4.1 Internal Potential

The gravitational potential at a point P in the interior of the body bounded by the surface S is

$$V(P) = \iiint G\frac{\rho}{l}\,dv = \iiint_{I_P} + \iiint_{E_P} = V_i(P) + V_e(P) \quad , \tag{4-6}$$

where I_P denotes the interior of the surface S_P of constant density labeled, as usual, by a parameter q, and E_P denotes its exterior, that is, the layer between S_P and S (Fig. 4.2).

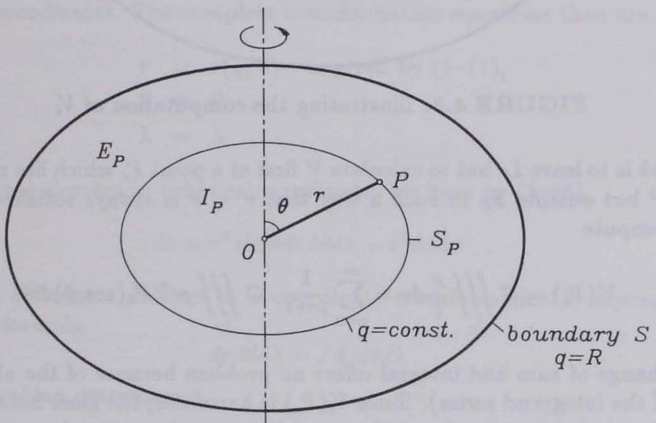

FIGURE 4.2: Illustrating the computation of $V(P)$

4.1.1 Potential of Interior I_P

Consider first only

$$V_i = G\iiint_{I_P} \frac{\rho}{l}\,dv \quad . \tag{4-7}$$

For $1/l$ we have the usual series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{4-8}$$

which converges if $r' < r$. *The problem* is that for $r = r_P = OP$ (Fig. 4.3), this convergence condition may be violated: r' may be greater than r.

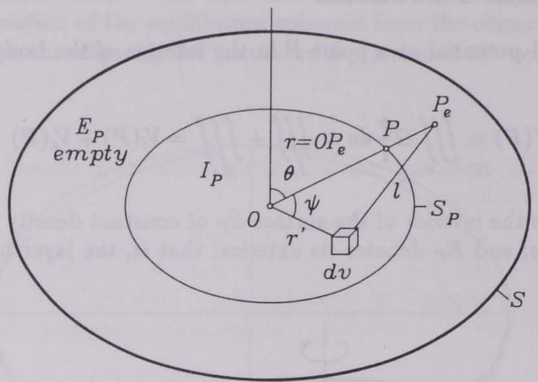

FIGURE 4.3: Illustrating the computation of V_i

The trick is to leave I_P but to calculate V first at a point P_e which lies on the radius vector of P but outside S_P in such a way that $r' < r$ is always satisfied (Fig. 4.3). Thus we compute

$$V_i(P_e) = G \iiint_{I_P} \frac{\rho}{l}\, dv = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot G \iiint_{I_P} \rho r'^n P_n(\cos\psi) dv \tag{4–9}$$

(the interchange of sum and integral offers no problem because of the absolute convergence of the integrand series). Since $V_i(P_e)$ is harmonic, the shell between S_P and S being disregarded for the time being, and because of rotational symmetry, (4–9) must necessarily have the form (1–37) with zonal harmonics only:

$$V_i(P_e) = \sum_{n=0}^{\infty} \frac{K_n}{r^{n+1}} P_n(\cos\theta)$$

or

$$V_i(P_e) = \frac{K_0(q)}{r} + \frac{K_2(q)}{r^3} P_2(\cos\theta) + \frac{K_4(q)}{r^5} P_4(\cos\theta) \quad , \tag{4–10}$$

neglecting higher-order terms. Here r, θ, λ are the spherical coordinates of P_e as usual; because of rotational symmetry there is no explicit dependence on longitude λ (no tesseral terms); and there are only even-degree zonal terms because of symmetry with respect to the equatorial plane. The coefficients K_n evidently depend on S_P and hence on its label q.

