by (1–49). The result is (4–10) with

$$K_0(q) = \frac{4\pi G}{3} \int_0^q \rho(q) \frac{d}{dq} \left[A_0(q) q^3\right] dq \quad ,$$

$$K_2(q) = \frac{4\pi G}{25} \int_0^q \rho(q) \frac{d}{dq} \left[B_2(q) q^5\right] dq \quad , \tag{4–26}$$

$$K_4(q) = \frac{4\pi G}{63} \int_0^q \rho(q) \frac{d}{dq} \left[C_4(q) q^7\right] dq \quad .$$

Here we have omitted the prime in the integration variable q' as we did before. The argument q of $K_i(q)$, of course, is identical with the upper limit of the integral (but not with the integration variable!).

4.1.3 Potential of Shell E_P

We now consider the potential of the "shell" E_P bounded by the surfaces S_P and S. We apply *the same trick* as before (sec. 4.1.1., Fig. 4.3). We calculate V_e first not at P, but at a point P_i situated on the radius vector of P in such a way that $r < r'$ is always satisfied and the series corresponding to (4–8),

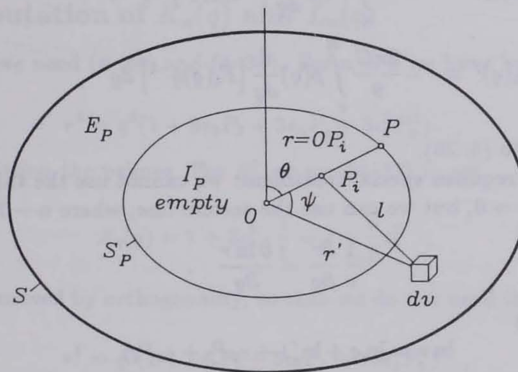

FIGURE 4.4: Illustrating the computation of V_e

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\psi) \quad , \tag{4–27}$$

always converges (Fig. 4.4). For this "harmless" point we have

$$V_e(P_i) = G \iiint_{E_P} \frac{\rho}{l}\, dv = \sum_{n=0}^{\infty} r^n \cdot G \iiint_{E_P} \frac{\rho}{r'^{n+1}} P_n(\cos\psi) dv \quad , \tag{4–28}$$

in analogy to (4–9). We again perform the change of variable of sec. 4.1.2, so that the integral in (4–28) becomes

$$G \iiint\limits_{E_P} \frac{\rho}{r'^{n+1}} P_n(\cos\psi) dv =$$

$$= G \int\limits_{q'=q}^{R} dq' \rho(q') \iint\limits_{\sigma} \frac{1}{r'^{n-1}} \frac{\partial r'}{\partial q'} P_n(\cos\psi) d\sigma$$

$$= \frac{G}{2-n} \int\limits_{q}^{R} dq' \rho(q') \iint\limits_{\sigma} \frac{\partial}{\partial q'} \left(r'^{2-n}\right) P_n(\cos\psi) d\sigma \quad . \tag{4–29}$$

In analogy to (4–24) we put

$$r'^{2-n} = q'^{2-n} \left[D_n(q') + E_n(q') P_2(\cos\theta') + F_n(q') P_4(\cos\theta')\right] \tag{4–30}$$

and substitute. Orthogonality will again remove most terms, and using (4–25) we get

$$V_e(P_i) = L_0(q) + L_2(q) r^2 P_2(\cos\theta) + L_4(q) r^4 P_4(\cos\theta) \tag{4–31}$$

with

$$\begin{aligned} L_0(q) &= 2\pi G \int\limits_{q}^{R} \rho(q) \frac{d}{dq}\left[D_0(q) q^2\right] dq \quad , \\ L_4(q) &= -\frac{2\pi G}{9} \int\limits_{q}^{R} \rho(q) \frac{d}{dq}\left[F_4(q) q^{-2}\right] dq \quad , \end{aligned} \tag{4–32}$$

in perfect analogy to (4–26).

