We further take from (2–57)

$$g(q) = \frac{4\pi G}{q^2} \int_0^q \rho' q'^2 dq' \tag{3–30}$$

and restore the subscript 2 to ζ. Then the comparison of (3–18) and (2–82), noting $r = OP = q + \zeta$ (Fig. 3.1) and $\zeta_0 = 0$, gives

$$\zeta = \zeta_2(q) = -\frac{2}{3}\, q f(q) \quad .$$

Thus (3–27) becomes, on omitting the prime on ρ so that $\rho = \rho(q')$ and similarly for f after the integral,

$$\begin{aligned} -\frac{2}{3}\frac{f}{q} \int_0^q \rho q'^2 dq' + \frac{2}{15}\frac{1}{q^3} \int_0^q \rho\, d(f q'^5) \, + \\ +\frac{2}{15} q^2 \int_q^R \rho\, df + \frac{\omega^2 q^2}{12\pi G} = 0 \quad , \end{aligned} \tag{3–31}$$

which is identical to (2–106) (up to a factor $15q^3/2$ which cancels), on noting, e.g.,

$$df = \frac{df}{dq'}\, dq' \quad .$$

Since Clairaut's equation (2–114), plus boundary condition (2–118), was a direct consequence of (2–106), it equally follows from (3–31).

This provides another method for deriving Clairaut's equation, which has the advantage of using an integral equation similar to the integral equations customary from Molodensky's approach to physical geodesy.

Therefore it is not surprising after all that even Molodensky (1988) occupied himself with the integral equation of Lichtenstein!

3.2 The Geometry of Equilibrium Surfaces

Clairaut's equation (2–114) for the basic geometric quantity, the flattening f, is a *homogeneous* differential equation.

Homogeneous differential equations (with right-hand side zero) with independent variable t, time, correspond to *free* motion, as opposed to forced motion. In the present case, the independent variable is the radius r rather than time, but the argument may indicate that the geometry of the equisurfaces for equilibrium figures seems to have a considerable autonomy.

This idea was thoroughly investigated in the fundamental book (Wavre, 1932). Since it is little known in the English-speaking scientific community, we shall outline Wavre's theory of stratification of equilibrium figures (which is *rigorous*).

3.2.1 Stratification of Equisurfaces

Let $S(t)$ denote the set of equisurfaces (surfaces of constant density and of constant potential), as a function of a parameter t (there is no danger of confusing it with time!). The parameter t thus "labels" the individual equisurfaces and could, in principle, be selected in many ways. Formerly, we have labeled the equisurface by its *mean radius* q, but in Wavre's theory it is more convenient instead to take the parameter t as the *semiminor axis* of the spheroidal equisurface under consideration. (This is well known since the ellipsoidal coordinate u also has this character, cf. sec. 5.1. For the limiting ("free") surface S we take $t = 1$, so that $S = S(1)$.

We again assume rotational symmetry about the z-axis, knowing already that the stratification must also be symmetric with respect to the equatorial plane (invariance for $z \to -z$). Thus we have no dependence on longitude λ; as latitudinal coordinate we take a parameter Θ that labels the plumb lines as indicated in Fig. 3.2.

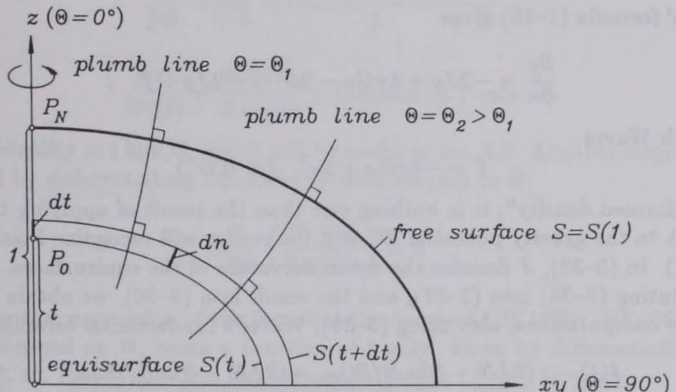

FIGURE 3.2: The geometry of stratification

Since the equisurfaces t = const. are not parallel, their infinitesimal distance dn differs, in general, from dt. We put

$$\frac{dn}{dt} = N(t, \Theta) \quad , \tag{3–32}$$

where the function N is unknown *a priori*. Note that N is always positive (from geometry), dimensionless (by our choice of units) and equals 1 on the rotation axis $\Theta = 0$. (The symbol N has also been used for the geoidal height and the ellipsoidal normal radius of curvature!)

