Eliminating S_1 between (4–93) and (4–94) yields

$$\dot{e}\left(1+\frac{4}{7}e-\frac{4}{21}m\right)=\frac{5}{2}m\left(1+\frac{8}{21}e\right)-2e\left(1+\frac{2}{7}e\right) \quad ,$$

which on multiplication by $(1-\frac{4}{7}e+\frac{4}{21}m)$ gives the desired boundary (or initial) condition

$$\dot{e}=\frac{5}{2}m-2e+\frac{4}{7}e^2-\frac{6}{7}em+\frac{10}{21}m^2 \quad . \tag{4–95}$$

This is the second-order equivalent of (2–118).

As the second boundary condition we may regard the surface flattening $f=f(1)$ as given. Furthermore, the ellipticity e must be finite at the earth's center, for $\beta=0$.

4.2.3 Radau's Transformation

Following sec. 2.6, we introduce Radau's parameter η by

$$\eta=\frac{\beta}{e}\frac{de}{d\beta}=\frac{\beta}{e}\dot{e} \quad . \tag{4–96}$$

Substituting

$$\dot{e}=\frac{\eta}{\beta}e, \qquad \ddot{e}=\left(\frac{1}{\beta}\frac{d\eta}{d\beta}+\frac{\eta^2-\eta}{\beta^2}\right)e \tag{4–97}$$

(by (2–123)) into (4–91) and dividing by e gives the second-order Radau equation

$$\beta\frac{d\eta}{d\beta}+\eta^2-\eta-6+6\frac{\delta}{D}(1+\eta)=\frac{4}{7}\left(1-\frac{\delta}{D}\right)\xi \quad , \tag{4–98}$$

where (4–92) takes the simpler form

$$\xi=7\mu(1+\eta)-3e(1+\eta)^2-4e \tag{4–99}$$

in view of (4–97). Following the derivation of sec. 2.6 formula by formula, we get (2–134):

$$\frac{d}{d\beta}\left(D\beta^5\sqrt{1+\eta}\right)=5D\beta^4\psi(\eta) \quad , \tag{4–100}$$

where now

$$\psi(\eta)=(1+\eta)^{-1/2}\left[1+\frac{1}{2}\eta-\frac{1}{10}\eta^2+\frac{2}{35}\left(1-\frac{\delta}{D}\right)\xi\right] \quad , \tag{4–101}$$

which is (2–132) with a small second-order correction. If $1+\lambda_1$ denotes an average value of $\psi(\eta)$ over the range $0\le\beta\le1$, then the integration of (4–100) gives

$$\int_0^1 D\beta^4 d\beta=\frac{1}{5}\frac{\sqrt{1+\eta_s}}{1+\lambda_1} \tag{4–102}$$

since $D(1) = 1$.

Moments of inertia. The sum of the three principal moments of inertia A, A, and C is, by (2–138) and (4–14)

$$2A + C = 2 \iiint (x^2 + y^2 + z^2) \rho \, dv = 2 \iiint r^4 \rho \, dr \, d\sigma \quad . \tag{4–103}$$

We perform the change of variables discussed in sec. 4.1.2 to get constant limits of integration, using (4–18):

$$2A + C = 2 \iiint r^4 \frac{\partial r}{\partial q} \rho(q) dq \, d\sigma \quad . \tag{4–104}$$

If we expand r by (4–50), we immediately see that the first-order terms are removed in view of (2–5), and there remains

$$2A + C = 8\pi \int_0^1 \delta \cdot \beta^4 d\beta + O(e^2) \tag{4–105}$$

in our usual new units. This may be written

$$C = \frac{8\pi}{3} \int_0^1 \delta \cdot \beta^4 d\beta + \frac{2}{3}(C - A) \quad . \tag{4–106}$$

The integral has form (2–141) and may be brought by integration by parts into the form (2–147), so that

$$C = \frac{2}{3} M - \frac{16\pi}{9} \int_0^1 D\beta^4 d\beta + \frac{2}{3}(C - A) \quad ; \tag{4–107}$$

note that we are using units in which, so to speak, $R = 1$ and $\rho_m = 1$. In these units the semimajor axis a is given by (4–46) for $q = 1$ as

$$a = 1 + \frac{1}{3} e + O(e^2) \quad . \tag{4–108}$$

Thus

$$Ma^2 = MR^2 \left(1 + \frac{2}{3} e\right) = \frac{4\pi}{3} \rho_m R^5 \left(1 + \frac{2}{3} e\right) \quad ,$$

which in our units reduces to

$$Ma^2 = \frac{4\pi}{3} \left(1 + \frac{2}{3} e\right) \quad . \tag{4–109}$$

Hence the ratio (2–152),

$$\frac{J_2}{H} = \frac{(C - A)/Ma^2}{(C - A)/C} = \frac{C}{Ma^2} = \frac{C}{MR^2} \left(1 - \frac{2}{3} e\right) = \frac{C}{M} \left(1 - \frac{2}{3} e\right) \tag{4–110}$$

becomes, using (4–107),

$$\frac{J_2}{H} = \frac{2}{3}\left(1 - \frac{2}{3}e\right) - \frac{4}{3}\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta + \frac{2}{3}J_2 + O(e^2) \quad , \tag{4–111}$$

noting that in our units,

$$M = \frac{4}{3}\pi R^3 \rho_m = \frac{4\pi}{3} \tag{4–112}$$

and

$$\frac{C-A}{M} = \frac{C-A}{MR^2} \doteq \frac{C-A}{Ma^2} = J_2 \quad .$$

To the same order we have, by (2–151)

$$J_2 = \frac{2}{3}e - \frac{1}{3}m \tag{4–113}$$

since $e = f + O(f^2)$. Thus (4–111) becomes

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - 2\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta\right] \quad , \tag{4–114}$$

from which we eliminate the integral by (4–102).

Hence

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - \frac{2}{5}\left(1 - \frac{2}{3}e\right)\frac{\sqrt{1+\eta_S}}{1+\lambda_1}\right] \quad . \tag{4–115}$$

For η_S we have by (4–95) and (4–96) with $\beta = 1$,

$$1 + \eta_S = \frac{5}{2}\frac{m}{e} - 1 + \frac{4}{7}e - \frac{6}{7}m + \frac{10}{21}\frac{m^2}{e} \quad . \tag{4–116}$$

Eqs. (4–115) and (4–116) provide the extension of (2–153) to second order (Jones, 1954).

4.2.4 Darwin's Equation

It is now not difficult to derive a differential equation for the deviation $\kappa = \kappa(\beta)$. We start from the equilibrium condition (4–70) with (4–68). This gives the identity

$$(3e^2 - 8\kappa)D - 6eS + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–117}$$

We eliminate S by means of (4–88):

$$S = De - \frac{1}{3}D\beta\dot{e} + O(e^2) \quad , \tag{4–118}$$

obtaining

$$(-3e^2 + 2\beta e\dot{e} - 8\kappa)D + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–119}$$