The comparison between (4–11) and (4–50) immediately gives

$$\epsilon_2 = -\frac{2}{3}\left(e + \frac{2}{3}e^2\right) , \qquad \epsilon_4 = \frac{4}{35}\left(3e^2 + 8\kappa\right) \quad . \tag{4–51}$$

This is substituted into the expressions (4–40) through (4–44), whence (4–26) and (4–32), as well as (4–39), become

$$\begin{aligned}
K_0(q) &= \frac{4\pi G}{3}\int_0^q \rho \frac{d}{dq}\left[\left(1+\frac{4}{15}e^2\right)q^3\right]dq \quad , \\
L_0(q) &= 2\pi G\int_q^R \rho \frac{d}{dq}\left[\left(1+\frac{4}{45}e^2\right)q^2\right]dq \quad , \\
K_2(q) &= -\frac{8\pi G}{15}\int_0^q \rho \frac{d}{dq}\left[\left(e+\frac{2}{7}e^2\right)q^5\right]dq \quad , \\
L_2(q) &= -\frac{8\pi G}{15}\int_q^R \rho \frac{d}{dq}\left(e+\frac{16}{21}e^2\right)dq \quad , \\
K_4(q) &= \frac{16\pi G}{9}\int_0^q \rho \frac{d}{dq}\left[\left(\frac{9}{35}e^2+\frac{8}{35}\kappa\right)q^7\right]dq \quad , \\
L_4(q) &= \frac{4\pi G}{9}\int_q^R \rho \frac{d}{dq}\left(\frac{32}{35}\kappa q^{-2}\right)dq \quad .
\end{aligned} \tag{4–52}$$

Note that $\rho = \rho(q)$, $e = e(q)$, and $\kappa = \kappa(q)$.

4.1.5 Gravitational Potential at P

The potential V consists of V_i and V_e according to (4–6). The *first part* of the trick was to compute V_i at a point P_e (Fig. 4.3) and the potential V_e at a point P_i (Fig. 4.4) for which the critical series (4–8) and (4–27) *always converge*. Thus we have satisfied the *desideratum of Tisserand* (Tisserand, 1891, p. 317; Wavre, 1932, p. 68) of working with convergent series only.

The result were the finite (truncated!) expressions (4–10) and (4–31); finite because the terms with $n > 4$ would already be $O(f^4)$ which we have agreed to neglect. These formulas represent functions which are *harmonic* and hence analytic in the "empty" regions E_P for V_i and I_P for V_e; see Figs. 4.3 and 4.4. Being analytic, these expressions hold *throughout* E_P for V_i and I_P for V_e; in view of the continuity of the potential they must hold also at the point P itself! This transition $P_e \to P$, $P_i \to P$ forms the *second part* of the trick.

This simple argument shows that we may use the expressions (4–10) and (4–31) also for P, so that the total gravitational potential V is their sum:

$$
\begin{aligned}
V(P) = V(q, \theta) \; = \; & \frac{K_0(q)}{r} + L_0(q) \; + \\
+ \; & \left[\frac{K_2(q)}{r^3} + r^2 L_2(q)\right] P_2(\cos\theta) \; + \\
+ \; & \left[\frac{K_4(q)}{r^5} + r^4 L_4(q)\right] P_4(\cos\theta) \quad .
\end{aligned} \tag{4–53}
$$

Here r and θ denote the spherical coordinates of the internal point P; the surface of constant density passing through P bears the label q (Fig. 4.2).

This reasoning also holds for $n > 4$: we are *working with convergent series only.* Thus we have achieved very simply the same result which Wavre has obtained by means of his very complicated "procédé uniforme". Quite another question is whether the *resulting* series is convergent. We have avoided this question by the simple (and usual) trick of limiting ourselves to the second-order (in f) approximation only, which automatically disregards higher-order terms.

Still the question remains open as a theoretical problem: the convergence of a spherical harmonic series at the boundary surface S_P. Nowadays we know much more about the convergence problem of spherical harmonic series than, say, twenty years ago; cf. (Moritz, 1980, secs. 6 and 7), especially the Runge–Krarup theorem. There may also be a relation to the existence proof by Liapunov and Lichtenstein mentioned in sec. 3.1. Another approach due to Trubitsyn is outlined in (Zharkov and Trubitsyn, 1978, sec. 38) and in (Denis, 1989).

The correctness of our second-order theory, however, is fully confirmed also by its derivation from Wavre's geometric theory to be treated in sec. 4.3, which is based on a completely different approach independent of any spherical-harmonic expansions.

4.2 Clairaut's and Darwin's Equations

4.2.1 Internal Gravity Potential

Following de Sitter (1924) we normalize the mean radius q and the density ρ by introducing the dimensionless quantities

$$\beta = \frac{q}{R} = \frac{\text{mean radius of } S_P}{\text{mean radius of earth}} \tag{4–54}$$

and

$$\delta = \frac{\rho}{\rho_m} = \frac{\text{density}}{\text{mean density of earth}} \quad . \tag{4–55}$$

The standard auxiliary expressions