

The small deformations $\delta f_k(\mathbf{x})$ being arbitrary, the integrand between brackets $\{\}$ must vanish:

$$\sum_{k=1}^n (\rho_k - \rho_{k+1})(W(\mathbf{x}) - \lambda_k) \delta[f_k(\mathbf{x})] = 0 \quad (3-116)$$

Now there is no more danger of confusion, so that now we were able to use the standard symbol δ instead of θ' for the Dirac delta function, but distinguish $\delta[f_k(\mathbf{x})]$ from $\delta f_k(\mathbf{x})$!

By the definition (3-100), the delta function $\delta[f_k(\mathbf{x})]$ vanishes everywhere except on the surface S_k , where it is different from zero (that it is even infinite there gives mathematicians a shudder but leaves physicists entirely cold). Thus since $\delta[f_k(\mathbf{x})] \neq 0$ on S_k , we must have $W(\mathbf{x}) - \lambda_k = 0$ or

$$W(\mathbf{x}) = \lambda_k = \text{constant on } S_k \quad (3-117)$$

which means that the boundary surfaces S_k of regions of constant density *must be equipotential surfaces*.

In the limit $n \rightarrow \infty$ of a continuous density we thus have recovered the basic fact that *the surfaces of constant density must be surfaces of constant potential*. This is our well-known condition for equilibrium figures.

What is new? Formerly, in sec. 2.5, we have derived this condition from (2-98) by means of the *pressure* p , a quantity which we have not used afterwards any more. For some people's taste, it is not very elegant to introduce an auxiliary concept which plays the role of a *deus ex machina* and disappears again. Here we have derived our basic condition $\rho = \text{const.} \iff W = \text{const.}$ from the principle of stationary energy, which is logically more satisfactory for many people, especially in view of the fact that maximum or minimum principles play a fundamental role in physics.

Another beautiful fact: the Lagrange multiplier λ_k admits a natural physical interpretation; it is nothing else than the constant value of the potential W on S_k , cf. (3-117).

3.3.5 A General Integral Equation

Now we are also in a position to give an explicit representation for the functions $f_k(\mathbf{x})$ which characterize the equisurfaces S_k : we may simply put

$$f_k(\mathbf{x}) = W(\mathbf{x}) - \lambda_k \quad (3-118)$$

In fact, on S_k we have $f_k(\mathbf{x}) = 0$ by (3-117), and inside S_k there is $f_k(\mathbf{x}) > 0$ since W increases monotonically towards the center. Thus (3-105) and (3-106) are satisfied.

Now in

$$W(\mathbf{x}) = G \int \frac{\rho}{l} dv + \frac{1}{2} \omega^2 (x^2 + y^2) \quad (= V + \Phi) \quad (3-119)$$

we may substitute (3-107) together with (3-118), obtaining

$$W(\mathbf{x}) = G \int \frac{dv}{l} \sum_{k=1}^n (\rho_k - \rho_{k+1}) \theta[W(\mathbf{x}') - \lambda_k] + \frac{1}{2} \omega^2 (x^2 + y^2) \quad (3-120)$$

with $l = \|\mathbf{x} - \mathbf{x}'\|$. This is a nonlinear integral equation for $W(\mathbf{x})$; the Lagrangian parameters λ_k are determined by the condition of constant volume (3-109):

$$v_k = \int \theta[W(\mathbf{x}') - \lambda_k] dv, \quad k = 1, 2, \dots, n. \quad (3-121)$$

It is straightforward to let $n \rightarrow \infty$ and perform the transition to a continuous (or piecewise continuous) mass distribution; then the index k becomes a continuous variable u :

$$W(\mathbf{x}) = -G \int \frac{dv}{l} \int_{u=0}^{\infty} d\rho(u) \theta[W(\mathbf{x}') - \lambda(u)] + \frac{1}{2} \omega^2 (x^2 + y^2), \quad (3-122)$$

$$v(u) = \int \theta[W(\mathbf{x}') - \lambda(u)] dv. \quad (3-123)$$

Doubtlessly, this formulation for hydrostatic equilibrium figures is extremely elegant: it generally and completely expresses the fact that the surfaces of constant density are also surfaces of constant potential. Voss (1965, 1966) has tried to solve this equation by a series expression in terms of f , f^2 , etc. (f is the flattening as usual). The linear terms again give Clairaut's theory, essentially the linear integral equation (3-15). Voss has also computed second-order terms, but the details are very complicated and these terms can be calculated much more directly as we shall see in the next section. Another application of the potential energy approach is found in (Chandrasekhar and Roberts, 1963).

Thus the main importance of the method described in the present section is theoretical and conceptual, but it is great indeed.