Since, by definition, the potential W depends on t only, we have for gravity

$$g = -\frac{\partial W}{\partial n} = -\frac{dW}{dt} \bigg/ \frac{dn}{dt} = -\frac{1}{N}\frac{dW}{dt} \quad . \tag{3–33}$$

Hence

$$\frac{dW}{dt} = -gN = W'(t) \tag{3-34}$$

is a function only of t, although g and N depend also on Θ. In other terms

$$(gN)_{\Theta_1} = (gN)_{\Theta_2} \quad ; \tag{3-35}$$

the product gN is independent of Θ along an equisurface $S(t)$.

Since (3–35) is an identity in t, it can be differentiated:

$$\left(\frac{\partial g}{\partial t} N + g\frac{\partial N}{\partial t}\right)_{\Theta_1} = \left(\frac{\partial g}{\partial t} N + g\frac{\partial N}{\partial t}\right)_{\Theta_2} . \tag{3-36}$$

Now by (3–32),

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial n}\frac{dn}{dt} = N\frac{\partial g}{\partial n} \quad , \tag{3-37}$$

and Bruns' formula (1–19) gives

$$\frac{\partial g}{\partial n} = -2Jg + 4\pi G\rho - 2\omega^2 = -2Jg - f \quad , \tag{3-38}$$

calling with Wavre

$$f = -4\pi G\rho + 2\omega^2 \quad (\equiv \Delta W!) \tag{3-39}$$

the "transformed density"; it is nothing else than the result of applying the Laplace operator Δ to the gravity potential W, and the reader will recognize Poisson's equation (1–14). In (3–38), J denotes the mean curvature of the equisurfaces.

Substituting (3–38) into (3–37), and the result into (3–36), we obtain after some elementary computations, also using (3–33), Wavre's fundamental formula

$$\frac{f(t)}{W'(t)} = \frac{(2JN - \partial \ln N/\partial t)_{\Theta_2} - (2JN - \partial \ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_2} - (N^2)_{\Theta_1}} \quad . \tag{3-40}$$

This equation is remarkable in that it provides a neat separation of the geometry and the physics of equilibrium surfaces: the left-hand side, containing physical quantities such as density ρ and potential W, depends only on t, whereas the right-hand side *depends only on the geometry of stratification* (J, N) and is independent of the density distribution!

Putting

$$X = X(t, \Theta) = N^2 \quad , \tag{3-41}$$

$$Y = Y(t, \Theta) = 2JN - \frac{\partial \ln N}{\partial t} = 2JN - \frac{1}{N}\frac{\partial N}{\partial t} \quad , \tag{3-42}$$

we may write (3–40) briefly as

$$\frac{f(t)}{W'(t)} = \frac{Y(t, \Theta_2) - Y(t, \Theta_1)}{X(t, \Theta_2) - X(t, \Theta_1)} \quad , \tag{3-43}$$

which holds for arbitrary Θ_1 and Θ_2. These equations are due to Wavre. Going one step further, we may put

$$\begin{aligned}\Theta_1 &= \Theta \quad , \\ \Theta_2 &= \Theta + h \quad ,\end{aligned}$$

so that (3–43) may be written as

$$\frac{f(t)}{W'(t)} = \frac{\dfrac{Y(t,\,\Theta+h) - Y(t,\,\Theta)}{h}}{\dfrac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h}} \quad .$$

Now, however, we may let $h \to 0$, obtaining according to the definition of the partial derivative

$$\frac{\partial X}{\partial \Theta} = \lim_{h \to 0} \frac{X(t,\,\Theta+h) - X(t,\,\Theta)}{h} \quad , \tag{3–44}$$

the form

$$\frac{f(t)}{W'(t)} = \frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta} = \text{function of } t \text{ only} \quad . \tag{3–45}$$

This is an identity in t and Θ, which will be useful in sec. 4.3. Another elegant formula is obtained by differentiating this identity with respect to Θ:

$$\frac{\partial}{\partial\Theta}\left(\frac{\partial Y/\partial\Theta}{\partial X/\partial\Theta}\right) = 0 \quad , \tag{3–46}$$

which is another expression of the fact that the quotient $(\partial Y/\partial\Theta)/(\partial X/\partial\Theta)$ does not explicitly depend on Θ, being a function of t only. Since by differentiation we lose $f(t)/W'(t)$, eq. (3–46) contains less information than (3–45), however.