4.1.2 Change of Variable

The equation of any surface of constant density may be written as

$$\begin{aligned} r &= q\left(1+\sum_{n=1}^{\infty}\epsilon_n P_n(\cos\theta)\right) = \\ &= q\left(1+\epsilon_2 P_2(\cos\theta)+\epsilon_4 P_4(\cos\theta)\right) \quad , \end{aligned} \tag{4–11}$$

again neglecting higher-order terms and taking into account the equatorial symmetry. This has the general form

$$r = r(q,\,\theta) \quad . \tag{4–12}$$

Considering both θ and q as variable, this may be regarded as a transformation equation between the triples $(r,\,\theta,\,\lambda)$ and $(q,\,\theta,\,\lambda)$, both triples being viewed as *spatial* curvilinear coordinates. The complete transformation equations then are

$$\begin{aligned} r &= r(q,\,\theta) \quad \text{as given by (4–11)}, \\ \theta &= \theta \quad , \\ \lambda &= \lambda \quad . \end{aligned} \tag{4–13}$$

For the volume element in spherical coordinates we have by (2–46)

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\lambda = r^2 dr\, d\sigma \quad . \tag{4–14}$$

The change of volume element in a coordinate transformation is expressed by the well-known formula

$$dr\,d\theta\,d\lambda = J\,dq\,d\theta\,d\lambda \quad , \tag{4–15}$$

with the Jacobian determinant

$$J = \begin{vmatrix} \dfrac{\partial r}{\partial q} & \dfrac{\partial r}{\partial \theta} & \dfrac{\partial r}{\partial \lambda} \\ \dfrac{\partial \theta}{\partial q} & \dfrac{\partial \theta}{\partial \theta} & \dfrac{\partial \theta}{\partial \lambda} \\ \dfrac{\partial \lambda}{\partial q} & \dfrac{\partial \lambda}{\partial \theta} & \dfrac{\partial \lambda}{\partial \lambda} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial r}{\partial q} & \dfrac{\partial r}{\partial \theta} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{4–16}$$

in view of (4–13). Working out the determinant gives

$$J = \frac{\partial r}{\partial q} \quad , \tag{4–17}$$

so that (4–14) becomes

$$dv = r^2 \frac{\partial r}{\partial q}\, dq\, d\sigma \quad . \tag{4–18}$$

This form is surprisingly simple, especially in view of the fact that the coordinate system $q,\,\theta,\,\lambda$ is easily seen to be non-orthogonal. In this transformation we have followed Kopal (1960, p. 9).

Now we can transform the integral

$$\iiint_{I_P} dv \tag{4-19}$$

as

$$\iint_{\sigma} \int_{r'=0}^{r(\theta,\lambda)} r'^2 dr' d\sigma = \iint_{\sigma} \int_{q'=0}^{q} r'^2 \frac{\partial r'}{\partial q'} dq' d\sigma \quad . \tag{4-20}$$

The integration variables are now r', θ', λ' or q', θ', λ' with

$$d\sigma = \sin\theta' d\theta' d\lambda' \quad . \tag{4-21}$$

The variable upper limit $r(\theta, \lambda)$ on the left-hand side of (4–20) denotes the equation of the surfaces S_P bounding I_P, for which q is constant (Fig. 4.2). The advantage of the transformation $(r, \theta, \lambda) \to (q, \theta, \lambda)$ thus consists in transforming the integral (4–19) into an integral with *constant* limits of integration. Then we can also invert the order of the integrals, writing

$$\iiint_{I_P} dv = \int_{q'=0}^{q} \iint_{\sigma} r'^2 \frac{\partial r'}{\partial q'} dq' d\sigma \quad . \tag{4-22}$$

Here, of course, $r' = r(q', \theta')$ as given by (4–11) with primed variables.

