$$\begin{aligned} V(P) = V(q, \theta) \ &= \ \frac{K_0(q)}{r} + L_0(q) \ + \\ &+ \ \left[\frac{K_2(q)}{r^3} + r^2 L_2(q)\right] P_2(\cos\theta) \ + \\ &+ \ \left[\frac{K_4(q)}{r^5} + r^4 L_4(q)\right] P_4(\cos\theta) \quad . \end{aligned} \tag{4–53}$$

Here r and θ denote the spherical coordinates of the internal point P; the surface of constant density passing through P bears the label q (Fig. 4.2).

This reasoning also holds for $n > 4$: we are *working with convergent series only*. Thus we have achieved very simply the same result which Wavre has obtained by means of his very complicated "procédé uniforme". Quite another question is whether the *resulting* series is convergent. We have avoided this question by the simple (and usual) trick of limiting ourselves to the second-order (in f) approximation only, which automatically disregards higher-order terms.

Still the question remains open as a theoretical problem: the convergence of a spherical harmonic series at the boundary surface S_P. Nowadays we know much more about the convergence problem of spherical harmonic series than, say, twenty years ago; cf. (Moritz, 1980, secs. 6 and 7), especially the Runge–Krarup theorem. There may also be a relation to the existence proof by Liapunov and Lichtenstein mentioned in sec. 3.1. Another approach due to Trubitsyn is outlined in (Zharkov and Trubitsyn, 1978, sec. 38) and in (Denis, 1989).

The correctness of our second-order theory, however, is fully confirmed also by its derivation from Wavre's geometric theory to be treated in sec. 4.3, which is based on a completely different approach independent of any spherical-harmonic expansions.

4.2 Clairaut's and Darwin's Equations

4.2.1 Internal Gravity Potential

Following de Sitter (1924) we normalize the mean radius q and the density ρ by introducing the dimensionless quantities

$$\beta = \frac{q}{R} = \frac{\text{mean radius of } S_P}{\text{mean radius of earth}} \tag{4–54}$$

and

$$\delta = \frac{\rho}{\rho_m} = \frac{\text{density}}{\text{mean density of earth}} \quad . \tag{4–55}$$

The standard auxiliary expressions

$$D = \beta^{-3} \int_0^\beta \delta \frac{d}{d\beta} \left[\left(1 + \frac{4}{15} e^2 \right) \beta^3 \right] d\beta \quad ,$$

$$S = \beta^{-5} \int_0^\beta \delta \frac{d}{d\beta} \left[\left(e + \frac{2}{7} e^2 \right) \beta^5 \right] d\beta \quad ,$$

$$T = \int_\beta^1 \delta \frac{d}{d\beta} \left[e + \frac{16}{21} e^2 \right] d\beta \quad , \tag{4–56}$$

$$P = \beta^{-7} \int_0^\beta \delta \frac{d}{d\beta} \left[\left(e^2 + \frac{8}{9} \kappa \right) \beta^7 \right] d\beta \quad ,$$

$$Q = \beta^2 \int_\beta^1 \delta \frac{d}{d\beta} \left[\kappa \beta^{-2} \right] d\beta$$

will then be very convenient. The symbol $D = D(\beta)$ now denotes the mean density (divided by ρ_m, dimensionless!) of the masses enclosed by the equidensity surface labeled by β; by definition

$$D(1) = 1 \tag{4–57}$$

(this is easily verified by specializing eq. (4–58) below for $\beta = 1$). Generally, the quantities (4–56) are identical to (4–52), up to conventional factors (L_0 is equivalent to E given below).

Using these expressions, we may write (4–53) in the form

$$\begin{aligned} V(P) &= \frac{4\pi}{3} G \rho_m R^2 \beta^3 \left[\frac{1}{r} D + \frac{3}{2} \frac{1}{\beta} E - \right. \\ &- \frac{2}{5} \left(\frac{\beta^2}{r^3} S + \frac{r^2}{\beta^3} T \right) P_2(\cos\theta) + \\ &+ \left. \frac{12}{35} \left(\frac{\beta^4}{r^5} P + \frac{8}{9} \frac{r^4}{\beta^5} Q \right) P_4(\cos\theta) \right] \quad , \end{aligned} \tag{4–58}$$

where, of course, r is also normalized or dimensionless with R as unit. The quantity

$$E = \frac{1}{\beta^2} \int_\beta^1 \delta \frac{d}{d\beta} \left[\left(1 + \frac{4}{45} e^2 \right) \beta^2 \right] d\beta \tag{4–59}$$

is less important and has, therefore, not been included in the standard list (4–56).