3.2.2 Wavre's Theorem

Put for the left-hand side of (3–40) or (3–45):

$$\Psi(t) = \frac{f(t)}{W'(t)} \quad . \tag{3–47}$$

Then (3–37), using (3–33), (3–38) and (3–47), can be brought into the form

$$\frac{1}{g}\frac{\partial g}{\partial t} = -2JN + \Psi N^2 \quad , \tag{3–48}$$

which again is a function of the geometrical stratification only and does not depend on the density! This is a direct consequence of the definition (3–47) and of the remarkable properties of (3–40) just pointed out.

Eq. (3–48) holds for any Θ, and in particular for $\Theta = 0$, on the rotation axis. Thus we may integrate it along this axis from P_N to P_0 (Fig. 3.2):

$$\int_{P_N}^{P_0} \frac{1}{g}\frac{\partial g}{\partial t}\,dt = \int_1^t (-2JN + \Psi N^2)dt = \ln g_0 - \ln g_N \quad , \tag{3–49}$$

so that

$$g_0 = g_N \exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] = g(t,\,0) \quad , \tag{3–50}$$

where $g_N = g(1,\,0)$ denotes gravity at the pole.

Now (3–35), with $\Theta_1 = 0$ and $\Theta_2 = \Theta$, together with (3–50), gives

$$g(t,\,\Theta) = \frac{1}{N(t,\,\Theta)}\,g(t,\,0) = \frac{g_N}{N(t,\,\Theta)}\exp\left[\int_1^t (-2JN + \Psi N^2)dt\right] \quad , \tag{3–51}$$

noting that $N(t,\,0) = 1$ as we have already remarked. Finally (3–47) and (3–34) yield

$$f(t) = -\Psi(t)N(t,\,\Theta)g(t,\,\Theta) \quad , \tag{3–52}$$

and hence the density $\rho(t)$ by (3–39).

Note the truly remarkable logical structure of these formulas: *the physics, especially the density distribution $\rho(t)$, is uniquely determined by the geometrical stratification.* In fact, given the geometry $(J,\,N)$, we can compute $\Psi(t)$ by (3–40) or (3–45), and (3–47). Then gravity $g(t,\,\Theta)$ is obtained by (3–51), and finally the density ρ by (3–52) and (3–39). The only constants that must be given in addition to the set of surfaces $S(t)$, are the angular velocity ω and polar gravity g_N, which, however, are uniquely determined by ω and the total mass M ("Stokes constants"), using the theory of the external gravity field; cf. sec. 2.1 for a first-order approximation, sec. 5.2 for the (nonequilibrium case of the) level ellipsoid, and sec. 7.7.5 for a general definition of Stokes' constants. Thus we have

Wavre's Theorem

> The physics of equilibrium figures (density ρ, gravity g) is completely determined by the geometrical stratification, i.e., the set of equisurfaces $S(t)$ $(0 \le t \le 1)$, together with the total mass M and the angular velocity ω.

3.2.3 Spherical Stratification as an Exception

For a spherical stratification, Wavre's theorem does not apply since the right-hand side of (3–40) becomes 0/0 here, so that $\Psi(t)$ is not defined.

In fact, we have seen that a nonrotating spherical equilibrium configuration admits arbitrary density laws (ρ positive and nondecreasing towards the center). The actual earth is close to a spherical stratification, so that Wavre's theorem, although theoretically applicable, is not "stable": a large change of the density law may go along with an unmeasurably small variation of the geometrical configuration.

Thus, of course, the density distribution of the earth can only be determined empirically: from seismology, free oscillations, etc.

3.2.4 Impossibility of a Purely Ellipsoidal Stratification

Consider the equation of an ellipsoid of revolution

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \quad . \tag{3–53}$$

Putting $A = 1/a^2$ and $B = 1/b^2$ we may write this as

$$A(x^2+y^2)+Bz^2-1=0 \quad . \tag{3–54}$$

To get a family of equisurfaces we must let A and B depend on a parameter, for which we may take the potential W:

$$A(W)(x^2+y^2)+B(W)z^2-1=0 \quad . \tag{3–55}$$

In fact, for any $W = $ const. we get some ellipsoid of the family.

