$$\begin{aligned} V(P) = V(q, \theta) &= \frac{K_0(q)}{r} + L_0(q) + \\ &+ \left[\frac{K_2(q)}{r^3} + r^2 L_2(q)\right] P_2(\cos\theta) + \\ &+ \left[\frac{K_4(q)}{r^5} + r^4 L_4(q)\right] P_4(\cos\theta) \quad . \end{aligned} \tag{4-53}$$

Here $r$ and $\theta$ denote the spherical coordinates of the internal point $P$; the surface of constant density passing through $P$ bears the label $q$ (Fig. 4.2).

This reasoning also holds for $n > 4$: we are *working with convergent series only*. Thus we have achieved very simply the same result which Wavre has obtained by means of his very complicated "procédé uniforme". Quite another question is whether the *resulting* series is convergent. We have avoided this question by the simple (and usual) trick of limiting ourselves to the second-order (in $f$) approximation only, which automatically disregards higher-order terms.

Still the question remains open as a theoretical problem: the convergence of a spherical harmonic series at the boundary surface $S_P$. Nowadays we know much more about the convergence problem of spherical harmonic series than, say, twenty years ago; cf. (Moritz, 1980, secs. 6 and 7), especially the Runge–Krarup theorem. There may also be a relation to the existence proof by Liapunov and Lichtenstein mentioned in sec. 3.1. Another approach due to Trubitsyn is outlined in (Zharkov and Trubitsyn, 1978, sec. 38) and in (Denis, 1989).

The correctness of our second-order theory, however, is fully confirmed also by its derivation from Wavre's geometric theory to be treated in sec. 4.3, which is based on a completely different approach independent of any spherical-harmonic expansions.

## 4.2 Clairaut's and Darwin's Equations

### 4.2.1 Internal Gravity Potential

Following de Sitter (1924) we normalize the mean radius $q$ and the density $\rho$ by introducing the dimensionless quantities

$$\beta = \frac{q}{R} = \frac{\text{mean radius of } S_P}{\text{mean radius of earth}} \tag{4-54}$$

and

$$\delta = \frac{\rho}{\rho_m} = \frac{\text{density}}{\text{mean density of earth}} \quad . \tag{4-55}$$

The standard auxiliary expressions

$$
\begin{aligned}
D &= \beta^{-3} \int_0^\beta \delta \frac{d}{d\beta} \left[ \left( 1 + \frac{4}{15} e^2 \right) \beta^3 \right] d\beta \quad , \\
S &= \beta^{-5} \int_0^\beta \delta \frac{d}{d\beta} \left[ \left( e + \frac{2}{7} e^2 \right) \beta^5 \right] d\beta \quad , \\
T &= \int_\beta^1 \delta \frac{d}{d\beta} \left[ e + \frac{16}{21} e^2 \right] d\beta \quad , \qquad (4\text{–}56) \\
P &= \beta^{-7} \int_0^\beta \delta \frac{d}{d\beta} \left[ \left( e^2 + \frac{8}{9} \kappa \right) \beta^7 \right] d\beta \quad , \\
Q &= \beta^2 \int_\beta^1 \delta \frac{d}{d\beta} \left[ \kappa \beta^{-2} \right] d\beta
\end{aligned}
$$

will then be very convenient. The symbol $D = D(\beta)$ now denotes the mean density (divided by $\rho_m$, dimensionless!) of the masses enclosed by the equidensity surface labeled by $\beta$; by definition

$$D(1) = 1 \qquad (4\text{–}57)$$

(this is easily verified by specializing eq. (4–58) below for $\beta = 1$). Generally, the quantities (4–56) are identical to (4–52), up to conventional factors ($L_0$ is equivalent to $E$ given below).

Using these expressions, we may write (4–53) in the form

$$
\begin{aligned}
V(P) &= \frac{4\pi}{3} G \rho_m R^2 \beta^3 \left[ \frac{1}{r} D + \frac{3}{2} \frac{1}{\beta} E - \right. \\
&- \frac{2}{5} \left( \frac{\beta^2}{r^3} S + \frac{r^2}{\beta^3} T \right) P_2(\cos\theta) + \\
&+ \left. \frac{12}{35} \left( \frac{\beta^4}{r^5} P + \frac{8}{9} \frac{r^4}{\beta^5} Q \right) P_4(\cos\theta) \right] \quad , \qquad (4\text{–}58)
\end{aligned}
$$

where, of course, $r$ is also normalized or dimensionless with $R$ as unit. The quantity

$$E = \frac{1}{\beta^2} \int_\beta^1 \delta \frac{d}{d\beta} \left[ \left( 1 + \frac{4}{45} e^2 \right) \beta^2 \right] d\beta \qquad (4\text{–}59)$$

is less important and has, therefore, not been included in the standard list (4–56).