Eq. (4–58) has a nice "pseudo-harmonic" form characterized by the powers $r^{-(n+1)}$ and r^n, but, of course, V is not harmonic but satisfies Poisson's equation $\Delta V = -4\pi G \rho$. It is therefore appropriate, to eliminate r by means of (4–50), also using (4–37):

$$\begin{aligned} \frac{1}{r} &= \beta^{-1}\left[1+\frac{4}{45}e^2+\frac{2}{3}\left(e+\frac{6}{7}e^2\right)P_2-\frac{4}{35}(e^2+8\kappa)P_4\right] \quad , \\ \frac{r^2}{\beta^3} &= \beta^{-1}\left(1-\frac{4}{3}eP_2\right)+O(e^2) \quad , \\ \frac{\beta^2}{r^3} &= \beta^{-1}(1+2eP_2)+O(e^2) \quad , \\ \frac{\beta^4}{r^5} &= \frac{r^4}{\beta^5}=\beta^{-1}+O(e) \quad . \end{aligned} \tag{4-60}$$

We have given these expressions only to the accuracy to which they are needed: S and T are $O(e)$, and P and Q are $O(e^2)$, as (4–56) shows. Eqs. (4–60) are substituted into (4–58), after adding the centrifugal potential

$$\Phi=\frac{1}{2}\omega^2(x^2+y^2)=\frac{1}{3}\omega^2R^2r^2\left[1-P_2(\cos\theta)\right] \tag{4-61}$$

by (2–101), with Rr instead of r because of normalization.

The result, after simplification, may be written as

$$W(\beta,\theta)=\frac{4\pi}{3}G\rho_mR^2\beta^2\left[A_0(\beta)+A_2(\beta)P_2(\cos\theta)+A_4(\beta)P_4(\cos\theta)\right] \tag{4-62}$$

where

$$\begin{aligned} A_0(\beta) &= D\left(1+\frac{1}{3}\mu+\frac{4}{45}e^2+\frac{4}{45}e\mu\right)+\frac{3}{2}E\ - \\ &- \frac{4}{25}eS+\frac{8}{75}eT \quad , \end{aligned} \tag{4-63}$$

$$\begin{aligned} A_2(\beta) &= -\frac{2}{3}\left[-D\left(e+\frac{6}{7}\cdot e^2\right)+\frac{3}{5}S\left(1+\frac{4}{7}e\right)+\right. \\ &+ \left.\frac{3}{5}T\left(1-\frac{8}{21}e\right)+\frac{1}{2}D\mu\left(1+\frac{20}{21}e\right)\right] \quad , \end{aligned} \tag{4-64}$$

$$\begin{aligned} A_4(\beta) &= \frac{8}{35}\left[-\left(\frac{1}{2}e^2+4\kappa\right)D-\frac{9}{5}eS+\right. \\ &+ \left.\frac{6}{5}eT+\frac{3}{2}P+\frac{4}{3}Q+e\mu D\right] \quad . \end{aligned} \tag{4-65}$$

Here μ denotes

$$\mu=\frac{\omega^2R^3}{GMD}=\frac{m}{D} \quad , \tag{4-66}$$

where

$$m=\frac{\omega^2R^3}{GM} \tag{4-67}$$

is the constant (1–83) since, by definition, $R=\sqrt[3]{a^2b}$, and $D=D(\beta)$ is the normalized mean density given by the system (4–56) which also furnishes S, T, P, and Q.

Eq. (4–63) will not be required later, but we shall need (4–64). For future reference we also calculate

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = \frac{8}{35}\left[\left(\frac{3}{2}\,e^2 - 4\kappa\right) D - 3eS + \frac{3}{2}\,P + \frac{4}{3}\,Q\right] \quad . \tag{4–68}$$

For *hydrostatic equilibrium*, W must be a function of β only, since the surfaces of constant potential are also surfaces of constant density (*equisurfaces*, cf. sec. 2.5). Thus the identities

$$A_2(\beta) = 0 \ , \qquad A_4(\beta) = 0 \ , \tag{4–69}$$

and hence also

$$A_4(\beta) + \frac{24}{35}\, eA_2(\beta) = 0 \tag{4–70}$$

must hold for equilibrium figures.

4.2.2 Clairaut's Equation to Second Order

The condition $A_2(\beta) = 0$ with (4–64) gives immediately

$$D\left(e + \frac{6}{7}\,e^2\right) - \frac{3}{5}S\left(1 + \frac{4}{7}\,e\right) - \frac{3}{5}T\left(1 - \frac{8}{21}\,e\right) = \frac{1}{2}D\mu\left(1 + \frac{20}{21}\,e\right) \quad . \tag{4–71}$$

Now there comes a trick which will be used several times and which should be kept in mind. To first order (4–71) becomes

$$De - \frac{3}{5}\,S - \frac{3}{5}\,T = \frac{1}{2}\,D\mu + O(e^2) \quad . \tag{4–72}$$

We multiply this expression by $(-4e/7)$ (this is why we need it only to first order!) and add it to (4–71), obtaining

$$D\left(e + \frac{2}{7}\,e^2\right) - \frac{1}{2}\,m - \frac{3}{5}\,(S+T) = \frac{4}{21}\,e(m - 3T) \quad , \tag{4–73}$$

where

$$m = \mu D = \text{const.} \tag{4–74}$$

is the constant (4–67).

Now we must eliminate the two integrals S and T defined by (4–56). This is done by two differentiations, similar but not identical to the procedure in sec. 2.5.

