Eq. (3–48) holds for any Θ, and in particular for $\Theta = 0$, on the rotation axis. Thus we may integrate it along this axis from P_N to P_0 (Fig. 3.2):

$$\int_{P_N}^{P_0} \frac{1}{g}\frac{\partial g}{\partial t}\,dt = \int_1^t (-2JN + \Psi N^2)dt = \ln g_0 - \ln g_N \quad , \tag{3–49}$$

so that

$$g_0 = g_N \exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] = g(t,\,0) \quad , \tag{3–50}$$

where $g_N = g(1,\,0)$ denotes gravity at the pole.

Now (3–35), with $\Theta_1 = 0$ and $\Theta_2 = \Theta$, together with (3–50), gives

$$g(t,\,\Theta) = \frac{1}{N(t,\,\Theta)}\,g(t,\,0) = \frac{g_N}{N(t,\,\Theta)}\exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] \quad , \tag{3–51}$$

noting that $N(t,\,0) = 1$ as we have already remarked. Finally (3–47) and (3–34) yield

$$f(t) = -\Psi(t)N(t,\,\Theta)g(t,\,\Theta) \quad , \tag{3–52}$$

and hence the density $\rho(t)$ by (3–39).

Note the truly remarkable logical structure of these formulas: *the physics, especially the density distribution $\rho(t)$, is uniquely determined by the geometrical stratification.* In fact, given the geometry (J, N), we can compute $\Psi(t)$ by (3–40) or (3–45), and (3–47). Then gravity $g(t,\,\Theta)$ is obtained by (3–51), and finally the density ρ by (3–52) and (3–39). The only constants that must be given in addition to the set of surfaces $S(t)$, are the angular velocity ω and polar gravity g_N, which, however, are uniquely determined by ω and the total mass M ("Stokes constants"), using the theory of the external gravity field; cf. sec. 2.1 for a first-order approximation, sec. 5.2 for the (nonequilibrium case of the) level ellipsoid, and sec. 7.7.5 for a general definition of Stokes' constants. Thus we have

Wavre's Theorem

> The physics of equilibrium figures (density ρ, gravity g) is completely determined by the geometrical stratification, i.e., the set of equisurfaces $S(t)$ $(0 \leq t \leq 1)$, together with the total mass M and the angular velocity ω.

3.2.3 Spherical Stratification as an Exception

For a spherical stratification, Wavre's theorem does not apply since the right-hand side of (3–40) becomes 0/0 here, so that $\Psi(t)$ is not defined.

In fact, we have seen that a nonrotating spherical equilibrium configuration admits arbitrary density laws (ρ positive and nondecreasing towards the center). The actual earth is close to a spherical stratification, so that Wavre's theorem, although theoretically applicable, is not "stable": a large change of the density law may go along with an unmeasurably small variation of the geometrical configuration.

Thus, of course, the density distribution of the earth can only be determined empirically: from seismology, free oscillations, etc.

3.2.4 Impossibility of a Purely Ellipsoidal Stratification

Consider the equation of an ellipsoid of revolution

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \quad . \tag{3–53}$$

Putting $A = 1/a^2$ and $B = 1/b^2$ we may write this as

$$A(x^2+y^2)+Bz^2-1=0 \quad . \tag{3–54}$$

To get a family of equisurfaces we must let A and B depend on a parameter, for which we may take the potential W:

$$A(W)(x^2+y^2)+B(W)z^2-1=0 \quad . \tag{3–55}$$

In fact, for any $W =$ const. we get some ellipsoid of the family.

An auxiliary formula. Eq. (3–55) has the form

$$F(x, y, z, W)=0 \quad . \tag{3–56}$$

If we express W as a function of the coordinates:

$$W=W(x, y, z) \tag{3–57}$$

and substitute into (3–56), we get an identity:

$$\bar{F}(x, y, z) \equiv F(x, y, z, W(x, y, z)) \equiv 0 \quad , \tag{3–58}$$

which may be differentiated (supposing smoothness) as often as we like. We differentiate twice ($F_x = \partial F/\partial x$, $F_W = \partial F/\partial W$, etc.)

$$\bar{F}_x = F_x + F_W W_x \equiv 0 \quad , \tag{3–59}$$

$$\bar{F}_{xx} = F_{xx} + 2F_{xW}W_x + F_{WW}W_x^2 + F_W W_{xx} \equiv 0 \quad . \tag{3–60}$$

Then we express W_x from (3–59):

$$W_x = -\frac{F_x}{F_W} \tag{3–61}$$

and substitute into (3–60), obtaining

$$F_{xx} - 2\frac{1}{F_W}F_x F_{xW} + \frac{1}{F_W^2}F_{WW}F_x^2 + F_W W_{xx} = 0 \quad . \tag{3–62}$$