Eq. (4–58) has a nice "pseudo-harmonic" form characterized by the powers $r^{-(n+1)}$ and $r^n$, but, of course, $V$ is not harmonic but satisfies Poisson's equation $\Delta V = -4\pi G \rho$. It is therefore appropriate, to eliminate $r$ by means of (4–50), also using (4–37):

$$
\begin{aligned}
\frac{1}{r} &= \beta^{-1}\left[1+\frac{4}{45}e^2+\frac{2}{3}\left(e+\frac{6}{7}e^2\right)P_2-\frac{4}{35}(e^2+8\kappa)P_4\right] \quad , \\
\frac{r^2}{\beta^3} &= \beta^{-1}\left(1-\frac{4}{3}eP_2\right)+O(e^2) \quad , \\
\frac{\beta^2}{r^3} &= \beta^{-1}(1+2eP_2)+O(e^2) \quad , \\
\frac{\beta^4}{r^5} &= \frac{r^4}{\beta^5}=\beta^{-1}+O(e) \quad .
\end{aligned}
\tag{4–60}
$$

We have given these expressions only to the accuracy to which they are needed: $S$ and $T$ are $O(e)$, and $P$ and $Q$ are $O(e^2)$, as (4–56) shows. Eqs. (4–60) are substituted into (4–58), after adding the centrifugal potential

$$\Phi = \frac{1}{2}\omega^2(x^2+y^2) = \frac{1}{3}\omega^2R^2r^2\left[1-P_2(\cos\theta)\right] \tag{4–61}$$

by (2–101), with $Rr$ instead of $r$ because of normalization.

The result, after simplification, may be written as

$$W(\beta,\theta) = \frac{4\pi}{3}G\rho_m R^2\beta^2\left[A_0(\beta)+A_2(\beta)P_2(\cos\theta)+A_4(\beta)P_4(\cos\theta)\right] \tag{4–62}$$

where

$$
\begin{aligned}
A_0(\beta) &= D\left(1+\frac{1}{3}\mu+\frac{4}{45}e^2+\frac{4}{45}e\mu\right)+\frac{3}{2}E \; - \\
&- \frac{4}{25}eS+\frac{8}{75}eT \quad ,
\end{aligned}
\tag{4–63}
$$

$$
\begin{aligned}
A_2(\beta) &= -\frac{2}{3}\left[-D\left(e+\frac{6}{7}e^2\right)+\frac{3}{5}S\left(1+\frac{4}{7}e\right)+\right. \\
&+ \left.\frac{3}{5}T\left(1-\frac{8}{21}e\right)+\frac{1}{2}D\mu\left(1+\frac{20}{21}e\right)\right] \quad ,
\end{aligned}
\tag{4–64}
$$

$$
\begin{aligned}
A_4(\beta) &= \frac{8}{35}\left[-\left(\frac{1}{2}e^2+4\kappa\right)D-\frac{9}{5}eS+\right. \\
&+ \left.\frac{6}{5}eT+\frac{3}{2}P+\frac{4}{3}Q+e\mu D\right] \quad .
\end{aligned}
\tag{4–65}
$$

Here $\mu$ denotes

$$\mu = \frac{\omega^2R^3}{GMD} = \frac{m}{D} \quad , \tag{4–66}$$

where

$$m = \frac{\omega^2R^3}{GM} \tag{4–67}$$

is the constant (1–83) since, by definition, $R=\sqrt[3]{a^2b}$, and $D=D(\beta)$ is the normalized mean density given by the system (4–56) which also furnishes $S$, $T$, $P$, and $Q$.

Eq. (4–63) will not be required later, but we shall need (4–64). For future reference we also calculate

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = \frac{8}{35}\left[\left(\frac{3}{2}\, e^2 - 4\kappa\right) D - 3eS + \frac{3}{2}\, P + \frac{4}{3}\, Q\right] \quad . \tag{4–68}$$

For *hydrostatic equilibrium*, $W$ must be a function of $\beta$ only, since the surfaces of constant potential are also surfaces of constant density (*equisurfaces*, cf. sec. 2.5). Thus the identities

$$A_2(\beta) = 0 \;, \qquad A_4(\beta) = 0 \;, \tag{4–69}$$

and hence also

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = 0 \tag{4–70}$$

must hold for equilibrium figures.

### 4.2.2 Clairaut's Equation to Second Order

The condition $A_2(\beta) = 0$ with (4–64) gives immediately

$$D\left(e + \frac{6}{7}\, e^2\right) - \frac{3}{5} S\left(1 + \frac{4}{7}\, e\right) - \frac{3}{5} T\left(1 - \frac{8}{21}\, e\right) = \frac{1}{2} D\mu\left(1 + \frac{20}{21}\, e\right) \quad . \tag{4–71}$$

Now there comes a trick which will be used several times and which should be kept in mind. To first order (4–71) becomes

$$De - \frac{3}{5}\, S - \frac{3}{5}\, T = \frac{1}{2}\, D\mu + O(e^2) \quad . \tag{4–72}$$

We multiply this expression by $(-4e/7)$ (this is why we need it only to first order!) and add it to (4–71), obtaining

$$D\left(e + \frac{2}{7}\, e^2\right) - \frac{1}{2}\, m - \frac{3}{5}\,(S + T) = \frac{4}{21}\, e(m - 3T) \quad , \tag{4–73}$$

where

$$m = \mu D = \text{const.} \tag{4–74}$$

is the constant (4–67).

Now we must eliminate the two integrals $S$ and $T$ defined by (4–56). This is done by two differentiations, similar but not identical to the procedure in sec. 2.5.

Differentiating (4–56) we easily find

$$\frac{dD}{d\beta} = -3\beta^{-1}(D - \delta) + O(e^2) \quad , \tag{4–75}$$

similar to (2–113) but with a different normalization (our present $D$ is $D/\rho_m$ in sec. 2.5), as well as

$$\frac{dS}{d\beta} = -5\beta^{-1} S + \delta\left[5\beta^{-1}\left(e + \frac{2}{7}\, e^2\right) + \dot{e} + \frac{4}{7}\, e\dot{e}\right] \quad , \tag{4–76}$$

$$\frac{dT}{d\beta} = -\delta\left(\dot{e} + \frac{32}{21}\, e\dot{e}\right) \quad , \tag{4–77}$$