or

$$(2t^2f' + 2tf)(4\pi G\rho - 2\omega^2) = (-t^2f'' + 6f)g_P(t) \quad . \tag{3-88}$$

Corresponding to our approximation, we neglect the product of $f\omega^2$ (this removes ω^2 from our further considerations), and take $g_p(t)$ spherical, using (2–62):

$$g_P(t) = \frac{4\pi G}{3} tD(t) \quad . \tag{3-89}$$

Thus (3–88) reduces to

$$3\rho(2t^2f' + 2tf) = tD(-t^2f'' + 6f) \quad , \tag{3-90}$$

from which Clairaut's formula (2–114) follows immediately (with $t \doteq q$ in our approximation).

Note that Wavre's theory gives only Clairaut's differential equation, but not the boundary condition (2–118)!

The corresponding second-order theory is considerably more involved and will be treated in sec. 4.3.

3.2.6 Concluding Remarks

Wavre's theory is very beautiful and deep. Its true significance lies below the relatively simple mathematical formulism and is not so easily understood as the formulas themselves. We shall, therefore, try now to put Wavre's results into a proper perspective.

Equilibrium figures may be fully characterized by three conditions:

(A) The surfaces of constant potential coincide with the surfaces of constant density (sec. 2.5). Mathematically this means that the density ρ is only a function of the potential W or, in view of (3–39),

$$\Delta W = F(W) \quad ; \tag{3-91}$$

the Laplacian of W is a function only of W! This condition clearly has a differential and hence *local* character.

(B) The density ρ is positive and does not decrease towards the center. This is a natural condition, as the density models of sec. 1.5 show.

(C) The boundary surface S_0 of the equilibrium figure is an equipotential surface W = const.; outside S_0 there are no masses, so that the corresponding external potential V is harmonic everywhere outside S_0 and goes to zero as GM/r for $r \to \infty$. This may be considered a *global* condition.

In addition, we have the *symmetry* conditions:

(D) There is symmetry with respect to the equatorial plane, and rotational symmetry, the first being necessary, the second being a natural assumption.

Now it is basic that Wavre only uses the *local* condition (A) and the symmetry (D). *The global condition* (C) *is not taken into account at all*! Thus Wavre's theory is essentially incomplete. His results continue to hold if the equilibrium figure were surrounded by a rotationally symmetric mass configuration, such as an equatorial ring of Saturn type. Then, however, we can no longer speak of *free* equilibrium figures.

The basic Poisson equation (3–91) is equivalent both to Wavre's fundamental equation (3–40) — Bruns' formula (3–38) is nothing else than a sophisticated form of Poisson's equation as we have remarked after eq. (1–19) — and to the auxiliary equation (3–66). It is truly remarkable that one is able to prove such important results as Wavre's theorem (sec. 3.2.2) and the impossibility of a strictly ellipsoidal stratification (sec. 3.2.4) on the basis of this local theory only. The global condition (C) is not even necessary for these purposes!

Thus Wavre's equation (3–40), leading to his theorem (sec. 3.2.2) is a necessary but by no means sufficient condition for a free equilibrium figure since the global condition (C) is not taken into account.

It might now be tempting to reason in the following way. Eq. (3–40) holds for arbitrary Θ_1 and Θ_2. If we replace Θ_2 by Θ_3, we get the *purely geometrical relation*

$$\frac{(2JN - \partial \ln N/\partial t)_{\Theta_3} - (2JN - \partial \ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_3} - (N^2)_{\Theta_1}} =$$

$$= \frac{(2JN - \partial \ln N/\partial t)_{\Theta_2} - (2JN - \partial \ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_2} - (N^2)_{\Theta_1}} \quad , \tag{3–92}$$

which is a *necessary* condition for all stratifications of equilibrium figures. An equivalent form of this condition, with differences replaced by derivatives, is (3–46).

Assume now that this condition were also sufficient. Then we could remove the layer above any internal equisurface $S(t)$, cf. Fig. 3.2. For the remaining "reduced" figure bounded by $S(t)$, eq. (3–92) continues to hold for any of its internal equisurfaces, and the reduced figure would also be a possible figure of equilibrium.

This is Ledersteger's (1969, p. 536) "Prinzip der Entblätterung" (principle of removing shells bounded by two equisurfaces). For homogeneous ellipsoidal equilibrium figures (Maclaurin ellipsoids), this principle indeed holds since in this case, such shells are bounded by geometrically similar ellipsoids, and it is well known (*Newton's theorem*) that such an "ellipsoidal homoeoid" exerts no attraction in its interior; cf. (Kellogg, 1929, p. 22) or (Chandrasekhar, 1969, p. 39). Furthermore, the centrifugal force reduces proportionally.

For heterogeneous ellipsoidal figures, however, this principle does not hold (Voss, 1965), not even in the linear approximation. In fact, if it holds, we could remove the layer above the equisurface labeled by q, so that $\rho = 0$ above it and the second integral in (2–109) would vanish. Thus the term

$$\int_q^R \rho \frac{df}{dq'}\, dq'$$

would have to vanish identically, which only holds if f = const., for a homothetic (geometrically similar) stratification, and this is only possible for homogeneous figures, as we have seen in sec. 3.2.4. (In fact, the layer between $S(t)$ and $S(1) = S_0$ in Fig. 3.2 has the function of an external mass, not unlike to Saturn's ring mentioned above, for the "reduced" equilibrium figure bounded by $S(t)$!)