The case $n = 2$ requires special treatment: we cannot use the third line of (4–29) because then $2 - n = 0$, but we can use the second line, where $n - 1 = 1$ and

$$\frac{1}{r}\frac{\partial r}{\partial q} = \frac{\partial \ln r}{\partial q} \quad . \tag{4–33}$$

From (4–11) we get

$$\ln r = \ln q + \ln(1 + \epsilon_2 P_2 + \epsilon_4 P_4) \quad . \tag{4–34}$$

Applying the well-known series

$$\ln(1+x) = x - \frac{1}{2}x^2 \cdots \tag{4–35}$$

we thus have

$$\ln r = \ln q + \epsilon_2 P_2 + \epsilon_4 P_4 - \frac{1}{2}\epsilon_2^2 P_2^2 \quad . \tag{4–36}$$

Here we note that $\epsilon_2 = O(f)$, $\epsilon_2^2 = O(f^2)$, $\epsilon_4 = O(f^2)$ where f is the flattening (this will be confirmed below). Hence ϵ_4^2 would already be $O(f^4)$ and thus is to be neglected.

For P_2^2 we have the formula

$$[P_2(t)]^2 = \frac{1}{5} + \frac{2}{7} P_2(t) + \frac{18}{35} P_4(t) \quad , \tag{4–37}$$

which expresses the square of the Legendre polynomial P_2 as a linear combination of P_2 and P_4. This formula, which can be verified immediately by substituting the defining expressions (1–33), will play a basic role in our second-order theory.

Since we are considering $L_2(q)$, we need only the coefficient of P_2 (all other terms are removed by orthogonality), so that (4–36) gives

$$\ln r = \cdots + (\epsilon_2 - \frac{1}{7}\epsilon_2^2)P_2(\cos\theta) + (\cdots)P_4(\cos\theta) \quad ; \tag{4–38}$$

$(-1/7)$ in (4–38) results as the product of $(-1/2)$ in (4–36) and $(2/7)$ in (4–37).

We take into account (4–38) and substitute (4–33) in the second line of (4–29). Orthogonality and (4–25) with $n = 2$ then give immediately

$$L_2(q) = \frac{4\pi G}{5} \int_q^R \rho(q) \frac{d}{dq}(\epsilon_2 - \frac{1}{7}\epsilon_2^2) dq \quad . \tag{4–39}$$

4.1.4 Computation of $K_n(q)$ and $L_n(q)$

For this purpose we need (4–24) and (4–30). For $n = 0$ we have by raising (4–11) to the third power:

$$r^3 = q^3(1 + 3\epsilon_2 P_2 + 3\epsilon_4 P_4 + 3\epsilon_2^2 P_2^2) \quad ,$$

to $O(f^2)$ and omitting the primes. For P_2^2 we use (4–37) to get

$$A_0(q) = 1 + 3\epsilon_2^2 \cdot \frac{1}{5} = 1 + \frac{3}{5}\epsilon_2^2 \quad ; \tag{4–40}$$

A_2 and A_4 are removed by orthogonality, so that we do not need them. For $n = 2$ we have

$$r^5 = q^5(1 + 5\epsilon_2 P_2 + 5\epsilon_4 P_4 + 10\epsilon_2^2 P_2^2) \quad ,$$

so the only required term in (4–24) is

$$B_2 = 5\epsilon_2 + 10\epsilon_2^2 \cdot \frac{2}{7} = 5\left(\epsilon_2 + \frac{4}{7}\epsilon_2^2\right) \quad . \tag{4–41}$$

For $n = 4$ we similarly find

$$\begin{aligned} r^7 &= q^7(1 + 7\epsilon_2 P_2 + 7\epsilon_4 P_4 + 21\epsilon_2^2 P_2^2) \quad , \\ C_4 &= 7\epsilon_4 + 21\epsilon_2^2 \cdot \frac{18}{35} = 7\left(\epsilon_4 + \frac{54}{35}\epsilon_2^2\right) \quad . \end{aligned} \tag{4–42}$$