Differentiating (4–56) we easily find

$$\frac{dD}{d\beta} = -3\beta^{-1}(D - \delta) + O(e^2) \quad , \tag{4–75}$$

similar to (2–113) but with a different normalization (our present D is D/ρ_m in sec. 2.5), as well as

$$\frac{dS}{d\beta} = -5\beta^{-1}S + \delta\left[5\beta^{-1}\left(e + \frac{2}{7}\,e^2\right) + \dot{e} + \frac{4}{7}\,e\dot{e}\right] \quad , \tag{4–76}$$

$$\frac{dT}{d\beta} = -\delta\left(\dot{e} + \frac{32}{21}\,e\dot{e}\right) \quad , \tag{4–77}$$

the dot denoting differentiation:

$$\dot{e} = \frac{de}{d\beta} \quad . \tag{4–78}$$

This is substituted into the differentiated equation (4–73), noting that many terms cancel, and multiplied by β. The result is

$$D\left(-3e - \frac{6}{7}e^2 + \beta\dot{e} + \frac{4}{7}\beta e\dot{e}\right) + 3S = \frac{4}{21}\beta\dot{e}(m - 3T) \quad . \tag{4–79}$$

We multiply by β^5 (to eliminate the integral $\beta^5 S$ by differentiation!) and differentiate. After division by β^4 and simplification we thus get

$$\begin{aligned}\beta^2\ddot{e}&\left[D\left(1 + \frac{4}{7}e\right) - \frac{4}{21}m + \frac{4}{7}T\right] + \\ &+6\beta\dot{e}\left[\delta\left(1 + \frac{4}{7}e\right) - \frac{4}{21}m + \frac{4}{7}T - \frac{2}{63}\beta^2\dot{D}\dot{e}\right] + \\ &+2\beta e\dot{D}\left(1 + \frac{2}{7}e\right) = 0 \quad .\end{aligned} \tag{4–80}$$

In the process of simplification, the relation (4–75)

$$\dot{D} = -3\beta^{-1}(D - \delta) \tag{4–81}$$

or equivalently,

$$D - \delta = -\frac{1}{3}\beta\dot{D} \; , \qquad D + \frac{1}{3}\beta\dot{D} = \delta \quad , \tag{4–82}$$

have played an essential role. The first-order approximation is sufficient since D is always multiplied by $O(e)$.

Now comes a variant of the trick applied at the very beginning of the present section: to first order, (4–80) reduces to

$$C(\beta) \equiv \beta^2\ddot{e}D + 6\beta\dot{e}\delta + 2\beta e\dot{D} = 0 \quad , \tag{4–83}$$

which, of course, is nothing else than the first-order Clairaut equation (2–114); note (4–82)! The first order is sufficient here for the same reason as above.

We write (4–80) in the form

$$C(\beta) + K(\beta) = 0 \quad , \tag{4–84}$$

$C(\beta)$ denoting Clairaut's equation (4–83) and $K(\beta)$ the remaining second-order terms in (4–80). By (4–83) we get

$$\beta^2\ddot{e}D = -6\beta\dot{e}\delta - 2\beta e\dot{D} \quad , \tag{4–85}$$

which permits us to eliminate $\ddot{e}$ in the second-order $K(\beta)$. The result is

$$K(\beta) = -\frac{4}{7}\beta\dot{D}e^2 - 2\beta\frac{\dot{D}}{D}(e + \beta\dot{e})\left(-\frac{4}{21}m + \frac{4}{7}T\right) - \frac{4}{21}\beta^3\dot{D}\dot{e}^2 \quad . \tag{4–86}$$

To eliminate T, we apply our trick again: (4–72) gives

$$T = \frac{5}{3} De - S - \frac{5}{6} m \quad , \tag{4–87}$$

and (4–79) reduces to first order to

$$-3De + D\beta\dot{e} + 3S = 0 \quad , \tag{4–88}$$

which we solve for S and substitute in (4–87), obtaining

$$T = \frac{2}{3} De + \frac{1}{3} D\beta\dot{e} - \frac{5}{6} m \tag{4–89}$$

to first order, which is sufficient for substitution in (4–86). Thus, after some laborious but straightforward computations we find simply

$$K(\beta) = \frac{4}{7}(D - \delta)\left[7e^2 + 6\beta e\dot{e} + 3\beta^2\dot{e}^2 - 7\mu(e + \beta\dot{e})\right] \quad , \tag{4–90}$$

so that (4–84), with (4–83) and (4–81), becomes

$$\beta^2\ddot{e} + 6\beta\frac{\delta}{D}\dot{e} - 6\left(1 - \frac{\delta}{D}\right)e = \frac{4}{7}\left(1 - \frac{\delta}{D}\right)e\xi \tag{4–91}$$

where, following (Jones, 1954),we have put

$$\xi = 7\mu\left(1 + \beta\frac{\dot{e}}{e}\right) - 3e\left(1 + \beta\frac{\dot{e}}{e}\right)^2 - 4e \quad . \tag{4–92}$$

Eq. (4–91) is the desired Clairaut equation to second order. It is solved iteratively, first solving Clairaut's equation (4–91) with right-hand side zero and then using $e(\beta) \doteq f(\beta)$ so obtained to compute the correction term (4–92) and hence the right-hand side of (4–91). Then the full equation (4–91) can be solved. Numerical methods for solving differential equations (Runge–Kutta etc.) are standard.