Hence the integral in (4–9) becomes

$$\begin{aligned} G \iiint_{I_P} \rho r'^n P_n(\cos\psi) dv &= \\ &= G \int_{q'=0}^{q} dq' \rho(q') \iint_{\sigma} r'^{n+2} \frac{\partial r'}{\partial q'} P_n(\cos\psi) d\sigma \\ &= \frac{G}{n+3} \int_{0}^{q} dq' \rho(q') \iint_{\sigma} \frac{\partial}{\partial q'} \left(r'^{n+3}\right) P_n(\cos\psi) d\sigma \quad . \end{aligned} \tag{4-23}$$

By raising (4–11) to the appropriate power we get an expression of the form

$$r'^{n+3} = q'^{n+3} \left[A_n(q') + B_n(q') P_2(\cos\theta') + C_n(q') P_4(\cos\theta')\right] \quad . \tag{4-24}$$

This form will be justified and the functions A_n, B_n and C_n will be explicitly given below. Substitute this into (4–23) and integrate over σ. Orthogonality will then remove all terms except certain terms with $n = 0, 2, 4$ for which

$$\iint_{\sigma} P_n(\cos\theta') P_n(\cos\psi) d\sigma = \frac{4\pi}{2n+1} P_n(\cos\theta) \tag{4-25}$$

by (1–49). The result is (4–10) with

$$K_0(q) = \frac{4\pi G}{3} \int_0^q \rho(q) \frac{d}{dq} \left[A_0(q) q^3\right] dq \quad ,$$

$$K_2(q) = \frac{4\pi G}{25} \int_0^q \rho(q) \frac{d}{dq} \left[B_2(q) q^5\right] dq \quad , \tag{4–26}$$

$$K_4(q) = \frac{4\pi G}{63} \int_0^q \rho(q) \frac{d}{dq} \left[C_4(q) q^7\right] dq \quad .$$

Here we have omitted the prime in the integration variable q' as we did before. The argument q of $K_i(q)$, of course, is identical with the upper limit of the integral (but not with the integration variable!).

4.1.3 Potential of Shell E_P

We now consider the potential of the "shell" E_P bounded by the surfaces S_P and S. We apply *the same trick* as before (sec. 4.1.1., Fig. 4.3). We calculate V_e first not at P, but at a point P_i situated on the radius vector of P in such a way that $r < r'$ is always satisfied and the series corresponding to (4–8),

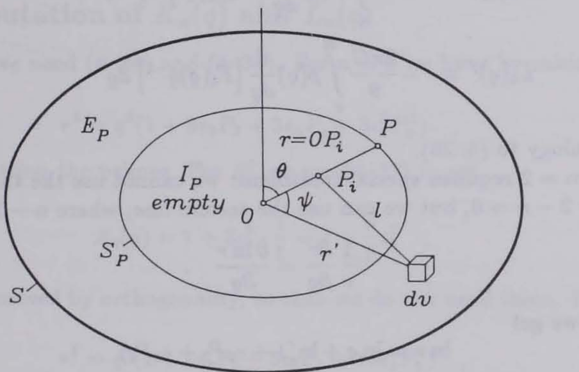

FIGURE 4.4: Illustrating the computation of V_e

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\psi) \quad , \tag{4–27}$$

always converges (Fig. 4.4). For this "harmless" point we have

$$V_e(P_i) = G \iiint_{E_P} \frac{\rho}{l}\, dv = \sum_{n=0}^{\infty} r^n \cdot G \iiint_{E_P} \frac{\rho}{r'^{n+1}} P_n(\cos\psi) dv \quad , \tag{4–28}$$

in analogy to (4–9). We again perform the change of variable of sec. 4.1.2, so that the integral in (4–28) becomes

$$G \iiint_{E_P} \frac{\rho}{r'^{n+1}} P_n(\cos\psi)dv =$$

$$= G \int_{q'=q}^{R} dq'\rho(q') \iint_{\sigma} \frac{1}{r'^{n-1}} \frac{\partial r'}{\partial q'} P_n(\cos\psi)d\sigma$$

$$= \frac{G}{2-n} \int_{q}^{R} dq'\rho(q') \iint_{\sigma} \frac{\partial}{\partial q'} \left(r'^{2-n}\right) P_n(\cos\psi)d\sigma \quad . \tag{4–29}$$

In analogy to (4–24) we put

$$r'^{2-n} = q'^{2-n} \left[D_n(q') + E_n(q')P_2(\cos\theta') + F_n(q')P_4(\cos\theta')\right] \tag{4–30}$$

and substitute. Orthogonality will again remove most terms, and using (4–25) we get

$$V_e(P_i) = L_0(q) + L_2(q)r^2 P_2(\cos\theta) + L_4(q)r^4 P_4(\cos\theta) \tag{4–31}$$

with

$$\begin{aligned} L_0(q) &= 2\pi G \int_{q}^{R} \rho(q) \frac{d}{dq} \left[D_0(q)q^2\right] dq \quad , \\ L_4(q) &= -\frac{2\pi G}{9} \int_{q}^{R} \rho(q) \frac{d}{dq} \left[F_4(q)q^{-2}\right] dq \quad , \end{aligned} \tag{4–32}$$

in perfect analogy to (4–26).

