For P_2^2 we have the formula

$$[P_2(t)]^2 = \frac{1}{5} + \frac{2}{7} P_2(t) + \frac{18}{35} P_4(t) \quad , \tag{4-37}$$

which expresses the square of the Legendre polynomial P_2 as a linear combination of P_2 and P_4. This formula, which can be verified immediately by substituting the defining expressions (1–33), will play a basic role in our second-order theory.

Since we are considering $L_2(q)$, we need only the coefficient of P_2 (all other terms are removed by orthogonality), so that (4–36) gives

$$\ln r = \cdots + (\epsilon_2 - \frac{1}{7}\epsilon_2^2)P_2(\cos\theta) + (\cdots)P_4(\cos\theta) \quad ; \tag{4-38}$$

$(-1/7)$ in (4–38) results as the product of $(-1/2)$ in (4–36) and $(2/7)$ in (4–37).

We take into account (4–38) and substitute (4–33) in the second line of (4–29). Orthogonality and (4–25) with $n = 2$ then give immediately

$$L_2(q) = \frac{4\pi G}{5} \int_q^R \rho(q) \frac{d}{dq}(\epsilon_2 - \frac{1}{7}\epsilon_2^2) dq \quad . \tag{4-39}$$

4.1.4 Computation of $K_n(q)$ and $L_n(q)$

For this purpose we need (4–24) and (4–30). For $n = 0$ we have by raising (4–11) to the third power:

$$r^3 = q^3(1 + 3\epsilon_2 P_2 + 3\epsilon_4 P_4 + 3\epsilon_2^2 P_2^2) \quad ,$$

to $O(f^2)$ and omitting the primes. For P_2^2 we use (4–37) to get

$$A_0(q) = 1 + 3\epsilon_2^2 \cdot \frac{1}{5} = 1 + \frac{3}{5}\epsilon_2^2 \quad ; \tag{4-40}$$

A_2 and A_4 are removed by orthogonality, so that we do not need them. For $n = 2$ we have

$$r^5 = q^5(1 + 5\epsilon_2 P_2 + 5\epsilon_4 P_4 + 10\epsilon_2^2 P_2^2) \quad ,$$

so the only required term in (4–24) is

$$B_2 = 5\epsilon_2 + 10\epsilon_2^2 \cdot \frac{2}{7} = 5\left(\epsilon_2 + \frac{4}{7}\epsilon_2^2\right) \quad . \tag{4-41}$$

For $n = 4$ we similarly find

$$\begin{aligned} r^7 &= q^7(1 + 7\epsilon_2 P_2 + 7\epsilon_4 P_4 + 21\epsilon_2^2 P_2^2) \quad , \\ C_4 &= 7\epsilon_4 + 21\epsilon_2^2 \cdot \frac{18}{35} = 7\left(\epsilon_4 + \frac{54}{35}\epsilon_2^2\right) \quad . \end{aligned} \tag{4-42}$$

In (4–30) we have for $n = 0$ and 4:

$$\begin{aligned} r^2 &= q^2(1 + 2\epsilon_2 P_2 + 2\epsilon_4 P_4 + \epsilon_2^2 P_2^2) \quad , \\ D_0 &= 1 + \frac{1}{5}\epsilon_2^2 \quad ; \end{aligned} \tag{4–43}$$

$$\begin{aligned} r^{-2} &= q^{-2}(1 - 2\epsilon_2 P_2 - 2\epsilon_4 P_4 + 3\epsilon_2^2 P_2^2) \quad , \\ F_4 &= -2\left(\epsilon_4 - \frac{27}{35}\epsilon_2^2\right) \quad . \end{aligned} \tag{4–44}$$

Finally we introduce the flattening f. In (4–3) we put

$$\begin{aligned} \cos^2\theta &= \frac{1}{3} + \frac{2}{3} P_2(\cos\theta) \quad , \\ \sin^2 2\theta &= \frac{8}{15} + \frac{8}{21} P_2 - \frac{32}{35} P_4 \quad , \end{aligned} \tag{4–45}$$

which is directly verified by inserting (1–33).

