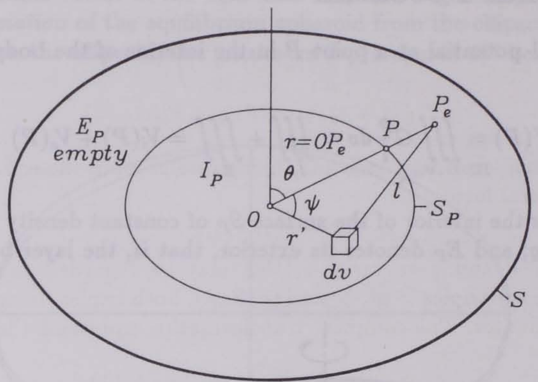

FIGURE 4.3: Illustrating the computation of V_i

The trick is to leave I_P but to calculate V first at a point P_e which lies on the radius vector of P but outside S_P in such a way that $r' < r$ is always satisfied (Fig. 4.3). Thus we compute

$$V_i(P_e) = G \iiint_{I_P} \frac{\rho}{l} \, dv = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot G \iiint_{I_P} \rho r'^n P_n(\cos\psi) dv \tag{4–9}$$

(the interchange of sum and integral offers no problem because of the absolute convergence of the integrand series). Since $V_i(P_e)$ is harmonic, the shell between S_P and S being disregarded for the time being, and because of rotational symmetry, (4–9) must necessarily have the form (1–37) with zonal harmonics only:

$$V_i(P_e) = \sum_{n=0}^{\infty} \frac{K_n}{r^{n+1}} P_n(\cos\theta)$$

or

$$V_i(P_e) = \frac{K_0(q)}{r} + \frac{K_2(q)}{r^3} P_2(\cos\theta) + \frac{K_4(q)}{r^5} P_4(\cos\theta) \quad , \tag{4–10}$$

neglecting higher-order terms. Here r, θ, λ are the spherical coordinates of P_e as usual; because of rotational symmetry there is no explicit dependence on longitude λ (no tesseral terms); and there are only even-degree zonal terms because of symmetry with respect to the equatorial plane. The coefficients K_n evidently depend on S_P and hence on its label q.

4.1.2 Change of Variable

The equation of any surface of constant density may be written as

$$\begin{aligned} r &= q\left(1+\sum_{n=1}^{\infty}\epsilon_n P_n(\cos\theta)\right) = \\ &= q\left(1+\epsilon_2 P_2(\cos\theta)+\epsilon_4 P_4(\cos\theta)\right) \quad , \end{aligned} \tag{4–11}$$

again neglecting higher-order terms and taking into account the equatorial symmetry. This has the general form

$$r = r(q,\,\theta) \quad . \tag{4–12}$$

Considering both θ and q as variable, this may be regarded as a transformation equation between the triples $(r,\,\theta,\,\lambda)$ and $(q,\,\theta,\,\lambda)$, both triples being viewed as *spatial* curvilinear coordinates. The complete transformation equations then are

$$\begin{aligned} r &= r(q,\,\theta) \quad \text{as given by (4–11),} \\ \theta &= \theta \quad , \\ \lambda &= \lambda \quad . \end{aligned} \tag{4–13}$$

For the volume element in spherical coordinates we have by (2–46)

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\lambda = r^2 dr\, d\sigma \quad . \tag{4–14}$$

The change of volume element in a coordinate transformation is expressed by the well-known formula

$$dr\, d\theta\, d\lambda = J\, dq\, d\theta\, d\lambda \quad , \tag{4–15}$$

with the Jacobian determinant

$$J = \begin{vmatrix} \dfrac{\partial r}{\partial q} & \dfrac{\partial r}{\partial \theta} & \dfrac{\partial r}{\partial \lambda} \\ \dfrac{\partial \theta}{\partial q} & \dfrac{\partial \theta}{\partial \theta} & \dfrac{\partial \theta}{\partial \lambda} \\ \dfrac{\partial \lambda}{\partial q} & \dfrac{\partial \lambda}{\partial \theta} & \dfrac{\partial \lambda}{\partial \lambda} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial r}{\partial q} & \dfrac{\partial r}{\partial \theta} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{4–16}$$

in view of (4–13). Working out the determinant gives

$$J = \frac{\partial r}{\partial q} \quad , \tag{4–17}$$

so that (4–14) becomes

$$dv = r^2 \frac{\partial r}{\partial q}\, dq\, d\sigma \quad . \tag{4–18}$$

This form is surprisingly simple, especially in view of the fact that the coordinate system $q,\,\theta,\,\lambda$ is easily seen to be non-orthogonal. In this transformation we have followed Kopal (1960, p. 9).