Boundary conditions. Two are needed. One is obtained by putting $\beta = 1$, $D = 1$, $T = 0$ in (4–79):

$$-3e - \frac{6}{7}e^2 + \dot{e} + \frac{4}{7}e\dot{e} + 3S_1 - \frac{4}{21}\dot{e}\,m = 0 \quad . \tag{4–93}$$

Now $S_1 = S(1)$ is found from (4–71) with $\beta = 1$:

$$e + \frac{6}{7}e^2 - \frac{3}{5}S_1\left(1 + \frac{4}{7}e\right) = \frac{1}{2}m\left(1 + \frac{20}{21}e\right) \quad .$$

We multiply by $\left(1 - \frac{4}{7}e\right)$ to obtain ($S = O(e)$!)

$$e + \frac{2}{7}e^2 - \frac{3}{5}S_1 = \frac{1}{2}m\left(1 + \frac{8}{21}e\right) \quad . \tag{4–94}$$

Eliminating S_1 between (4–93) and (4–94) yields

$$\dot{e}\left(1+\frac{4}{7}e-\frac{4}{21}m\right)=\frac{5}{2}m\left(1+\frac{8}{21}e\right)-2e\left(1+\frac{2}{7}e\right) \quad ,$$

which on multiplication by $(1-\frac{4}{7}e+\frac{4}{21}m)$ gives the desired boundary (or initial) condition

$$\dot{e}=\frac{5}{2}m-2e+\frac{4}{7}e^2-\frac{6}{7}em+\frac{10}{21}m^2 \quad . \tag{4–95}$$

This is the second-order equivalent of (2–118).

As the second boundary condition we may regard the surface flattening $f=f(1)$ as given. Furthermore, the ellipticity e must be finite at the earth's center, for $\beta=0$.

4.2.3 Radau's Transformation

Following sec. 2.6, we introduce Radau's parameter η by

$$\eta=\frac{\beta}{e}\frac{de}{d\beta}=\frac{\beta}{e}\dot{e} \quad . \tag{4–96}$$

Substituting

$$\dot{e}=\frac{\eta}{\beta}e, \qquad \ddot{e}=\left(\frac{1}{\beta}\frac{d\eta}{d\beta}+\frac{\eta^2-\eta}{\beta^2}\right)e \tag{4–97}$$

(by (2–123)) into (4–91) and dividing by e gives the second-order Radau equation

$$\beta\frac{d\eta}{d\beta}+\eta^2-\eta-6+6\frac{\delta}{D}(1+\eta)=\frac{4}{7}\left(1-\frac{\delta}{D}\right)\xi \quad , \tag{4–98}$$

where (4–92) takes the simpler form

$$\xi=7\mu(1+\eta)-3e(1+\eta)^2-4e \tag{4–99}$$

in view of (4–97). Following the derivation of sec. 2.6 formula by formula, we get (2–134):

$$\frac{d}{d\beta}\left(D\beta^5\sqrt{1+\eta}\right)=5D\beta^4\psi(\eta) \quad , \tag{4–100}$$

where now

$$\psi(\eta)=(1+\eta)^{-1/2}\left[1+\frac{1}{2}\eta-\frac{1}{10}\eta^2+\frac{2}{35}\left(1-\frac{\delta}{D}\right)\xi\right] \quad , \tag{4–101}$$

which is (2–132) with a small second-order correction. If $1+\lambda_1$ denotes an average value of $\psi(\eta)$ over the range $0\le\beta\le1$, then the integration of (4–100) gives

$$\int_0^1 D\beta^4 d\beta=\frac{1}{5}\frac{\sqrt{1+\eta_s}}{1+\lambda_1} \tag{4–102}$$

since $D(1) = 1$.

Moments of inertia. The sum of the three principal moments of inertia A, A, and C is, by (2–138) and (4–14)

$$2A + C = 2 \iiint (x^2 + y^2 + z^2)\rho \, dv = 2 \iiint r^4 \rho \, dr \, d\sigma \quad . \tag{4–103}$$

We perform the change of variables discussed in sec. 4.1.2 to get constant limits of integration, using (4–18):

$$2A + C = 2 \iiint r^4 \frac{\partial r}{\partial q} \rho(q) dq \, d\sigma \quad . \tag{4–104}$$

If we expand r by (4–50), we immediately see that the first–order terms are removed in view of (2–5), and there remains

$$2A + C = 8\pi \int_0^1 \delta \cdot \beta^4 d\beta + O(e^2) \tag{4–105}$$

in our usual new units. This may be written

$$C = \frac{8\pi}{3} \int_0^1 \delta \cdot \beta^4 d\beta + \frac{2}{3}(C - A) \quad . \tag{4–106}$$

