Eq. (4–63) will not be required later, but we shall need (4–64). For future reference we also calculate

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = \frac{8}{35}\left[\left(\frac{3}{2}\, e^2 - 4\kappa\right) D - 3eS + \frac{3}{2}\, P + \frac{4}{3}\, Q\right] \quad . \tag{4–68}$$

For *hydrostatic equilibrium*, W must be a function of β only, since the surfaces of constant potential are also surfaces of constant density (*equisurfaces*, cf. sec. 2.5). Thus the identities

$$A_2(\beta) = 0 \ , \qquad A_4(\beta) = 0 \ , \tag{4–69}$$

and hence also

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = 0 \tag{4–70}$$

must hold for equilibrium figures.

4.2.2 Clairaut's Equation to Second Order

The condition $A_2(\beta) = 0$ with (4–64) gives immediately

$$D\left(e + \frac{6}{7}e^2\right) - \frac{3}{5}S\left(1 + \frac{4}{7}e\right) - \frac{3}{5}T\left(1 - \frac{8}{21}e\right) = \frac{1}{2}D\mu\left(1 + \frac{20}{21}e\right) \quad . \tag{4–71}$$

Now there comes a trick which will be used several times and which should be kept in mind. To first order (4–71) becomes

$$De - \frac{3}{5}\, S - \frac{3}{5}\, T = \frac{1}{2}\, D\mu + O(e^2) \quad . \tag{4–72}$$

We multiply this expression by $(-4e/7)$ (this is why we need it only to first order!) and add it to (4–71), obtaining

$$D\left(e + \frac{2}{7}\, e^2\right) - \frac{1}{2}\, m - \frac{3}{5}\,(S+T) = \frac{4}{21}\, e(m - 3T) \quad , \tag{4–73}$$

where

$$m = \mu D = \text{const.} \tag{4–74}$$

is the constant (4–67).

Now we must eliminate the two integrals S and T defined by (4–56). This is done by two differentiations, similar but not identical to the procedure in sec. 2.5.

Differentiating (4–56) we easily find

$$\frac{dD}{d\beta} = -3\beta^{-1}(D - \delta) + O(e^2) \quad , \tag{4–75}$$

similar to (2–113) but with a different normalization (our present D is D/ρ_m in sec. 2.5), as well as

$$\frac{dS}{d\beta} = -5\beta^{-1} S + \delta\left[5\beta^{-1}\left(e + \frac{2}{7}\, e^2\right) + \dot{e} + \frac{4}{7}\, e\dot{e}\right] \quad , \tag{4–76}$$

$$\frac{dT}{d\beta} = -\delta\left(\dot{e} + \frac{32}{21}\, e\dot{e}\right) \quad , \tag{4–77}$$

the dot denoting differentiation:

$$\dot{e} = \frac{de}{d\beta} \quad . \tag{4–78}$$

This is substituted into the differentiated equation (4–73), noting that many terms cancel, and multiplied by β. The result is

$$D\left(-3e - \frac{6}{7}e^2 + \beta\dot{e} + \frac{4}{7}\beta e\dot{e}\right) + 3S = \frac{4}{21}\beta\dot{e}(m - 3T) \quad . \tag{4–79}$$

We multiply by β^5 (to eliminate the integral $\beta^5 S$ by differentiation!) and differentiate. After division by β^4 and simplification we thus get

$$\begin{aligned}
&\beta^2\ddot{e}\left[D\left(1 + \frac{4}{7}e\right) - \frac{4}{21}m + \frac{4}{7}T\right] + \\
&+6\beta\dot{e}\left[\delta\left(1 + \frac{4}{7}e\right) - \frac{4}{21}m + \frac{4}{7}T - \frac{2}{63}\beta^2\dot{D}\dot{e}\right] + \\
&+2\beta e\dot{D}\left(1 + \frac{2}{7}e\right) = 0 \quad .
\end{aligned} \tag{4–80}$$

In the process of simplification, the relation (4–75)

$$\dot{D} = -3\beta^{-1}(D - \delta) \tag{4–81}$$

or equivalently,

$$D - \delta = -\frac{1}{3}\beta\dot{D} \ , \qquad D + \frac{1}{3}\beta\dot{D} = \delta \quad , \tag{4–82}$$

have played an essential role. The first-order approximation is sufficient since D is always multiplied by $O(e)$.

Now comes a variant of the trick applied at the very beginning of the present section: to first order, (4–80) reduces to

$$C(\beta) \equiv \beta^2\ddot{e}D + 6\beta\dot{e}\delta + 2\beta e\dot{D} = 0 \quad , \tag{4–83}$$

which, of course, is nothing else than the first-order Clairaut equation (2–114); note (4–82)! The first order is sufficient here for the same reason as above.