An auxiliary formula. Eq. (3–55) has the form

$$F(x, y, z, W)=0 \quad . \tag{3–56}$$

If we express W as a function of the coordinates:

$$W=W(x, y, z) \tag{3–57}$$

and substitute into (3–56), we get an identity:

$$\bar{F}(x, y, z) \equiv F(x, y, z, W(x, y, z)) \equiv 0 \quad , \tag{3–58}$$

which may be differentiated (supposing smoothness) as often as we like. We differentiate twice ($F_x = \partial F/\partial x$, $F_W = \partial F/\partial W$, etc.)

$$\bar{F}_x = F_x + F_W W_x \equiv 0 \quad , \tag{3–59}$$

$$\bar{F}_{xx} = F_{xx} + 2F_{xW}W_x + F_{WW}W_x^2 + F_W W_{xx} \equiv 0 \quad . \tag{3–60}$$

Then we express W_x from (3–59):

$$W_x = -\frac{F_x}{F_W} \tag{3–61}$$

and substitute into (3–60), obtaining

$$F_{xx} - 2\frac{1}{F_W}F_xF_{xW} + \frac{1}{F_W^2}F_{WW}F_x^2 + F_W W_{xx} = 0 \quad . \tag{3–62}$$

Now we replace x by y and then by z and add the three equations thus obtained, considering the Laplacian

$$F_{xx} + F_{yy} + F_{zz} = \Delta F \quad . \tag{3-63}$$

The result is

$$\Delta F - \frac{1}{F_W}\frac{\partial}{\partial W}(F_x^2 + F_y^2 + F_z^2) + \frac{1}{F_W^2} F_{WW}(F_x^2 + F_y^2 + F_z^2) + F_W \Delta W = 0 \quad . \tag{3-64}$$

Now, by (3–39)

$$\Delta W = -4\pi G\rho + 2\omega^2 = f = f(W) \tag{3-65}$$

must be a function of the potential W only, if the equilibrium condition (2–100) is to hold. Thus (3–64) becomes

$$\frac{1}{F_W^3}\left[-F_W^2 \Delta F + F_W \frac{\partial}{\partial W}(F_x^2 + F_y^2 + F_z^2) - F_{WW}(F_x^2 + F_y^2 + F_z^2)\right] = f(W) \quad , \tag{3-66}$$

which is our auxiliary equation which any family of equisurfaces must satisfy.

The ellipsoidal case. Comparing (3–56) with (3–55) we have for ellipsoidal equisurfaces

$$F(x, y, z, W) = A(W)(x^2 + y^2) + B(W)z^2 - 1 \quad . \tag{3-67}$$

Putting $A' = A'(W) = dA/dW$, we find

$$\begin{aligned} F_W &= \frac{\partial F}{\partial W} = A'(x^2 + y^2) + B'z^2 \ , \\ F_{WW} &= A''(x^2 + y^2) + B''z^2 \ , \end{aligned} \tag{3-68}$$

and

$$F_x = 2Ax \ , \quad F_{xx} = 2A \ , \ \text{etc.} \quad ,$$

so that

$$\begin{aligned} \Delta F &= 2(2A + B) \ , \\ F_x^2 + F_y^2 + F_z^2 &= 4\left[A^2(x^2 + y^2) + B^2 z^2\right] \ , \\ \frac{\partial}{\partial W}(F_x^2 + F_y^2 + F_z^2) &= 8\left[AA'(x^2 + y^2) + BB'z^2\right] \ . \end{aligned} \tag{3-69}$$

On expressing $x^2 + y^2$ by (3–55):

$$x^2 + y^2 = \frac{1}{A} - \frac{B}{A}z^2 \quad , \tag{3-70}$$

all equations (3–68) and (3–69) with the exception of ΔF take the form

$$P + Qz^2$$

linear in z^2, with appropriate functions $P = P(W)$ and $Q = Q(W)$. Thus our basic equation (3–66) is readily seen to take the form

$$\frac{\text{polynomial of fourth degree in } z}{\text{polynomial of sixth degree in } z} = f(W) \quad , \tag{3–71}$$

where the function $f(W)$ is independent of z.