The integral has form (2–141) and may be brought by integration by parts into the form (2–147), so that

$$C = \frac{2}{3} M - \frac{16\pi}{9} \int_0^1 D\beta^4 d\beta + \frac{2}{3}(C - A) \quad ; \tag{4–107}$$

note that we are using units in which, so to speak, $R = 1$ and $\rho_m = 1$. In these units the semimajor axis a is given by (4–46) for $q = 1$ as

$$a = 1 + \frac{1}{3} e + O(e^2) \quad . \tag{4–108}$$

Thus

$$Ma^2 = MR^2 \left(1 + \frac{2}{3} e\right) = \frac{4\pi}{3} \rho_m R^5 \left(1 + \frac{2}{3} e\right) \quad ,$$

which in our units reduces to

$$Ma^2 = \frac{4\pi}{3} \left(1 + \frac{2}{3} e\right) \quad . \tag{4–109}$$

Hence the ratio (2–152),

$$\frac{J_2}{H} = \frac{(C - A)/Ma^2}{(C - A)/C} = \frac{C}{Ma^2} = \frac{C}{MR^2} \left(1 - \frac{2}{3} e\right) = \frac{C}{M} \left(1 - \frac{2}{3} e\right) \tag{4–110}$$

becomes, using (4–107),

$$\frac{J_2}{H} = \frac{2}{3}\left(1 - \frac{2}{3}e\right) - \frac{4}{3}\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta + \frac{2}{3}J_2 + O(e^2) \quad , \tag{4–111}$$

noting that in our units,

$$M = \frac{4}{3}\pi R^3 \rho_m = \frac{4\pi}{3} \tag{4–112}$$

and

$$\frac{C-A}{M} = \frac{C-A}{MR^2} \doteq \frac{C-A}{Ma^2} = J_2 \quad .$$

To the same order we have, by (2–151)

$$J_2 = \frac{2}{3}e - \frac{1}{3}m \tag{4–113}$$

since $e = f + O(f^2)$. Thus (4–111) becomes

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - 2\left(1 - \frac{2}{3}e\right)\int_0^1 D\beta^4 d\beta\right] \quad , \tag{4–114}$$

from which we eliminate the integral by (4–102).

Hence

$$\frac{J_2}{H} = \frac{2}{3}\left[1 - \frac{1}{3}m - \frac{2}{5}\left(1 - \frac{2}{3}e\right)\frac{\sqrt{1+\eta_S}}{1+\lambda_1}\right] \quad . \tag{4–115}$$

For η_S we have by (4–95) and (4–96) with $\beta = 1$,

$$1 + \eta_S = \frac{5}{2}\frac{m}{e} - 1 + \frac{4}{7}e - \frac{6}{7}m + \frac{10}{21}\frac{m^2}{e} \quad . \tag{4–116}$$

Eqs. (4–115) and (4–116) provide the extension of (2–153) to second order (Jones, 1954).

4.2.4 Darwin's Equation

It is now not difficult to derive a differential equation for the deviation $\kappa = \kappa(\beta)$. We start from the equilibrium condition (4–70) with (4–68). This gives the identity

$$(3e^2 - 8\kappa)D - 6eS + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–117}$$

We eliminate S by means of (4–88):

$$S = De - \frac{1}{3}D\beta\dot{e} + O(e^2) \quad , \tag{4–118}$$

obtaining

$$(-3e^2 + 2\beta e\dot{e} - 8\kappa)D + 3P + \frac{8}{3}Q = 0 \quad . \tag{4–119}$$

D, P and Q are given by (4–56). To eliminate P, multiply by β^7 and differentiate. The result, using (4–81), is

$$\begin{aligned}&(2\beta e\ddot{e} - 4e\dot{e} + 2\beta\dot{e}^2 - 8\dot{\kappa})\beta^7 D + \\ &+(2\beta e\dot{e} - 3e^2 - 8\kappa)(4\beta^6 D + 3\beta^6\delta) + \\ &+3\delta\left[7\beta^6\left(e^2 + \frac{8}{9}\kappa\right) + \beta^7\left(2e\dot{e} + \frac{8}{9}\dot{\kappa}\right)\right] + \\ &+\frac{8}{3}\beta^9\delta(2\beta^{-3}\kappa - \beta^{-2}\dot{\kappa}) + 24\beta^6 Q = 0 \quad . \end{aligned} \qquad (4\text{–}120)$$

Again we eliminate $\ddot{e}$ by Clairaut's equation (4–83). The rest is elementary but cumbersome algebra, leading to the surprisingly simple result

$$2\beta e\dot{e} + \beta^2\dot{e}^2 - 16\kappa - 4\beta\dot{\kappa} + 12D^{-1}Q = 0 \quad , \qquad (4\text{–}121)$$

which in view of (4–56) gives the beautiful integro-differential *equation of Wavre* (1932, eq. (177)):

$$4(4\kappa + \beta\dot{\kappa}) = \beta\dot{e}(2e + \beta\dot{e}) + 12\frac{\beta^2}{D}\int_\beta^1 \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right)d\beta \quad . \qquad (4\text{–}122)$$

This equation is extensively studied in Wavre (1932, pp. 109–113).

