which holds for arbitrary Θ_1 and Θ_2. These equations are due to Wavre. Going one step further, we may put

$$\begin{aligned} \Theta_1 &= \Theta \quad , \\ \Theta_2 &= \Theta + h \quad , \end{aligned}$$

so that (3–43) may be written as

$$\frac{f(t)}{W'(t)} = \frac{\dfrac{Y(t,\,\Theta+h) - Y(t,\,\Theta)}{h}}{\dfrac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h}} \quad .$$

Now, however, we may let $h \to 0$, obtaining according to the definition of the partial derivative

$$\frac{\partial X}{\partial \Theta} = \lim_{h \to 0} \frac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h} \quad , \tag{3–44}$$

the form

$$\frac{f(t)}{W'(t)} = \frac{\partial Y/\partial \Theta}{\partial X/\partial \Theta} = \text{function of } t \text{ only} \quad . \tag{3–45}$$

This is an identity in t and Θ, which will be useful in sec. 4.3. Another elegant formula is obtained by differentiating this identity with respect to Θ:

$$\frac{\partial}{\partial \Theta}\left(\frac{\partial Y/\partial \Theta}{\partial X/\partial \Theta}\right) = 0 \quad , \tag{3–46}$$

which is another expression of the fact that the quotient $(\partial Y/\partial \Theta)/(\partial X/\partial \Theta)$ does not explicitly depend on Θ, being a function of t only. Since by differentiation we lose $f(t)/W'(t)$, eq. (3–46) contains less information than (3–45), however.

3.2.2 Wavre's Theorem

Put for the left-hand side of (3–40) or (3–45):

$$\Psi(t) = \frac{f(t)}{W'(t)} \quad . \tag{3–47}$$

Then (3–37), using (3–33), (3–38) and (3–47), can be brought into the form

$$\frac{1}{g}\frac{\partial g}{\partial t} = -2JN + \Psi N^2 \quad , \tag{3–48}$$

which again is a function of the geometrical stratification only and does not depend on the density! This is a direct consequence of the definition (3–47) and of the remarkable properties of (3–40) just pointed out.

Eq. (3–48) holds for any Θ, and in particular for $\Theta = 0$, on the rotation axis. Thus we may integrate it along this axis from P_N to P_0 (Fig. 3.2):

$$\int_{P_N}^{P_0} \frac{1}{g}\frac{\partial g}{\partial t}\,dt = \int_1^t (-2JN + \Psi N^2)dt = \ln g_0 - \ln g_N \quad , \tag{3–49}$$

so that

$$g_0 = g_N \exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] = g(t,\,0) \quad , \tag{3–50}$$

where $g_N = g(1,\,0)$ denotes gravity at the pole.

Now (3–35), with $\Theta_1 = 0$ and $\Theta_2 = \Theta$, together with (3–50), gives

$$g(t,\,\Theta) = \frac{1}{N(t,\,\Theta)}\,g(t,\,0) = \frac{g_N}{N(t,\,\Theta)}\exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] \quad , \tag{3–51}$$

noting that $N(t,\,0) = 1$ as we have already remarked. Finally (3–47) and (3–34) yield

$$f(t) = -\Psi(t)N(t,\,\Theta)g(t,\,\Theta) \quad , \tag{3–52}$$

and hence the density $\rho(t)$ by (3–39).

Note the truly remarkable logical structure of these formulas: *the physics, especially the density distribution $\rho(t)$, is uniquely determined by the geometrical stratification.* In fact, given the geometry (J, N), we can compute $\Psi(t)$ by (3–40) or (3–45), and (3–47). Then gravity $g(t,\,\Theta)$ is obtained by (3–51), and finally the density ρ by (3–52) and (3–39). The only constants that must be given in addition to the set of surfaces $S(t)$, are the angular velocity ω and polar gravity g_N, which, however, are uniquely determined by ω and the total mass M ("Stokes constants"), using the theory of the external gravity field; cf. sec. 2.1 for a first-order approximation, sec. 5.2 for the (nonequilibrium case of the) level ellipsoid, and sec. 7.7.5 for a general definition of Stokes' constants. Thus we have

Wavre's Theorem

> The physics of equilibrium figures (density ρ, gravity g) is completely determined by the geometrical stratification, i.e., the set of equisurfaces $S(t)$ $(0 \le t \le 1)$, together with the total mass M and the angular velocity ω.

3.2.3 Spherical Stratification as an Exception

For a spherical stratification, Wavre's theorem does not apply since the right-hand side of (3–40) becomes 0/0 here, so that $\Psi(t)$ is not defined.

In fact, we have seen that a nonrotating spherical equilibrium configuration admits arbitrary density laws (ρ positive and nondecreasing towards the center). The