Now we can transform the integral

$$\iiint_{I_P} dv \tag{4-19}$$

as

$$\iint_{\sigma} \int_{r'=0}^{r(\theta,\lambda)} r'^2 dr' d\sigma = \iint_{\sigma} \int_{q'=0}^{q} r'^2 \frac{\partial r'}{\partial q'} dq' d\sigma \quad . \tag{4-20}$$

The integration variables are now r', θ', λ' or q', θ', λ' with

$$d\sigma = \sin\theta' d\theta' d\lambda' \quad . \tag{4-21}$$

The variable upper limit $r(\theta, \lambda)$ on the left-hand side of (4–20) denotes the equation of the surfaces S_P bounding I_P, for which q is constant (Fig. 4.2). The advantage of the transformation $(r, \theta, \lambda) \to (q, \theta, \lambda)$ thus consists in transforming the integral (4–19) into an integral with *constant* limits of integration. Then we can also invert the order of the integrals, writing

$$\iiint_{I_P} dv = \int_{q'=0}^{q} \iint_{\sigma} r'^2 \frac{\partial r'}{\partial q'} dq' d\sigma \quad . \tag{4-22}$$

Here, of course, $r' = r(q', \theta')$ as given by (4–11) with primed variables.

Hence the integral in (4–9) becomes

$$\begin{aligned} G \iiint_{I_P} \rho r'^n P_n(\cos\psi) dv &= \\ &= G \int_{q'=0}^{q} dq' \rho(q') \iint_{\sigma} r'^{n+2} \frac{\partial r'}{\partial q'} P_n(\cos\psi) d\sigma \\ &= \frac{G}{n+3} \int_{0}^{q} dq' \rho(q') \iint_{\sigma} \frac{\partial}{\partial q'} \left(r'^{n+3}\right) P_n(\cos\psi) d\sigma \quad . \end{aligned} \tag{4-23}$$

By raising (4–11) to the appropriate power we get an expression of the form

$$r'^{n+3} = q'^{n+3} \left[A_n(q') + B_n(q') P_2(\cos\theta') + C_n(q') P_4(\cos\theta')\right] \quad . \tag{4-24}$$

This form will be justified and the functions A_n, B_n and C_n will be explicitly given below. Substitute this into (4–23) and integrate over σ. Orthogonality will then remove all terms except certain terms with $n = 0, 2, 4$ for which

$$\iint_{\sigma} P_n(\cos\theta') P_n(\cos\psi) d\sigma = \frac{4\pi}{2n+1} P_n(\cos\theta) \tag{4-25}$$

by (1–49). The result is (4–10) with

$$K_0(q) = \frac{4\pi G}{3} \int_0^q \rho(q) \frac{d}{dq} \left[A_0(q) q^3\right] dq \quad ,$$

$$K_2(q) = \frac{4\pi G}{25} \int_0^q \rho(q) \frac{d}{dq} \left[B_2(q) q^5\right] dq \quad , \tag{4–26}$$

$$K_4(q) = \frac{4\pi G}{63} \int_0^q \rho(q) \frac{d}{dq} \left[C_4(q) q^7\right] dq \quad .$$

Here we have omitted the prime in the integration variable q' as we did before. The argument q of $K_i(q)$, of course, is identical with the upper limit of the integral (but not with the integration variable!).

4.1.3 Potential of Shell E_P

We now consider the potential of the "shell" E_P bounded by the surfaces S_P and S. We apply *the same trick* as before (sec. 4.1.1., Fig. 4.3). We calculate V_e first not at P, but at a point P_i situated on the radius vector of P in such a way that $r < r'$ is always satisfied and the series corresponding to (4–8),

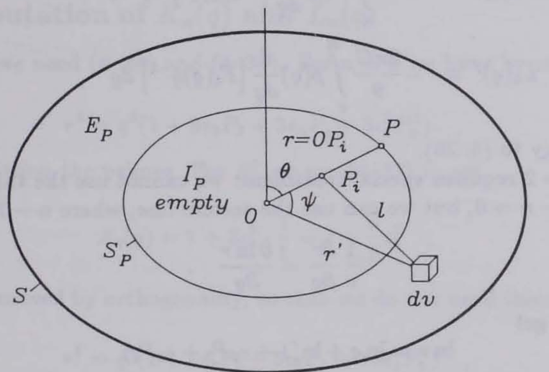

FIGURE 4.4: Illustrating the computation of V_e

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\psi) \quad , \tag{4–27}$$

always converges (Fig. 4.4). For this "harmless" point we have

$$V_e(P_i) = G \iiint_{E_P} \frac{\rho}{l}\, dv = \sum_{n=0}^{\infty} r^n \cdot G \iiint_{E_P} \frac{\rho}{r'^{n+1}} P_n(\cos\psi) dv \quad , \tag{4–28}$$