We shall, however, eliminate also Q. For this purpose we multiply (4–121) by $\beta^{-2}D$ and differentiate. Again we take (4-81) into account and eliminate $\ddot{e}$ by (4–83). The result is *Darwin's equation*

$$\begin{aligned}\beta^2\ddot{\kappa} + 6\frac{\delta}{D}\beta\dot{\kappa} + \left(-20 + 6\frac{\delta}{D}\right)\kappa &= 3\left(1 - \frac{\delta}{D}\right)e^2 + \\ &+\left(1 - \frac{9}{2}\frac{\delta}{D}\right)\beta e\dot{e} - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\beta^2\dot{e}^2 \quad . \end{aligned} \qquad (4\text{–}123)$$

This equation is not unlike the simple Clairaut equation

$$\beta^2\ddot{e} + 6\frac{\delta}{D}\beta\dot{e} + \left(-6 + 6\frac{\delta}{D}\right)e = 0 \quad , \qquad (4\text{–}124)$$

but in contrast to (4–124), the right-hand side of (4–123) is not zero: Darwin's equation is *inhomogeneous*. Using Radau's parameter (4–96), the right-hand side of (4–123) takes the slightly simpler form

$$e^2\left[3\left(1 - \frac{\delta}{D}\right) + \left(1 - \frac{9}{2}\frac{\delta}{D}\right)\eta - \frac{1}{4}\left(1 + 9\frac{\delta}{D}\right)\eta^2\right] \qquad (4\text{–}125)$$

(Bullard, 1948; Jones, 1954, p. 12).

Boundary conditions. One boundary condition we get from Wavre's equation (4–122) with $\beta = 1$:

$$\dot{\kappa} = -4\kappa + \frac{1}{2}e\dot{e} + \frac{1}{4}\dot{e}^2 \quad , \qquad (4\text{–}126)$$

whence by (2–118) with $R = 1$ and $f = e$ on the surface:

$$\dot{\kappa} = -4\kappa - \frac{5}{4}\, e\, m + \frac{25}{16}\, m^2 \quad . \tag{4–127}$$

The second boundary condition refers to the earth's center $\beta = 0$:

$$\kappa(0) = 0 \quad . \tag{4–128}$$

This is also a result of Wavre's equation (4–122), in which the integral may be written

$$Q = \beta^2 \int_\beta^1 \delta \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta = Q_1 - \beta^2 \int_0^\beta \delta \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta \quad . \tag{4–129}$$

Since Q and Q_1 are finite by definition, the last integral must also be finite. Assume δ and κ expandable by Taylor's theorem

$$\begin{aligned} \delta &= \delta_0 + \delta_1\beta + \delta_2\beta^2 + \delta_3\beta^3 + O(\beta^4) \quad , \\ \kappa &= \kappa_0 + \kappa_1\beta + \kappa_2\beta^2 + \kappa_3\beta^3 + O(\beta^4) \quad . \end{aligned} \tag{4–130}$$

Then

$$\begin{aligned} \frac{\kappa}{\beta^2} &= \kappa_0\beta^{-2} + \kappa_1\beta^{-1} + \kappa_2 + \kappa_3\beta + O(\beta^2) \quad , \\ \frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) &= -2\kappa_0\beta^{-3} - \kappa_1\beta^{-2} + \kappa_3 + O(\beta) \quad , \\ \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) &= -2\kappa_0\delta_0\beta^{-3} - (\kappa_1\delta_0 + 2\kappa_0\delta_1)\beta^{-2} - (\kappa_1\delta_1 + 2\kappa_0\delta_2)\beta^{-1} + \\ &+ (\kappa_3\delta_0 - \kappa_1\delta_2 - 2\kappa_0\delta_3) + O(\beta) \end{aligned}$$

and

$$\begin{aligned} \int \delta\frac{d}{d\beta}\left(\frac{\kappa}{\beta^2}\right) d\beta &= \kappa_0\delta_0\beta^{-2} + (\kappa_1\delta_0 + 2\kappa_0\delta_1)\beta^{-1} - (\kappa_1\delta_1 + 2\kappa_0\delta_2)\ln\beta + \\ &+ (\kappa_3\delta_0 - \kappa_1\delta_2 - 2\kappa_0\delta_3)\beta + O(\beta^2) \quad . \end{aligned} \tag{4–131}$$

Now the first three terms become infinite at the center $\beta = 0$, which is impossible. This gives $\kappa_0 = 0$ or (4–128) and even

$$\kappa_1 = 0 \quad , \tag{4–132}$$

so that the expansion (4–130) must begin with $\kappa_2\beta^2$:

$$\kappa = \kappa_2\beta^2 + O(\beta^3) \quad . \tag{4–133}$$

Note that the boundary conditions for Darwin's equation: $\kappa(0)$ and $\dot{\kappa}(1)$, have a character different from those for Clairaut's equation: $f(1)$ and $\dot{f}(1)$.