Substituting into (4–3) and putting $P_2 = P_4 = 0$ (the average of P_n is zero!) we get the mean radius

$$q = a\left(1 - \frac{1}{3}f - \frac{1}{5}f^2 - \frac{8}{15}\kappa\right) \quad . \tag{4–46}$$

This is solved for a and substituted into (4–3), together with (4–45). The result is

$$r = q\left[1 - \frac{2}{3}\left(f + \frac{23}{42}f^2 + \frac{4}{7}\kappa\right)P_2 + \frac{4}{35}(3f^2 + 8\kappa)P_4\right] \tag{4–47}$$

with $P_n = P_n(\cos\theta)$, up to $O(f^2)$.

Following de Sitter, we introduce, instead of f, the auxiliary quantity

$$e = f - \frac{5}{42}f^2 + \frac{4}{7}\kappa \quad , \tag{4–48}$$

which we shall call *ellipticity*. (The ellipticity e is not to be confused with the first excentricity (1–55)!) To our approximation we may put

$$e^2 \doteq f^2 \quad ; \tag{4–49}$$

note also that $\kappa = O(f^2) = O(e^2)$.

In terms of e, (4–47) simplifies to

$$r = q\left[1 - \frac{2}{3}\left(e + \frac{2}{3}e^2\right)P_2(\cos\theta) + \frac{4}{35}(3e^2 + 8\kappa)P_4(\cos\theta)\right] \quad . \tag{4–50}$$

We notice that the second-order coefficient ***no longer contains the deviation*** κ: remember that κ represents the deviation of our spheroid from the ellipsoid (cf. Fig. 4.1), which holds for the internal equidensity surfaces ($q < R$) as well as for the bounding surface $q = R$.

The comparison between (4–11) and (4–50) immediately gives

$$\epsilon_2 = -\frac{2}{3}(e + \frac{2}{3}e^2)\,, \qquad \epsilon_4 = \frac{4}{35}(3e^2 + 8\kappa) \quad . \tag{4–51}$$

This is substituted into the expressions (4–40) through (4–44), whence (4–26) and (4–32), as well as (4–39), become

$$\begin{aligned}
K_0(q) &= \frac{4\pi G}{3}\int_0^q \rho \frac{d}{dq}\left[\left(1 + \frac{4}{15}e^2\right)q^3\right] dq \quad , \\
L_0(q) &= 2\pi G \int_q^R \rho \frac{d}{dq}\left[\left(1 + \frac{4}{45}e^2\right)q^2\right] dq \quad , \\
K_2(q) &= -\frac{8\pi G}{15}\int_0^q \rho \frac{d}{dq}\left[\left(e + \frac{2}{7}e^2\right)q^5\right] dq \quad , \\
L_2(q) &= -\frac{8\pi G}{15}\int_q^R \rho \frac{d}{dq}\left(e + \frac{16}{21}e^2\right) dq \quad , \\
K_4(q) &= \frac{16\pi G}{9}\int_0^q \rho \frac{d}{dq}\left[\left(\frac{9}{35}e^2 + \frac{8}{35}\kappa\right)q^7\right] dq \quad , \\
L_4(q) &= \frac{4\pi G}{9}\int_q^R \rho \frac{d}{dq}\left(\frac{32}{35}\kappa q^{-2}\right) dq \quad .
\end{aligned} \tag{4–52}$$

Note that $\rho = \rho(q)$, $e = e(q)$, and $\kappa = \kappa(q)$.

4.1.5 Gravitational Potential at P

The potential V consists of V_i and V_e according to (4–6). The *first part* of the trick was to compute V_i at a point P_e (Fig. 4.3) and the potential V_e at a point P_i (Fig. 4.4) for which the critical series (4–8) and (4–27) *always converge*. Thus we have satisfied the *desideratum of Tisserand* (Tisserand, 1891, p. 317; Wavre, 1932, p. 68) of working with convergent series only.

The result were the finite (truncated!) expressions (4–10) and (4–31); finite because the terms with $n > 4$ would already be $O(f^4)$ which we have agreed to neglect. These formulas represent functions which are *harmonic* and hence analytic in the "empty" regions E_P for V_i and I_P for V_e; see Figs. 4.3 and 4.4. Being analytic, these expressions hold *throughout* E_P for V_i and I_P for V_e; in view of the continuity of the potential they must hold also at the point P itself! This transition $P_e \to P$, $P_i \to P$ forms the *second part* of the trick.

This simple argument shows that we may use the expressions (4–10) and (4–31) also for P, so that the total gravitational potential V is their sum: