Lichtenstein determines the geometry from the physics. Also, for Lichtenstein, the spherical configuration is the starting point, whereas for Wavre it is a singularity (0/0)!

Wavre's approach is also described in the books (Baeschlin, 1948) and (Ledersteger, 1969), whereas the basic book in English, (Jardetzky, 1958), does not present it, although it outlines an approximation method also due to Wavre ("procédé uniforme") which intends, by an ingenious but complicated trick, to circumvent the convergence problem of certain series of spherical harmonics. We shall not treat this here because the author believes that this problem can be tackled in a much simpler way as we shall see in sec. 4.1.5.

3.3 Stationary Potential Energy

In various domains of physics, equilibrium is associated with a stationary (maximum or minimum, depending on the sign) value of potential energy. Liapunov and Poincaré have treated *homogeneous* equilibrium figures from this point of view; a modern approach is found in the book (Macke, 1967, p. 543). Macke's method has been generalized to heterogeneous (terrestrial) equilibrium figures (Macke et al., 1964; Voss, 1965, 1966). This approach is interesting because it reflects the typical thinking and mathematical methods of modern theoretical physics.

3.3.1 Potential Energy

The gravitational energy of a material particle of mass m in a field of potential V is mV, and that of a system of particles thus

$$E = \sum m_i V_i \quad ; \tag{3–93}$$

the sign (+ or −) is conventional.

This holds for an *external* potential field V. If the field is produced by the mutual gravitational attraction of the particles themselves:

$$V_i = G \sum_j \frac{m_j}{l_{ij}} \qquad (j \neq i) \quad , \tag{3–94}$$

then (3–93) gives

$$G \sum_{i,j} \frac{m_i m_j}{l_{ij}} \quad .$$

Each term occurs twice, however (interchange i and j), so that we must divide by 2, obtaining

$$E_V = \frac{1}{2} G \sum_i \sum_j \frac{m_i m_j}{l_{ij}} \qquad (j \neq i) \quad ; \tag{3–95}$$

cf. also (Kellogg, 1929, pp. 79–81) or (Poincaré, 1902, pp. 7–8).

The continuous analogue of (3–95) is

$$E_V = \frac{1}{2} G \iiint_v \iiint_v \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{||\mathbf{x}-\mathbf{x}'||} dv dv' \tag{3–96}$$

with obvious notations: $\mathbf{x}$, $\mathbf{x}'$ position vectors; v volume of the body; dv, dv' volume elements; and $l = ||\mathbf{x}-\mathbf{x}'||$. Another form of (3–96) is

$$E_V = \frac{1}{2} \iiint_v V \rho dv \quad , \tag{3–97}$$

where V is the usual gravitational potential. Comparing with (3–93) note the factor 1/2, reflecting the fact that E_V is produced by an *internal field* created by the mass elements $dm = \rho dv$ themselves.

For the centrifugal part we have

$$E_\Phi = \sum m_i \Phi_i = \iiint_v \Phi \rho dv \quad , \tag{3–98}$$

in agreement with (3–93), since the centrifugal potential Φ acts as an *external* field.

The potential energy of the gravity potential $W = V + \Phi$ thus is the sum of (3–97) and (3–98):

$$E_W = \int \left(\frac{1}{2} V + \Phi \right) \rho dv \quad , \tag{3–99}$$

using only a simple integral sign for notational convenience.

3.3.2 Dirac's and Heaviside's Functions

We recall the basic definition of *Dirac's delta function* (cf. Moritz, 1980, pp. 28–30):

$$\begin{aligned} \delta(x) &= 0 && \text{except for } x = 0 \quad , \\ \delta(0) &= \infty && \text{in such a way that} \end{aligned} \tag{3–100}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \quad . \tag{3–101}$$

It is a somewhat strange "function" but is extremely useful and popular in physics.

Its integral is Heaviside's step function:

$$\theta(x) = \int_{-\infty}^{x} \delta(x') dx' \quad . \tag{3–102}$$

From (3–100) and (3–101) it immediately follows that

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \quad , \\ 1 & \text{for } x > 0 \quad . \end{cases} \tag{3–103}$$

For $\theta(0)$ we may take the value 1/2.

From (3–102) there follows the basic relation

$$\delta(x) = \frac{d\theta(x)}{dx} = \theta'(x) \quad . \tag{3–104}$$