This is impossible unless the sixth–degree polynomial

$$F_W^3 = \left[\frac{A'}{A} + \left(B' - B\,\frac{A'}{A}\right) z^2\right]^3 \tag{3–72}$$

degenerates, which requires that the coefficient of z^2 vanishes:

$$B' - B\,\frac{A'}{A} = 0 \quad . \tag{3–73}$$

This immediately leads to the differential equation

$$\frac{dB}{B} = \frac{dA}{A} \quad ,$$

which can be directly integrated:

$$\begin{aligned} \ln B &= \ln A + \ln k \quad , \\ B(W) &= k\,A(W) \quad , \end{aligned} \tag{3–74}$$

k being a constant of integration (independent of W).

The equation of our equisurface (3–55) thus becomes

$$A(W)(x^2 + y^2 + k\,z^2) = 1 \quad . \tag{3–75}$$

This is an ellipsoid for which the semiaxes have the ratio

$$\frac{a(W)}{b(W)} = \sqrt{\frac{k\,A(W)}{A(W)}} = \sqrt{k} = \text{const.} \tag{3–76}$$

independent of W. This means that *all ellipsoids of our family of equisurfaces are geometrically similar* (homothetic). This is our *first main conclusion.*

Let us now substitute (3–70) and

$$B = k\,A\ , \quad B' = k\,A'\ , \quad B'' = k\,A''$$

into (3–68) and (3–69), and the result into (3–66). The calculations are direct, and we get

$$\frac{A^3}{A'^3}\left[-2\frac{A'^2}{A}(2+k) + 4(2A'^2 - A\,A'')\,\frac{1}{A} + 4(k^2 - k)(2A'^2 - A\,A'')z^2\right] = f(W)\ . \tag{3–77}$$

Since the right–hand side does not depend on z^2, the left–hand side must be independent of z^2 as well: the coefficient of z^2 must be zero. The case

$$k = 1$$

is excluded, since then the ellipsoids would degenerate to spheres. There remains

$$2A'^2 - A\,A'' = 0 \quad ; \tag{3–78}$$

(remember $A' = A'(W) = dA/dW$). This is an ordinary differential equation of the second order for the unknown function $A(W)$. Its general solution may be found by standard methods to be

$$A = \frac{1}{a_1^2(W_1 - W)} \quad , \tag{3–79}$$

with two integration constants a_1 and W_1 as customary for second–order differential equations. The solution (3–79) is easily verified by substitution into (3–78).

Then (3–74) gives

$$B = \frac{1}{b_1^2(W_1 - W)} \quad , \tag{3–80}$$

where

$$b_1 = a_1/\sqrt{k} \tag{3–81}$$

denotes another constant. Then our family of equisurfaces (3–55) takes the simple form

$$\frac{x^2 + y^2}{a_1^2} + \frac{z^2}{b_1^2} = W_1 - W \quad , \tag{3–82}$$

for which the kind — a family of geometrically similar ellipsoids — and the simple dependence on the potential W are evident; obviously W_1 is the potential at the center of the figure.

Finally we calculate from (3–79)

$$\begin{aligned} A' &= \frac{1}{a_1^2(W_1 - W)^2} = a_1^2 A^2 \quad , \\ A'' &= 2a_1^2\, A\, A' = 2a_1^4 A^3 \quad , \end{aligned}$$

which, of course, satisfies (3–78), and substitute into (3–77). The function A and its derivatives cancel completely, and there remains

$$-\frac{4 + 2k}{a_1^2} = f = -4\pi G\rho + 2\omega^2 \quad . \tag{3–83}$$

Now there comes our *second main conclusion*. Since the left–hand side of (3–83) is a constant, also *the density* ρ in this equation *must be constant*: the ellipsoidal figure of equilibrium *must be homogeneous*. Such ellipsoidal figures of equilibrium exist: they are the Maclaurin ellipsoids to be discussed in sec. 5.4, but the earth obviously is not homogeneous.

Let us repeat our argument. Eq. (3–73) leads necessarily to (3–76) and thus excludes any ellipsoidal stratification that is not homothetic, i.e., that does not consist of geometrically similar ellipsoids. Then (3–83) shows that the density must be homogeneous, which excludes heterogeneous equilibrium figures with ellipsoidal stratification. This proves the

Theorem of Hamy–Pizzetti

> An ellipsoidal stratification is impossible for heterogeneous, rotationally symmetric figures of equilibrium.