The case $n = 2$ requires special treatment: we cannot use the third line of (4–29) because then $2 - n = 0$, but we can use the second line, where $n - 1 = 1$ and

$$\frac{1}{r}\frac{\partial r}{\partial q} = \frac{\partial \ln r}{\partial q} \quad . \tag{4–33}$$

From (4–11) we get

$$\ln r = \ln q + \ln(1 + \epsilon_2 P_2 + \epsilon_4 P_4) \quad . \tag{4–34}$$

Applying the well-known series

$$\ln(1+x) = x - \frac{1}{2}x^2 \cdots \tag{4–35}$$

we thus have

$$\ln r = \ln q + \epsilon_2 P_2 + \epsilon_4 P_4 - \frac{1}{2}\epsilon_2^2 P_2^2 \quad . \tag{4–36}$$

Here we note that $\epsilon_2 = O(f)$, $\epsilon_2^2 = O(f^2)$, $\epsilon_4 = O(f^2)$ where f is the flattening (this will be confirmed below). Hence ϵ_4^2 would already be $O(f^4)$ and thus is to be neglected.

For P_2^2 we have the formula

$$[P_2(t)]^2 = \frac{1}{5} + \frac{2}{7} P_2(t) + \frac{18}{35} P_4(t) \quad , \tag{4-37}$$

which expresses the square of the Legendre polynomial P_2 as a linear combination of P_2 and P_4. This formula, which can be verified immediately by substituting the defining expressions (1–33), will play a basic role in our second-order theory.

Since we are considering $L_2(q)$, we need only the coefficient of P_2 (all other terms are removed by orthogonality), so that (4–36) gives

$$\ln r = \cdots + (\epsilon_2 - \frac{1}{7}\epsilon_2^2) P_2(\cos\theta) + (\cdots) P_4(\cos\theta) \quad ; \tag{4-38}$$

$(-1/7)$ in (4–38) results as the product of $(-1/2)$ in (4–36) and $(2/7)$ in (4–37).

We take into account (4–38) and substitute (4–33) in the second line of (4–29). Orthogonality and (4–25) with $n = 2$ then give immediately

$$L_2(q) = \frac{4\pi G}{5} \int_q^R \rho(q) \frac{d}{dq} (\epsilon_2 - \frac{1}{7}\epsilon_2^2) dq \quad . \tag{4-39}$$

4.1.4 Computation of $K_n(q)$ and $L_n(q)$

For this purpose we need (4–24) and (4–30). For $n = 0$ we have by raising (4–11) to the third power:

$$r^3 = q^3(1 + 3\epsilon_2 P_2 + 3\epsilon_4 P_4 + 3\epsilon_2^2 P_2^2) \quad ,$$

to $O(f^2)$ and omitting the primes. For P_2^2 we use (4–37) to get

$$A_0(q) = 1 + 3\epsilon_2^2 \cdot \frac{1}{5} = 1 + \frac{3}{5}\epsilon_2^2 \quad ; \tag{4-40}$$

A_2 and A_4 are removed by orthogonality, so that we do not need them. For $n = 2$ we have

$$r^5 = q^5(1 + 5\epsilon_2 P_2 + 5\epsilon_4 P_4 + 10\epsilon_2^2 P_2^2) \quad ,$$

so the only required term in (4–24) is

$$B_2 = 5\epsilon_2 + 10\epsilon_2^2 \cdot \frac{2}{7} = 5\left(\epsilon_2 + \frac{4}{7}\epsilon_2^2\right) \quad . \tag{4-41}$$