Level ellipsoid. If the bounding surface of the equilibrium figure is an ellipsoid of revolution, then

$$\kappa(1) = 0 \quad .$$

Adding this as a boundary condition would result in three boundary conditions: $\kappa(0)$, $\kappa(1)$ and $\dot{\kappa}(1)$, which in general are incompatible for a second-order differential equation. This gives the

Theorem of Ledersteger

A level ellipsoid cannot in general be an equilibrium figure.

An exception is the Maclaurin ellipsoid (sec. 5.4) which, however, is homogeneous and in no way similar to the real earth.

This theorem was shown in second-order approximation only, but it will hold *a forteriori* for a rigorous ellipsoid.

The argument is very simple and intuitively convincing, especially in the light of later developments (Chapter 5 and secs. 6.2 and 6.4), which show that the earth is certainly not another exceptional case. A direct proof, going beyond the second-order approximation, would be desirable but seems to be very difficult.

Note that, as a first-order approximation (Clairaut's theory), heterogeneous ellipsoidal earth-like equilibrium figures do exist, but deviations start already in the second order.

4.2.5 Practical Comments and Results

The most important and recent pre-satellite determination of the flattening f, related to the ellipticity e by (4–48):

$$f = e + \frac{5}{42}e^2 - \frac{4}{7}\kappa \quad , \tag{4–134}$$

and of the deviation κ by solving Clairaut's and Darwin's equations was made by Bullard (1948), with modifications by Jones (1954).

Bullard gets the value (4–1), and Jones the closely similar value

$$f^{-1} = 297.300 \pm 0.065 \quad . \tag{4–135}$$

Bullard finds for de Sitter's numerical constants λ_1 and η_S the values

$$\lambda_1 = 0.00016 \pm 0.00018 \quad (!) \quad , \tag{4–136}$$

$$\eta_S = 0.565 \quad , \tag{4–137}$$

and for the surface value of κ, $\kappa_1 = \kappa(1)$ (not to be confused with (4–132)):

$$\kappa_1 = 68 \times 10^{-8} \quad , \tag{4–138}$$

corresponding to a deviation of the spheroid from the ellipsoid of 4.3 meters at latitude 45° (see Fig. 4.1).

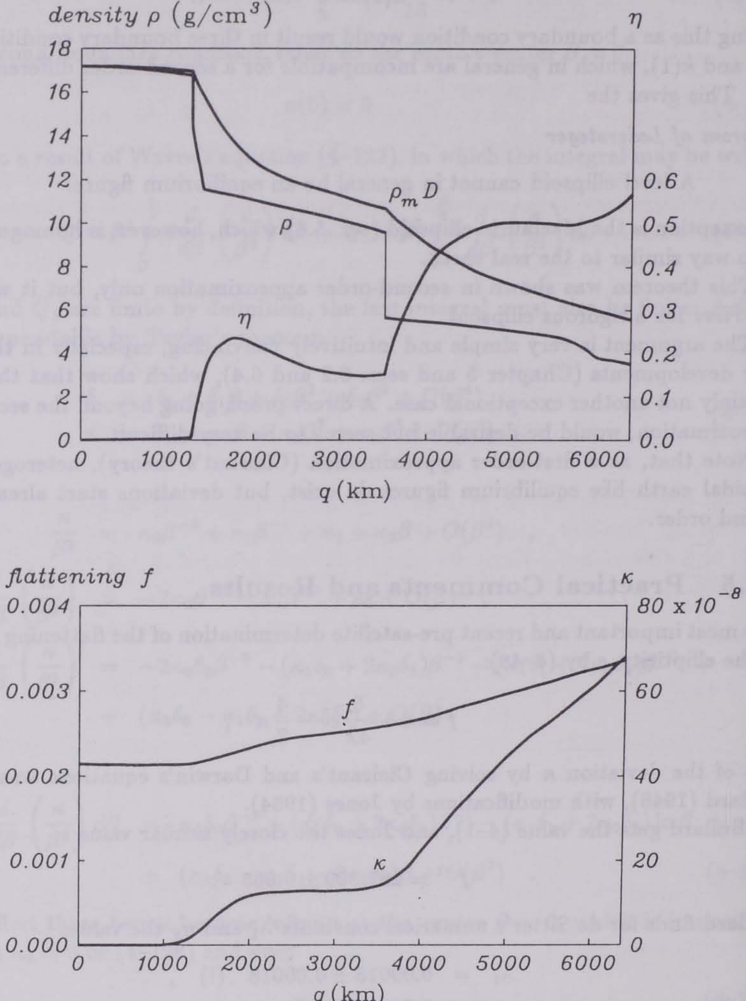

FIGURE 4.5: Density ρ, mean density $\rho_m D$, and η (above) and flattening f and deviation κ (below) as a function of the average radius $q = R\beta$ (in kilometers)

Fig. 4.5 shows the distribution of density ρ, mean density $\rho_m D$, Radau's parameter η, flattening f, and deviation κ in the earth's interior, following Bullard (1948) and Jones (1954). The density model is now obsolete, cf. Fig. 1.7, as well as the surface value for f, but the diagrams are nevertheless extremely instructive.