This is an extremely important "no–go theorem". The history of the subject starts with Hamy in 1887 and continues with work by Volterra in 1903 and Véronnet in 1912. The present method of proof was given by Pizzetti (1913, pp. 190–193) and essentially also used by Wavre (1932, pp. 60–61). We have tried to streamline it and to make every step explicit.

Later (secs. 4.2.4 and 6.4) we shall see that the terrestrial level ellipsoid, even with an arbitrary non–ellipsoidal internal stratification, cannot be an exact equilibrium figure, although it is extremely close to such a figure (Ledersteger's theorem).

3.2.5 Another Derivation of Clairaut's Equation

Although *rigorously*, the spheroidal equisurfaces are not ellipsoids, they are so *in linear approximation* (in f). Thus Wavre has used his equation (3–40) for a very elegant derivation of Clairaut' s equation. We put $\Theta_1 = 0$ (Pole P), $\Theta_2 = 90°$ (Equator E), and write, noting $N(t, 0) = 1$,

$$\begin{aligned} g(t, 0) &= g_P(t) , & N(t, 90°) &= N_E(t) , \\ J(t, 0) &= J_P(t) , & J(t, 90°) &= J_E(t) . \end{aligned} \tag{3–84}$$

The equisurfaces are (approximately!) ellipsoids of semiaxes $a(t)$ and $b(t) = t$, so that

$$a(t) = \frac{t}{1-f} = t\,(1 + f(t)) + O(f^2) \quad . \tag{3–85}$$

We further have

$$N_E(t) = \frac{da}{dt} = 1 + f(t) + tf'(t) \quad , \tag{3–86}$$

always disregarding $O(f^2)$. The ellipsoidal formulas of sec. 1.4 give the mean curvatures to our linear approximation:

$$J_P = \frac{1}{t}\,(1 - 2f) \; , \quad J_E = \frac{1}{t} \; , \tag{3–87}$$

so that (3–40), with (3–39), readily becomes

$$\frac{4\pi G\rho - 2\omega^2}{g_P(t)} = \frac{-t^2 f'' + 6f}{2t^2 f' + 2tf}$$

or

$$(2t^2 f' + 2tf)(4\pi G\rho - 2\omega^2) = (-t^2 f'' + 6f) g_P(t) \quad . \tag{3-88}$$

Corresponding to our approximation, we neglect the product of $f\omega^2$ (this removes ω^2 from our further considerations), and take $g_p(t)$ spherical, using (2–62):

$$g_P(t) = \frac{4\pi G}{3} tD(t) \quad . \tag{3-89}$$

Thus (3–88) reduces to

$$3\rho(2t^2 f' + 2tf) = tD(-t^2 f'' + 6f) \quad , \tag{3-90}$$

from which Clairaut's formula (2–114) follows immediately (with $t \doteq q$ in our approximation).

Note that Wavre's theory gives only Clairaut's differential equation, but not the boundary condition (2–118)!

The corresponding second-order theory is considerably more involved and will be treated in sec. 4.3.

3.2.6 Concluding Remarks

Wavre's theory is very beautiful and deep. Its true significance lies below the relatively simple mathematical formulism and is not so easily understood as the formulas themselves. We shall, therefore, try now to put Wavre's results into a proper perspective.

Equilibrium figures may be fully characterized by three conditions:

(A) The surfaces of constant potential coincide with the surfaces of constant density (sec. 2.5). Mathematically this means that the density ρ is only a function of the potential W or, in view of (3–39),

$$\Delta W = F(W) \quad ; \tag{3-91}$$

the Laplacian of W is a function only of W! This condition clearly has a differential and hence *local* character.

(B) The density ρ is positive and does not decrease towards the center. This is a natural condition, as the density models of sec. 1.5 show.

(C) The boundary surface S_0 of the equilibrium figure is an equipotential surface W = const.; outside S_0 there are no masses, so that the corresponding external potential V is harmonic everywhere outside S_0 and goes to zero as GM/r for $r \to \infty$. This may be considered a *global* condition.