We write (4–80) in the form

$$C(\beta) + K(\beta) = 0 \quad , \tag{4–84}$$

$C(\beta)$ denoting Clairaut's equation (4–83) and $K(\beta)$ the remaining second-order terms in (4–80). By (4–83) we get

$$\beta^2\ddot{e}D = -6\beta\dot{e}\delta - 2\beta e\dot{D} \quad , \tag{4–85}$$

which permits us to eliminate $\ddot{e}$ in the second-order $K(\beta)$. The result is

$$K(\beta) = -\frac{4}{7}\beta\dot{D}e^2 - 2\beta\frac{\dot{D}}{D}(e + \beta\dot{e})\left(-\frac{4}{21}m + \frac{4}{7}T\right) - \frac{4}{21}\beta^3\dot{D}\dot{e}^2 \quad . \tag{4–86}$$

To eliminate T, we apply our trick again: (4–72) gives

$$T = \frac{5}{3} De - S - \frac{5}{6} m \quad , \tag{4-87}$$

and (4–79) reduces to first order to

$$-3De + D\beta\dot{e} + 3S = 0 \quad , \tag{4-88}$$

which we solve for S and substitute in (4–87), obtaining

$$T = \frac{2}{3} De + \frac{1}{3} D\beta\dot{e} - \frac{5}{6} m \tag{4-89}$$

to first order, which is sufficient for substitution in (4–86). Thus, after some laborious but straightforward computations we find simply

$$K(\beta) = \frac{4}{7}(D - \delta)\left[7e^2 + 6\beta e\dot{e} + 3\beta^2\dot{e}^2 - 7\mu(e + \beta\dot{e})\right] \quad , \tag{4-90}$$

so that (4–84), with (4–83) and (4–81), becomes

$$\beta^2\ddot{e} + 6\beta\frac{\delta}{D}\dot{e} - 6\left(1 - \frac{\delta}{D}\right)e = \frac{4}{7}\left(1 - \frac{\delta}{D}\right)e\xi \tag{4-91}$$

where, following (Jones, 1954),we have put

$$\xi = 7\mu\left(1 + \beta\frac{\dot{e}}{e}\right) - 3e\left(1 + \beta\frac{\dot{e}}{e}\right)^2 - 4e \quad . \tag{4-92}$$

Eq. (4–91) is the desired Clairaut equation to second order. It is solved iteratively, first solving Clairaut's equation (4–91) with right-hand side zero and then using $e(\beta) \doteq f(\beta)$ so obtained to compute the correction term (4–92) and hence the right-hand side of (4–91). Then the full equation (4–91) can be solved. Numerical methods for solving differential equations (Runge–Kutta etc.) are standard.

Boundary conditions. Two are needed. One is obtained by putting $\beta = 1$, $D = 1$, $T = 0$ in (4–79):

$$-3e - \frac{6}{7}e^2 + \dot{e} + \frac{4}{7}e\dot{e} + 3S_1 - \frac{4}{21}\dot{e}m = 0 \quad . \tag{4-93}$$

Now $S_1 = S(1)$ is found from (4–71) with $\beta = 1$:

$$e + \frac{6}{7}e^2 - \frac{3}{5}S_1\left(1 + \frac{4}{7}e\right) = \frac{1}{2}m\left(1 + \frac{20}{21}e\right) \quad .$$

We multiply by $\left(1 - \frac{4}{7}e\right)$ to obtain ($S = O(e)$!)

$$e + \frac{2}{7}e^2 - \frac{3}{5}S_1 = \frac{1}{2}m\left(1 + \frac{8}{21}e\right) \quad . \tag{4-94}$$

Eliminating S_1 between (4–93) and (4–94) yields

$$\dot{e}\left(1+\frac{4}{7}e-\frac{4}{21}m\right)=\frac{5}{2}m\left(1+\frac{8}{21}e\right)-2e\left(1+\frac{2}{7}e\right) \quad ,$$

which on multiplication by $(1-\frac{4}{7}e+\frac{4}{21}m)$ gives the desired boundary (or initial) condition

$$\dot{e}=\frac{5}{2}m-2e+\frac{4}{7}e^2-\frac{6}{7}em+\frac{10}{21}m^2 \quad . \tag{4–95}$$

This is the second-order equivalent of (2–118).

As the second boundary condition we may regard the surface flattening $f=f(1)$ as given. Furthermore, the ellipticity e must be finite at the earth's center, for $\beta=0$.

4.2.3 Radau's Transformation

Following sec. 2.6, we introduce Radau's parameter η by

$$\eta=\frac{\beta}{e}\frac{de}{d\beta}=\frac{\beta}{e}\dot{e} \quad . \tag{4–96}$$

Substituting

$$\dot{e}=\frac{\eta}{\beta}e, \qquad \ddot{e}=\left(\frac{1}{\beta}\frac{d\eta}{d\beta}+\frac{\eta^2-\eta}{\beta^2}\right)e \tag{4–97}$$

(by (2–123)) into (4–91) and dividing by e gives the second-order Radau equation

$$\beta\frac{d\eta}{d\beta}+\eta^2-\eta-6+6\frac{\delta}{D}(1+\eta)=\frac{4}{7}\left(1-\frac{\delta}{D}\right)\xi \quad , \tag{4–98}$$

where (4–92) takes the simpler form

$$\xi=7\mu(1+\eta)-3e(1+\eta)^2-4e \tag{4–99}$$

in view of (4–97). Following the derivation of sec. 2.6 formula by formula, we get (2–134):

$$\frac{d}{d\beta}\left(D\beta^5\sqrt{1+\eta}\right)=5D\beta^4\psi(\eta) \quad , \tag{4–100}$$

where now

$$\psi(\eta)=(1+\eta)^{-1/2}\left[1+\frac{1}{2}\eta-\frac{1}{10}\eta^2+\frac{2}{35}\left(1-\frac{\delta}{D}\right)\xi\right] \quad , \tag{4–101}$$

which is (2–132) with a small second-order correction. If $1+\lambda_1$ denotes an average value of $\psi(\eta)$ over the range $0\le\beta\le1$, then the integration of (4–100) gives

$$\int_0^1 D\beta^4 d\beta=\frac{1}{5}\frac{\sqrt{1+\eta_s}}{1+\lambda_1} \tag{4–102}$$