Recent determinations are extensively and carefully discussed in (Denis, 1989). As we have already remarked, instead of solving Clairaut's and Darwin's differential equations, we may also solve corresponding integro-differential equations such as (4–79) and (4–122) by iterative procedures described in (Zharkov and Trubitsyn, 1978, secs. 36 and 37) and in (Denis, 1989); the latter work is an excellent complement of the present book, especially as regards numerical aspects and results; it also contains extensive additional references. A modern counterpart of Fig. 4.5 is Fig. 4.6, following the preprint (Denis, 1985) which was available when the present book was written. The dependence of f on the underlying density model is remarkable.

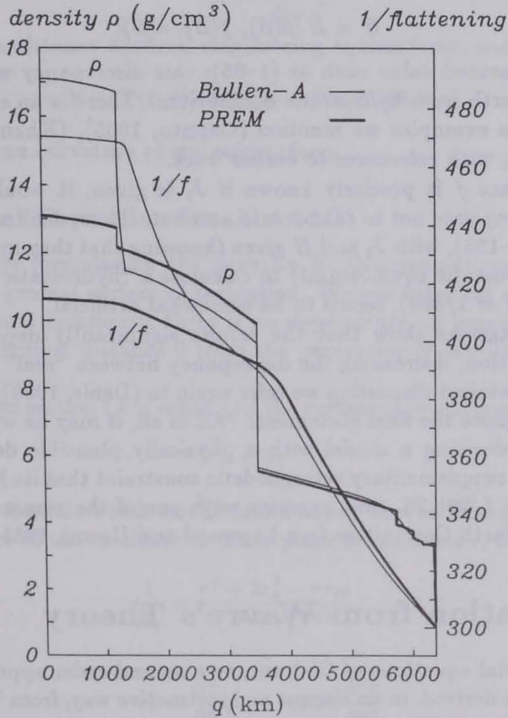

FIGURE 4.6: Inverse flattening f^{-1} for two different models of density ρ

Modern determinations of κ_1, comparable to (4–138), lie between 64 and 78×10^{-8}. So it may be expected that the plot of κ in Fig. 4.5 is still reasonably representative.

A final word on the determination of the flattening may be in order. For conceptual clarity we base our discussion on the first-order theory of sec. 2.7, but the more precise second-order theory of sec. 4.2.3 may also be considered.

In the pre-satellite era, J_2 was unknown, so the derivation had to be based on the known dynamical ellipticity H, solving (2–154) for the surface value of f.

From satellite determinations we now know J_2 very accurately and can use it directly, only applying the theory of the external field (of the equipotential ellipsoid, say), to determine the flattening

$$f = f(J_2) \quad ; \tag{4–139}$$

cf. (1–77) and (1–79). This value of $f = f(1)$ may now be used as a boundary condition for the determination of the function $f = f(\beta)$ by Clairaut's equation (4–91), at the risk that the value of H calculated on the basis of the distributions $\rho(\beta)$, $f(\beta)$, and $\kappa(\beta)$:

$$H = H\left[\rho(\beta),\, f(\beta),\, \kappa(\beta)\right] \tag{4–140}$$

differs from a measured value such as (1–85); this discrepancy will then indicate a deviation of the earth from hydrostatic equilibrium. There is an enormous literature on this subject; as examples we mention (Caputo, 1965), (Khan, 1968, 1969), and (Nakiboglu, 1979), with references to earlier work.

Since the surface f is precisely known if J_2 is given, it would, in the author's opinion, be inappropriate not to take it into account. Thus, deliberately ignoring this value and using (2–153), with J_2 and H given (knowing that they may be incompatible in the case of hydrostatic equilibrium!) to calculate a "hydrostatic flattening" f_H (on the order of 1/299 or 1/300), seems to be somewhat artificial.

Recent computations show that the results significantly depend on the choice of density distribution, decreasing the discrepancy between "real" and "hydrostatic" flattening. For a detailed discussion we refer again to (Denis, 1989); from the preprint (Denis, 1985) we quote the final statement: "All in all, it may be worthwhile to study the possibility of deriving a model with a physically plausible density distribution which satisfies the supplementary astrogeodetic constraint that its hydrostatic surface flattening is about 1/298.25, thus agreeing with one of the recommendations issued by the Standard Earth Committee (see Lapwood and Usami, 1981, p. 213)."

4.3 Derivation from Wavre's Theory

The basic differential equations of Clairaut, to a second-order approximation, and of Darwin can also be derived, in an elegant and instructive way, from Wavre's geometric theory described in sec. 3.2.

We start from eq. (3–45) with (3–47):

$$\Psi(t) = \frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta} \quad , \tag{4–141}$$