In addition, we have the *symmetry* conditions:

(D) There is symmetry with respect to the equatorial plane, and rotational symmetry, the first being necessary, the second being a natural assumption.

Now it is basic that Wavre only uses the *local* condition (A) and the symmetry (D). *The global condition* (C) *is not taken into account at all*! Thus Wavre's theory is essentially incomplete. His results continue to hold if the equilibrium figure were surrounded by a rotationally symmetric mass configuration, such as an equatorial ring of Saturn type. Then, however, we can no longer speak of *free* equilibrium figures.

The basic Poisson equation (3–91) is equivalent both to Wavre's fundamental equation (3–40) — Bruns' formula (3–38) is nothing else than a sophisticated form of Poisson's equation as we have remarked after eq. (1–19) — and to the auxiliary equation (3–66). It is truly remarkable that one is able to prove such important results as Wavre's theorem (sec. 3.2.2) and the impossibility of a strictly ellipsoidal stratification (sec. 3.2.4) on the basis of this local theory only. The global condition (C) is not even necessary for these purposes!

Thus Wavre's equation (3–40), leading to his theorem (sec. 3.2.2) is a necessary but by no means sufficient condition for a free equilibrium figure since the global condition (C) is not taken into account.

It might now be tempting to reason in the following way. Eq. (3–40) holds for arbitrary Θ_1 and Θ_2. If we replace Θ_2 by Θ_3, we get the *purely geometrical relation*

$$\frac{(2JN-\partial\ln N/\partial t)_{\Theta_3}-(2JN-\partial\ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_3}-(N^2)_{\Theta_1}}=$$

$$=\frac{(2JN-\partial\ln N/\partial t)_{\Theta_2}-(2JN-\partial\ln N/\partial t)_{\Theta_1}}{(N^2)_{\Theta_2}-(N^2)_{\Theta_1}}\,, \tag{3–92}$$

which is a *necessary* condition for all stratifications of equilibrium figures. An equivalent form of this condition, with differences replaced by derivatives, is (3–46).

Assume now that this condition were also sufficient. Then we could remove the layer above any internal equisurface $S(t)$, cf. Fig. 3.2. For the remaining "reduced" figure bounded by $S(t)$, eq. (3–92) continues to hold for any of its internal equisurfaces, and the reduced figure would also be a possible figure of equilibrium.

This is Ledersteger's (1969, p. 536) "Prinzip der Entblätterung" (principle of removing shells bounded by two equisurfaces). For homogeneous ellipsoidal equilibrium figures (Maclaurin ellipsoids), this principle indeed holds since in this case, such shells are bounded by geometrically similar ellipsoids, and it is well known (*Newton's theorem*) that such an "ellipsoidal homoeoid" exerts no attraction in its interior; cf. (Kellogg, 1929, p. 22) or (Chandrasekhar, 1969, p. 39). Furthermore, the centrifugal force reduces proportionally.

For heterogeneous ellipsoidal figures, however, this principle does not hold (Voss, 1965), not even in the linear approximation. In fact, if it holds, we could remove the layer above the equisurface labeled by q, so that $\rho = 0$ above it and the second integral in (2–109) would vanish. Thus the term

$$\int_q^R \rho \frac{df}{dq'}\,dq'$$

would have to vanish identically, which only holds if $f = \text{const.}$, for a homothetic (geometrically similar) stratification, and this is only possible for homogeneous figures, as we have seen in sec. 3.2.4. (In fact, the layer between $S(t)$ and $S(1) = S_0$ in Fig. 3.2 has the function of an external mass, not unlike to Saturn's ring mentioned above, for the "reduced" equilibrium figure bounded by $S(t)$!)

This confirms that (3–92) is only necessary but not sufficient. Hence, before applying Wavre's procedure described by (3–51) and (3–52), we must first make sure that *the given stratification corresponds to a possible figure of equilibrium*, which is by no means a simple matter, as already the counterexample of sec. 3.2.4 (non-homothetic ellipsoidal stratification) has shown.

To find such a possible stratification is a highly nontrivial problem indeed. In fact, no *rigorous* solution for a *heterogeneous* earthlike figure of equilibrium is known to the author. Heterogeneous solutions can only be constructed by a process of successive iteration or expansions with respect to powers of the flattening, the convergence for "small" values of the flattening f being guaranteed by the theorem of Liapunov–Lichtenstein mentioned at the beginning of sec. 3.1.