For $n = 4$ we similarly find

$$\begin{aligned} r^7 &= q^7(1 + 7\epsilon_2 P_2 + 7\epsilon_4 P_4 + 21\epsilon_2^2 P_2^2) \quad , \\ C_4 &= 7\epsilon_4 + 21\epsilon_2^2 \cdot \frac{18}{35} = 7\left(\epsilon_4 + \frac{54}{35}\epsilon_2^2\right) \quad . \end{aligned} \tag{4-42}$$

In (4–30) we have for $n = 0$ and 4:

$$\begin{aligned} r^2 &= q^2(1 + 2\epsilon_2 P_2 + 2\epsilon_4 P_4 + \epsilon_2^2 P_2^2) \quad , \\ D_0 &= 1 + \frac{1}{5}\epsilon_2^2 \quad ; \end{aligned} \tag{4–43}$$

$$\begin{aligned} r^{-2} &= q^{-2}(1 - 2\epsilon_2 P_2 - 2\epsilon_4 P_4 + 3\epsilon_2^2 P_2^2) \quad , \\ F_4 &= -2\left(\epsilon_4 - \frac{27}{35}\epsilon_2^2\right) \quad . \end{aligned} \tag{4–44}$$

Finally we introduce the flattening f. In (4–3) we put

$$\begin{aligned} \cos^2\theta &= \frac{1}{3} + \frac{2}{3}P_2(\cos\theta) \quad , \\ \sin^2 2\theta &= \frac{8}{15} + \frac{8}{21}P_2 - \frac{32}{35}P_4 \quad , \end{aligned} \tag{4–45}$$

which is directly verified by inserting (1–33).

Substituting into (4–3) and putting $P_2 = P_4 = 0$ (the average of P_n is zero!) we get the mean radius

$$q = a\left(1 - \frac{1}{3}f - \frac{1}{5}f^2 - \frac{8}{15}\kappa\right) \quad . \tag{4–46}$$

This is solved for a and substituted into (4–3), together with (4–45). The result is

$$r = q\left[1 - \frac{2}{3}\left(f + \frac{23}{42}f^2 + \frac{4}{7}\kappa\right)P_2 + \frac{4}{35}(3f^2 + 8\kappa)P_4\right] \tag{4–47}$$

with $P_n = P_n(\cos\theta)$, up to $O(f^2)$.

Following de Sitter, we introduce, instead of f, the auxiliary quantity

$$e = f - \frac{5}{42}f^2 + \frac{4}{7}\kappa \quad , \tag{4–48}$$

which we shall call *ellipticity*. (The ellipticity e is not to be confused with the first excentricity (1–55)!) To our approximation we may put

$$e^2 \doteq f^2 \quad ; \tag{4–49}$$

note also that $\kappa = O(f^2) = O(e^2)$.

In terms of e, (4–47) simplifies to

$$r = q\left[1 - \frac{2}{3}\left(e + \frac{2}{3}e^2\right)P_2(\cos\theta) + \frac{4}{35}(3e^2 + 8\kappa)P_4(\cos\theta)\right] \quad . \tag{4–50}$$

We notice that the second-order coefficient ***no longer contains the deviation*** κ: remember that κ represents the deviation of our spheroid from the ellipsoid (cf. Fig. 4.1), which holds for the internal equidensity surfaces ($q < R$) as well as for the bounding surface $q = R$.

The comparison between (4–11) and (4–50) immediately gives

$$\epsilon_2 = -\frac{2}{3}(e + \frac{2}{3}e^2) , \qquad \epsilon_4 = \frac{4}{35}(3e^2 + 8\kappa) \quad . \tag{4–51}$$

