Let us repeat our argument. Eq. (3–73) leads necessarily to (3–76) and thus excludes any ellipsoidal stratification that is not homothetic, i.e., that does not consist of geometrically similar ellipsoids. Then (3–83) shows that the density must be homogeneous, which excludes heterogeneous equilibrium figures with ellipsoidal stratification. This proves the

Theorem of Hamy–Pizzetti

> An ellipsoidal stratification is impossible for heterogeneous, rotationally symmetric figures of equilibrium.

This is an extremely important "no–go theorem". The history of the subject starts with Hamy in 1887 and continues with work by Volterra in 1903 and Véronnet in 1912. The present method of proof was given by Pizzetti (1913, pp. 190–193) and essentially also used by Wavre (1932, pp. 60–61). We have tried to streamline it and to make every step explicit.

Later (secs. 4.2.4 and 6.4) we shall see that the terrestrial level ellipsoid, even with an arbitrary non–ellipsoidal internal stratification, cannot be an exact equilibrium figure, although it is extremely close to such a figure (Ledersteger's theorem).

3.2.5 Another Derivation of Clairaut's Equation

Although *rigorously*, the spheroidal equisurfaces are not ellipsoids, they are so *in linear approximation* (in f). Thus Wavre has used his equation (3–40) for a very elegant derivation of Clairaut' s equation. We put $\Theta_1 = 0$ (Pole P), $\Theta_2 = 90°$ (Equator E), and write, noting $N(t, 0) = 1$,

$$\begin{aligned} g(t,0) &= g_P(t) , & N(t,90°) &= N_E(t) , \\ J(t,0) &= J_P(t) , & J(t,90°) &= J_E(t) . \end{aligned} \tag{3–84}$$

The equisurfaces are (approximately!) ellipsoids of semiaxes $a(t)$ and $b(t) = t$, so that

$$a(t) = \frac{t}{1-f} = t\,(1 + f(t)) + O(f^2) \quad . \tag{3–85}$$

We further have

$$N_E(t) = \frac{da}{dt} = 1 + f(t) + tf'(t) \quad , \tag{3–86}$$

always disregarding $O(f^2)$. The ellipsoidal formulas of sec. 1.4 give the mean curvatures to our linear approximation:

$$J_P = \frac{1}{t}(1-2f) , \quad J_E = \frac{1}{t} , \tag{3–87}$$

so that (3–40), with (3–39), readily becomes

$$\frac{4\pi G\rho - 2\omega^2}{g_P(t)} = \frac{-t^2 f'' + 6f}{2t^2 f' + 2tf}$$

or

$$(2t^2 f' + 2tf)(4\pi G\rho - 2\omega^2) = (-t^2 f'' + 6f)g_P(t) \quad . \tag{3-88}$$

Corresponding to our approximation, we neglect the product of $f\omega^2$ (this removes ω^2 from our further considerations), and take $g_p(t)$ spherical, using (2–62):

$$g_P(t) = \frac{4\pi G}{3} tD(t) \quad . \tag{3-89}$$

Thus (3–88) reduces to

$$3\rho(2t^2 f' + 2tf) = tD(-t^2 f'' + 6f) \quad , \tag{3-90}$$

from which Clairaut's formula (2–114) follows immediately (with $t \doteq q$ in our approximation).

Note that Wavre's theory gives only Clairaut's differential equation, but not the boundary condition (2–118)!

The corresponding second-order theory is considerably more involved and will be treated in sec. 4.3.

3.2.6 Concluding Remarks

Wavre's theory is very beautiful and deep. Its true significance lies below the relatively simple mathematical formulism and is not so easily understood as the formulas themselves. We shall, therefore, try now to put Wavre's results into a proper perspective.

Equilibrium figures may be fully characterized by three conditions:

(A) The surfaces of constant potential coincide with the surfaces of constant density (sec. 2.5). Mathematically this means that the density ρ is only a function of the potential W or, in view of (3–39),

$$\Delta W = F(W) \quad ; \tag{3-91}$$

the Laplacian of W is a function only of W! This condition clearly has a differential and hence *local* character.

(B) The density ρ is positive and does not decrease towards the center. This is a natural condition, as the density models of sec. 1.5 show.

(C) The boundary surface S_0 of the equilibrium figure is an equipotential surface W = const.; outside S_0 there are no masses, so that the corresponding external potential V is harmonic everywhere outside S_0 and goes to zero as GM/r for $r \to \infty$. This may be considered a *global* condition.

In addition, we have the *symmetry* conditions:

(D) There is symmetry with respect to the equatorial plane, and rotational symmetry, the first being necessary, the second being a natural assumption.

Now it is basic that Wavre only uses the *local* condition (A) and the symmetry (D). *The global condition* (C) *is not taken into account at all*! Thus Wavre's theory