This confirms that (3–92) is only necessary but not sufficient. Hence, before applying Wavre's procedure described by (3–51) and (3–52), we must first make sure that *the given stratification corresponds to a possible figure of equilibrium*, which is by no means a simple matter, as already the counterexample of sec. 3.2.4 (non-homothetic ellipsoidal stratification) has shown.

To find such a possible stratification is a highly nontrivial problem indeed. In fact, no *rigorous* solution for a *heterogeneous* earthlike figure of equilibrium is known to the author. Heterogeneous solutions can only be constructed by a process of successive iteration or expansions with respect to powers of the flattening, the convergence for "small" values of the flattening f being guaranteed by the theorem of Liapunov–Lichtenstein mentioned at the beginning of sec. 3.1.

The "local" character of Wavre's theory is also expressed by the fact that it permits us to derive Clairaut's differential equation (2–114) but not the boundary condition (2–118), as we have pointed out at the end of sec. 3.2.5 and shall see again in more detail in sec. 4.3. Boundary conditions are typically *global.*

The theory of equilibrium figures is extremely subtle and full of unexpected pitfalls. There are "no–go theorems" such as the impossibility of a purely ellipsoidal stratification for heterogeneous equilibrium figures (sec. 3.2.4) and the fact that the terrestrial level ellipsoid cannot be an equilibrium figure, as we shall see in sec. 4.2.4 and later throughout Chapter 5 and then again in secs. 6.2 and 6.4. The latter fact was clearly recognized and repeatedly emphasized by Karl Ledersteger. It should be noticed here that Ledersteger was the last great geodesist who seriously and deeply engaged himself in Wavre's theory of equilibrium figures. This should be acknowledged even if one is not prepared to follow him all the way (see his "Prinzip der Entblätterung" as mentioned above).

Still, to first order in the flattening f, the level ellipsoid is an equilibrium figure with an approximately ellipsoidal stratification: this is Clairaut's theory, cf. sec. 3.2.5. Deviations from an ellipsoidal stratification start only in the second-order approximation (sec. 4.2.4) and are thus very small. Hence a very small change is sufficient to destroy equilibrium, which means that the property of being an equilibrium figure is extremely sensitive with respect to small perturbations: in a very special sense, it is an "unstable" property (this has nothing to do with the problem of instability of equilibrium figures which is important for stellar figures but not for the figure of the earth!). For another such "special instability" cf. sec. 3.2.3.

A final word on the relationship between Wavre's approach and the approach by Clairaut–Liapunov–Lichtenstein described in sec. 3.1. In a sense, the two approaches are "dialectical opposites". Wavre starts from a given stratification (the geometry) and determines the corresponding density distribution (the physics), whereas Lichtenstein starts from a given density distribution (which is initially spherical) and determines the configuration or stratification which results from a "small" rotation ω. Hence Wavre determines the physics of the problem from its geometry, whereas

Lichtenstein determines the geometry from the physics. Also, for Lichtenstein, the spherical configuration is the starting point, whereas for Wavre it is a singularity (0/0)!

Wavre's approach is also described in the books (Baeschlin, 1948) and (Ledersteger, 1969), whereas the basic book in English, (Jardetzky, 1958), does not present it, although it outlines an approximation method also due to Wavre ("procédé uniforme") which intends, by an ingenious but complicated trick, to circumvent the convergence problem of certain series of spherical harmonics. We shall not treat this here because the author believes that this problem can be tackled in a much simpler way as we shall see in sec. 4.1.5.

3.3 Stationary Potential Energy

In various domains of physics, equilibrium is associated with a stationary (maximum or minimum, depending on the sign) value of potential energy. Liapunov and Poincaré have treated *homogeneous* equilibrium figures from this point of view; a modern approach is found in the book (Macke, 1967, p. 543). Macke's method has been generalized to heterogeneous (terrestrial) equilibrium figures (Macke et al., 1964; Voss, 1965, 1966). This approach is interesting because it reflects the typical thinking and mathematical methods of modern theoretical physics.

3.3.1 Potential Energy

The gravitational energy of a material particle of mass m in a field of potential V is mV, and that of a system of particles thus

$$E = \sum m_i V_i \quad ; \tag{3–93}$$

the sign (+ or −) is conventional.

This holds for an *external* potential field V. If the field is produced by the mutual gravitational attraction of the particles themselves:

$$V_i = G \sum_j \frac{m_j}{l_{ij}} \qquad (j \neq i) \quad , \tag{3–94}$$

then (3–93) gives

$$G \sum_{i,j} \frac{m_i m_j}{l_{ij}} \quad .$$

Each term occurs twice, however (interchange i and j), so that we must divide by 2, obtaining

$$E_V = \frac{1}{2} G \sum_i \sum_j \frac{m_i m_j}{l_{ij}} \qquad (j \neq i) \quad ; \tag{3–95}$$

cf. also (Kellogg, 1929, pp. 79–81) or (Poincaré, 1902, pp. 7–8).