3.2.1 Stratification of Equisurfaces

Let $S(t)$ denote the set of equisurfaces (surfaces of constant density and of constant potential), as a function of a parameter t (there is no danger of confusing it with time!). The parameter t thus "labels" the individual equisurfaces and could, in principle, be selected in many ways. Formerly, we have labeled the equisurface by its *mean radius* q, but in Wavre's theory it is more convenient instead to take the parameter t as the *semiminor axis* of the spheroidal equisurface under consideration. (This is well known since the ellipsoidal coordinate u also has this character, cf. sec. 5.1. For the limiting ("free") surface S we take $t = 1$, so that $S = S(1)$.

We again assume rotational symmetry about the z-axis, knowing already that the stratification must also be symmetric with respect to the equatorial plane (invariance for $z \to -z$). Thus we have no dependence on longitude λ; as latitudinal coordinate we take a parameter Θ that labels the plumb lines as indicated in Fig. 3.2.

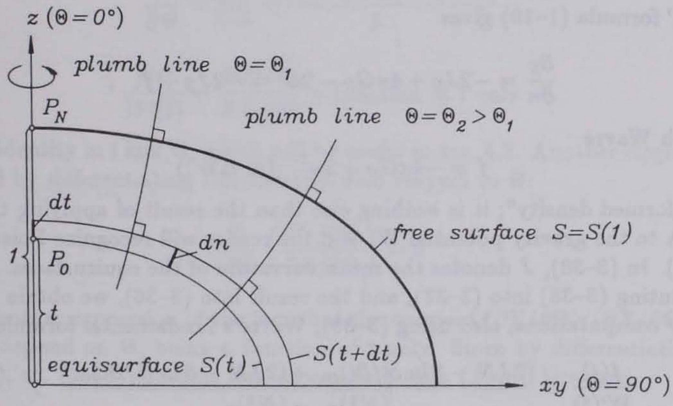

FIGURE 3.2: The geometry of stratification

Since the equisurfaces t = const. are not parallel, their infinitesimal distance dn differs, in general, from dt. We put

$$\frac{dn}{dt} = N(t, \Theta) \quad , \tag{3–32}$$

where the function N is unknown *a priori*. Note that N is always positive (from geometry), dimensionless (by our choice of units) and equals 1 on the rotation axis $\Theta = 0$. (The symbol N has also been used for the geoidal height and the ellipsoidal normal radius of curvature!)

Since, by definition, the potential W depends on t only, we have for gravity

$$g = -\frac{\partial W}{\partial n} = -\frac{dW}{dt} \Big/ \frac{dn}{dt} = -\frac{1}{N}\frac{dW}{dt} \quad . \tag{3–33}$$

Hence

$$\frac{dW}{dt} = -gN = W'(t) \tag{3–34}$$

is a function only of t, although g and N depend also on Θ. In other terms

$$(gN)_{\Theta_1} = (gN)_{\Theta_2} \quad ; \tag{3–35}$$

the product gN is independent of Θ along an equisurface $S(t)$.

Since (3–35) is an identity in t, it can be differentiated:

$$\left(\frac{\partial g}{\partial t} N + g\frac{\partial N}{\partial t}\right)_{\Theta_1} = \left(\frac{\partial g}{\partial t} N + g\frac{\partial N}{\partial t}\right)_{\Theta_2} . \tag{3–36}$$

Now by (3–32),

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial n}\frac{dn}{dt} = N\frac{\partial g}{\partial n} \quad , \tag{3–37}$$

and Bruns' formula (1–19) gives

$$\frac{\partial g}{\partial n} = -2Jg + 4\pi G\rho - 2\omega^2 = -2Jg - f \quad , \tag{3–38}$$

calling with Wavre

$$f = -4\pi G\rho + 2\omega^2 \quad (\equiv \Delta W!) \tag{3–39}$$

the "transformed density"; it is nothing else than the result of applying the Laplace operator Δ to the gravity potential W, and the reader will recognize Poisson's equation (1–14). In (3–38), J denotes the mean curvature of the equisurfaces.

Substituting (3–38) into (3–37), and the result into (3–36), we obtain after some elementary computations, also using (3–33), Wavre's fundamental formula

$$\frac{f(t)}{W'(t)} = \frac{(2JN - \partial \ln N/\partial t)_{\Theta_2} - (2JN - \partial \ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_2} - (N^2)_{\Theta_1}} . \tag{3–40}$$

This equation is remarkable in that it provides a neat separation of the geometry and the physics of equilibrium surfaces: the left-hand side, containing physical quantities such as density ρ and potential W, depends only on t, whereas the right-hand side *depends only on the geometry of stratification* (J, N) and is independent of the density distribution!

Putting

$$X = X(t, \Theta) = N^2 \quad , \tag{3–41}$$

$$Y = Y(t, \Theta) = 2JN - \frac{\partial \ln N}{\partial t} = 2JN - \frac{1}{N}\frac{\partial N}{\partial t} \quad , \tag{3–42}$$

we may write (3–40) briefly as

$$\frac{f(t)}{W'(t)} = \frac{Y(t, \Theta_2) - Y(t, \Theta_1)}{X(t, \Theta_2) - X(t, \Theta_1)} \quad , \tag{3–43}$$

which holds for arbitrary Θ_1 and Θ_2. These equations are due to Wavre. Going one step further, we may put

$$\begin{aligned}\Theta_1 &= \Theta \quad ,\\ \Theta_2 &= \Theta + h \quad ,\end{aligned}$$

so that (3–43) may be written as

$$\frac{f(t)}{W'(t)} = \frac{\dfrac{Y(t,\,\Theta+h) - Y(t,\,\Theta)}{h}}{\dfrac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h}} \quad .$$

Now, however, we may let $h \to 0$, obtaining according to the definition of the partial derivative

$$\frac{\partial X}{\partial \Theta} = \lim_{h\to 0} \frac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h} \quad , \tag{3–44}$$

the form

$$\frac{f(t)}{W'(t)} = \frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta} = \text{function of } t \text{ only} \quad . \tag{3–45}$$

This is an identity in t and Θ, which will be useful in sec. 4.3. Another elegant formula is obtained by differentiating this identity with respect to Θ:

$$\frac{\partial}{\partial\Theta}\left(\frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta}\right) = 0 \quad , \tag{3–46}$$

which is another expression of the fact that the quotient $(\partial Y/\partial\Theta)/(\partial X/\partial\Theta)$ does not explicitly depend on Θ, being a function of t only. Since by differentiation we lose $f(t)/W'(t)$, eq. (3–46) contains less information than (3–45), however.

3.2.2 Wavre's Theorem

Put for the left-hand side of (3–40) or (3–45):

$$\Psi(t) = \frac{f(t)}{W'(t)} \quad . \tag{3–47}$$

Then (3–37), using (3–33), (3–38) and (3–47), can be brought into the form

$$\frac{1}{g}\frac{\partial g}{\partial t} = -2JN + \Psi N^2 \quad , \tag{3–48}$$

which again is a function of the geometrical stratification only and does not depend on the density! This is a direct consequence of the definition (3–47) and of the remarkable properties of (3–40) just pointed out.