The "local" character of Wavre's theory is also expressed by the fact that it permits us to derive Clairaut's differential equation (2–114) but not the boundary condition (2–118), as we have pointed out at the end of sec. 3.2.5 and shall see again in more detail in sec. 4.3. Boundary conditions are typically *global*.

The theory of equilibrium figures is extremely subtle and full of unexpected pitfalls. There are "no–go theorems" such as the impossibility of a purely ellipsoidal stratification for heterogeneous equilibrium figures (sec. 3.2.4) and the fact that the terrestrial level ellipsoid cannot be an equilibrium figure, as we shall see in sec. 4.2.4 and later throughout Chapter 5 and then again in secs. 6.2 and 6.4. The latter fact was clearly recognized and repeatedly emphasized by Karl Ledersteger. It should be noticed here that Ledersteger was the last great geodesist who seriously and deeply engaged himself in Wavre's theory of equilibrium figures. This should be acknowledged even if one is not prepared to follow him all the way (see his "Prinzip der Entblätterung" as mentioned above).

Still, to first order in the flattening f, the level ellipsoid is an equilibrium figure with an approximately ellipsoidal stratification: this is Clairaut's theory, cf. sec. 3.2.5. Deviations from an ellipsoidal stratification start only in the second-order approximation (sec. 4.2.4) and are thus very small. Hence a very small change is sufficient to destroy equilibrium, which means that the property of being an equilibrium figure is extremely sensitive with respect to small perturbations: in a very special sense, it is an "unstable" property (this has nothing to do with the problem of instability of equilibrium figures which is important for stellar figures but not for the figure of the earth!). For another such "special instability" cf. sec. 3.2.3.

A final word on the relationship between Wavre's approach and the approach by Clairaut–Liapunov–Lichtenstein described in sec. 3.1. In a sense, the two approaches are "dialectical opposites". Wavre starts from a given stratification (the geometry) and determines the corresponding density distribution (the physics), whereas Lichtenstein starts from a given density distribution (which is initially spherical) and determines the configuration or stratification which results from a "small" rotation ω. Hence Wavre determines the physics of the problem from its geometry, whereas

Lichtenstein determines the geometry from the physics. Also, for Lichtenstein, the spherical configuration is the starting point, whereas for Wavre it is a singularity (0/0)!

Wavre's approach is also described in the books (Baeschlin, 1948) and (Ledersteger, 1969), whereas the basic book in English, (Jardetzky, 1958), does not present it, although it outlines an approximation method also due to Wavre ("procédé uniforme") which intends, by an ingenious but complicated trick, to circumvent the convergence problem of certain series of spherical harmonics. We shall not treat this here because the author believes that this problem can be tackled in a much simpler way as we shall see in sec. 4.1.5.

3.3 Stationary Potential Energy

In various domains of physics, equilibrium is associated with a stationary (maximum or minimum, depending on the sign) value of potential energy. Liapunov and Poincaré have treated *homogeneous* equilibrium figures from this point of view; a modern approach is found in the book (Macke, 1967, p. 543). Macke's method has been generalized to heterogeneous (terrestrial) equilibrium figures (Macke et al., 1964; Voss, 1965, 1966). This approach is interesting because it reflects the typical thinking and mathematical methods of modern theoretical physics.

3.3.1 Potential Energy

The gravitational energy of a material particle of mass m in a field of potential V is mV, and that of a system of particles thus

$$E = \sum m_i V_i \quad ; \tag{3–93}$$

the sign (+ or −) is conventional.

This holds for an *external* potential field V. If the field is produced by the mutual gravitational attraction of the particles themselves:

$$V_i = G \sum_j \frac{m_j}{l_{ij}} \qquad (j \neq i) \quad , \tag{3–94}$$

then (3–93) gives

$$G \sum_{i,j} \frac{m_i m_j}{l_{ij}} \quad .$$

Each term occurs twice, however (interchange i and j), so that we must divide by 2, obtaining

$$E_V = \frac{1}{2} G \sum_i \sum_j \frac{m_i m_j}{l_{ij}} \qquad (j \neq i) \quad ; \tag{3–95}$$

cf. also (Kellogg, 1929, pp. 79–81) or (Poincaré, 1902, pp. 7–8).