid of 4.3 meters at latitude 45° (see Fig. 4.1).