This is substituted into the expressions (4–40) through (4–44), whence (4–26) and (4–32), as well as (4–39), become

$$\begin{aligned}
K_0(q) &= \frac{4\pi G}{3}\int_0^q \rho \frac{d}{dq}\left[\left(1+\frac{4}{15}e^2\right)q^3\right]dq \quad , \\
L_0(q) &= 2\pi G\int_q^R \rho \frac{d}{dq}\left[\left(1+\frac{4}{45}e^2\right)q^2\right]dq \quad , \\
K_2(q) &= -\frac{8\pi G}{15}\int_0^q \rho \frac{d}{dq}\left[\left(e+\frac{2}{7}e^2\right)q^5\right]dq \quad , \\
L_2(q) &= -\frac{8\pi G}{15}\int_q^R \rho \frac{d}{dq}\left(e+\frac{16}{21}e^2\right)dq \quad , \\
K_4(q) &= \frac{16\pi G}{9}\int_0^q \rho \frac{d}{dq}\left[\left(\frac{9}{35}e^2+\frac{8}{35}\kappa\right)q^7\right]dq \quad , \\
L_4(q) &= \frac{4\pi G}{9}\int_q^R \rho \frac{d}{dq}\left(\frac{32}{35}\kappa q^{-2}\right)dq \quad .
\end{aligned} \tag{4–52}$$

Note that $\rho = \rho(q)$, $e = e(q)$, and $\kappa = \kappa(q)$.

4.1.5 Gravitational Potential at P

The potential V consists of V_i and V_e according to (4–6). The *first part* of the trick was to compute V_i at a point P_e (Fig. 4.3) and the potential V_e at a point P_i (Fig. 4.4) for which the critical series (4–8) and (4–27) *always converge*. Thus we have satisfied the *desideratum of Tisserand* (Tisserand, 1891, p. 317; Wavre, 1932, p. 68) of working with convergent series only.

The result were the finite (truncated!) expressions (4–10) and (4–31); finite because the terms with $n > 4$ would already be $O(f^4)$ which we have agreed to neglect. These formulas represent functions which are *harmonic* and hence analytic in the "empty" regions E_P for V_i and I_P for V_e; see Figs. 4.3 and 4.4. Being analytic, these expressions hold *throughout* E_P for V_i and I_P for V_e; in view of the continuity of the potential they must hold also at the point P itself! This transition $P_e \to P$, $P_i \to P$ forms the *second part* of the trick.

This simple argument shows that we may use the expressions (4–10) and (4–31) also for P, so that the total gravitational potential V is their sum:

$$
\begin{aligned}
V(P) = V(q,\theta) \;=\; & \frac{K_0(q)}{r} + L_0(q) + \\
+ \; & \left[\frac{K_2(q)}{r^3} + r^2 L_2(q)\right] P_2(\cos\theta) + \\
+ \; & \left[\frac{K_4(q)}{r^5} + r^4 L_4(q)\right] P_4(\cos\theta) \quad . \qquad (4\text{–}53)
\end{aligned}
$$

Here r and θ denote the spherical coordinates of the internal point P; the surface of constant density passing through P bears the label q (Fig. 4.2).

This reasoning also holds for $n > 4$: we are *working with convergent series only*. Thus we have achieved very simply the same result which Wavre has obtained by means of his very complicated "procédé uniforme". Quite another question is whether the *resulting* series is convergent. We have avoided this question by the simple (and usual) trick of limiting ourselves to the second-order (in f) approximation only, which automatically disregards higher-order terms.

Still the question remains open as a theoretical problem: the convergence of a spherical harmonic series at the boundary surface S_P. Nowadays we know much more about the convergence problem of spherical harmonic series than, say, twenty years ago; cf. (Moritz, 1980, secs. 6 and 7), especially the Runge–Krarup theorem. There may also be a relation to the existence proof by Liapunov and Lichtenstein mentioned in sec. 3.1. Another approach due to Trubitsyn is outlined in (Zharkov and Trubitsyn, 1978, sec. 38) and in (Denis, 1989).

The correctness of our second-order theory, however, is fully confirmed also by its derivation from Wavre's geometric theory to be treated in sec. 4.3, which is based on a completely different approach independent of any spherical-harmonic expansions.

4.2 Clairaut's and Darwin's Equations

4.2.1 Internal Gravity Potential

Following de Sitter (1924) we normalize the mean radius q and the density ρ by introducing the dimensionless quantities

$$\beta = \frac{q}{R} = \frac{\text{mean radius of } S_P}{\text{mean radius of earth}} \qquad (4\text{–}54)$$

and

$$\delta = \frac{\rho}{\rho_m} = \frac{\text{density}}{\text{mean density of earth}} \quad . \qquad (4\text{–}55)$$

The standard auxiliary expressions