### 3.3.3 A Remarkable Expression for the Density

Assume the body to consist of $n$ layers bounded by surfaces $S_k$ and $S_{k+1}$ (Fig. 3.3). The density within each layer is constant, denoted in our case by $\rho_{k+1}$.

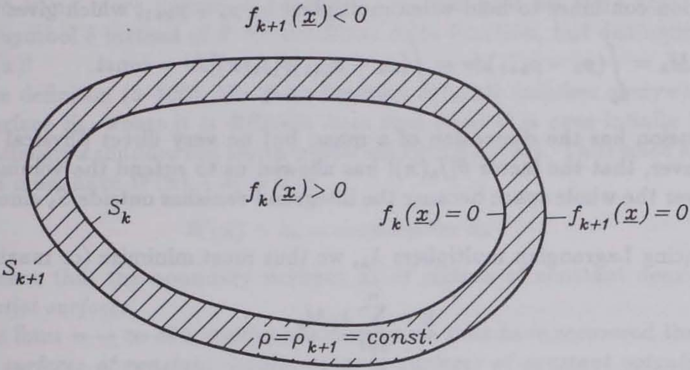


**FIGURE 3.3**: A layer of constant density ($\underline{x}$ denotes $\mathbf{x}$)

Let the surface $S_k$ have the equation

$$f_k(\mathbf{x}) = 0 \quad , \tag{3–105}$$

and let $f_k$ be monotonic with

$$f_k(\mathbf{x}) > 0 \qquad \text{inside } S_k \tag{3–106}$$

(otherwise change the sign of $f_k$!).

Then the density everywhere within the stratified body can be described by the single expression

$$\rho(\mathbf{x}) = \sum_{k=1}^{n} (\rho_k - \rho_{k+1})\theta[f_k(\mathbf{x})] \quad . \tag{3–107}$$

The reader is invited to verify this on the basis of (3–103) and (3–106). Eq. (3–107) holds with the understanding that $\rho_{n+1} = 0$ since the density is zero outside the boundary surface $S = S_n$.

### 3.3.4 Variation of the Potential Energy

Let us find the extremum of the potential energy $E = E_W$ as given by (3–99):

$$E = \int \left(\frac{1}{2} V + \Phi\right) \rho dv \quad , \tag{3–108}$$

where $\rho$ is expressed by (3–107); since $\rho = 0$ outside $S$, we may extend the integral formally over the whole space. The *side condition* is that the volume enclosed by any surface $S_k$ (Fig. 3.3) remains unchanged during the variation $\delta$ to be performed below:

$$v_k = \int_{S_k} dv = \text{const.} \tag{3–109}$$

This equation continues to hold when multiplied by $\rho_k - \rho_{k+1}$, which gives

$$M_k = \int_{S_k} (\rho_k - \rho_{k+1}) dv = \int (\rho_k - \rho_{k+1}) \theta[f_k(\mathbf{x})] dv = \text{const.} \tag{3–110}$$

This expression has the dimension of a mass, but no very direct physical meaning. Note, however, that the factor $\theta[f_k(\mathbf{x})]$ has allowed us to extend the volume integral formally over the whole space because the integrand vanishes outside $S_k$ since $f_k(\mathbf{x}) < 0$ there.

Introducing Lagrangian multipliers $\lambda_k$, we thus must minimize (or maximize)

$$E - \sum_{k=1}^{n} \lambda_k M_k \quad .$$

This leads to the variational condition ($\delta$ is now the sign for variation and has nothing to do with the Delta function!):

$$\delta \left[ E - \sum_{k=1}^{n} \lambda_k M_k \right] = 0 \tag{3–111}$$

or

$$\int (V + \Phi) \delta \rho dv - \sum_{k=1}^{n} \lambda_k \delta M_k = 0 \quad . \tag{3–112}$$

Note that we are varying the density $\rho$ by $\delta\rho$ and that, as compared to (3–108) the factor 1/2 seems to be missing. However, by (3–96), $E_V$ is a *quadratic* functional of $\rho$. This introduces the usual factor of 2 on differentiation, which combines with 1/2 to 1. With the gravity potential $W = V + \Phi$ this reduces to

$$\int W \delta \rho dv - \sum_{k=1}^{n} \lambda_k \delta M_k = 0 \quad . \tag{3–113}$$

Now we must express the density variations $\delta\rho$ by $\delta f_k(\mathbf{x})$ since $\delta\rho$ is caused by a change in the boundary surfaces only. Now our expression (3–107) comes in handy: we have

$$\delta\theta[f_k(\mathbf{x})] = \theta'[f_k(\mathbf{x})] \delta f_k(\mathbf{x}) \quad , \tag{3–114}$$

where $\theta'(x) = \delta(x)$ is the delta function by (3–104); we prefer the notation $\theta'$ to avoid confusion with the variation $\delta$.

With (3–114) everything is straightforward: (3–107) gives $\delta\rho$, and (3–110) similarly gives $\delta M_k$. Thus (3–113) becomes

$$\int dv \left\{ \sum_{k=1}^{n} (\rho_k - \rho_{k+1})(W(\mathbf{x}) - \lambda_k) \theta'[f_k(\mathbf{x})] \right\} \delta f_k(\mathbf{x}) = 0 \quad . \tag{3–115}$$

The small deformations $\delta f_k(\mathbf{x})$ being arbitrary, the integrand between brackets {} must vanish:

$$\sum_{k=1}^{n}(\rho_k - \rho_{k+1})(W(\mathbf{x}) - \lambda_k)\delta[f_k(\mathbf{x})] = 0 \quad . \tag{3-116}$$

Now there is no more danger of confusion, so that now we were able to use the standard symbol $\delta$ instead of $\theta'$ for the Dirac delta function, but distinguish $\delta[f_k(\mathbf{x})]$ from $\delta f_k(\mathbf{x})$!

By the definiton (3–100), the delta function $\delta[f_k(\mathbf{x})]$ vanishes everywhere except on the surface $S_k$, where it is different from zero (that it is even infinite there gives mathematicians a shudder but leaves physicists entirely cold). Thus since $\delta[f_k(\mathbf{x})] \neq 0$ on $S_k$, we must have $W(\mathbf{x}) - \lambda_k = 0$ or

$$W(\mathbf{x}) = \lambda_k = \text{constant on } S_k \quad , \tag{3-117}$$

which means that the boundary surfaces $S_k$ of regions of constant density *must be equipotential surfaces.*

In the limit $n \to \infty$ of a continuous density we thus have recovered the basic fact that *the surfaces of constant density must be surfaces of constant potential.* This is our well-known condition for equilibrium figures.

What is new? Formerly, in sec. 2.5, we have derived this condition from (2–98) by means of the *pressure* $p$, a quantity which we have not used afterwards any more. For some people's taste, it is not very elegant to introduce an auxiliary concept which plays the role of a *deus ex machina* and disappears again. Here we have derived our basic condition $\rho = \text{const.} \Longleftrightarrow W = \text{const.}$ from the principle of stationary energy, which is logically more satisfactory for many people, especially in view of the fact that maximum or minimum principles play a fundamental role in physics.

Another beautiful fact: the Lagrange multiplier $\lambda_k$ admits a natural physical interpretation; it is nothing else than the constant value of the potential $W$ on $S_k$, cf. (3–117).

### 3.3.5 A General Integral Equation

Now we are also in a position to give an explicit representation for the functions $f_k(\mathbf{x})$ which characterize the equisurfaces $S_k$: we may simply put

$$f_k(\mathbf{x}) = W(\mathbf{x}) - \lambda_k \quad . \tag{3-118}$$

In fact, on $S_k$ we have $f_k(\mathbf{x}) = 0$ by (3–117), and inside $S_k$ there is $f_k(\mathbf{x}) > 0$ since $W$ increases monotonically towards the center. Thus (3–105) and (3–106) are satisfied.

Now in

$$W(\mathbf{x}) = G\int \frac{\rho}{l}\,dv + \frac{1}{2}\omega^2(x^2 + y^2) \qquad (= V + \Phi) \tag{3-119}$$

we may substitute (3–107) together with (3–118), obtaining

$$W(\mathbf{x}) = G\int \frac{dv}{l}\sum_{k=1}^{n}(\rho_k - \rho_{k+1})\theta[W(\mathbf{x}') - \lambda_k] + \frac{1}{2}\omega^2(x^2 + y^2) \tag{3-120}$$