Chapter 3

Equilibrium Figures: Alternative Approaches

Besides the standard theory of Clairaut, Laplace, and Radau, described in Chapter 2, and its second-order improvement to be treated in Chapter 4, there are several other approaches to equilibrium figures which are of considerable theoretical interest. Although they may not offer new computational formulas (eventually, all will lead to Clairaut's equation), they essentially broaden our understanding of the problem, permit us to look at it from various sides, and throw new light on it, much in the same way as the various methods of analytical dynamics act for Newtonian mechanics.

The integral equation method (sec. 3.1) is particularly appealing to the physical geodesist so familiar with integral equations, cf. (Heiskanen and Moritz, 1967, Chapter 8; Moritz, 1980, Part D). On the other hand, the reader who does not like integral equations may skip sec. 3.1 and pass directly to the next section.

The beautiful geometric theory of equilibrium figures due to Wavre (sec. 3.2) is particularly important: besides offering an essentially different derivation of Clairaut's equation and the corresponding second-order theory to be considered in Chapter 4, it allows us to deduce important "no-go theorems" such as the impossibility of a rigorously ellipsoidal stratification.

Finally, the method of stationary potential energy (sec. 3.3) is very close to the general trend in physics and may permit generalizations to non-equilibrium figures, as we shall see in section 5.12.

3.1 The Method of Integral Equations

The mathematically rigorous treatment of equilibrium figures goes back to the French mathematician Poincaré (1885) and to his contemporary, the Russian mathematician Liapunov. Whereas Poincaré concentrated his research on homogeneous equilibrium figures (density $\rho = \text{const.}$), Liapunov (Liapounoff, 1904) studied heterogeneous figures as well, thus providing a rigorous justification of Clairaut's theory.

Lichtenstein (1933) continued Liapunov's work and tried to simplify it, but his attempt to achieve perfect mathematical rigor still makes his book extremely difficult to read, so that his work, also because it is written in German, has shared the fate of Liapunov's researches of being largely ignored by the geodetic and geophysical community.

Their result may be described as follows: Consider a nonrotating heterogeneous mass in hydrostatic equilibrium of arbitrary density distribution in the absence of external forces. The density is subject only to the natural condition of being positive and non-decreasing towards the interior. In this case it can be proved that the configuration must be spherically symmetric: $\rho = \rho(r)$ is an arbitrary (in the above sense) function of the radius r of the spherical equisurfaces.

If this configuration is subjected to a "sufficiently slow" rotation, then a spheroidal equilibrium figure exists which is "close" to the original spherical configuration and possesses the same density law $\rho(q)$, q denoting the mean radius of the equisurfaces. In other words, Liapunov and Lichtenstein proved the existence and uniqueness of an equilibrium figure "in the neighborhood" of a given spherical mass configuration. To be sure, "smallness" of the angular velocity ω is to be considered in the mathematical sense, without implying that the earth's actual rotational velocity is "sufficiently small" in this sense. The author does not know whether the required extremely laborious estimates for this purpose have ever been performed numerically.

In a sense, Liapunov and Lichtenstein achieved for Clairaut's problem essentially what Hörmander in 1976 did for Molodensky's gravimetric boundary value problem (cf. Moritz, 1980, sec. 51): a proof of existence and uniqueness under certain mathematical restrictions.

It would be presumptuous in this context to even give a mathematical description of the proof, so the reader is referred to Lichtenstein's book.

The basis of the proof, however, is a linear integral equation, which has a certain analogy with Molodensky's famous integral equation and may, therefore, interest the geodetic reader. Hence we shall attempt to sketch a simple geometric derivation of Lichtenstein's fundamental integral equation (valid *to first order*).

Consider a non-rotating spherically symmetric mass S, and submit it to a rotation with angular velocity ω which deforms it into the spheroid E (which, at least aproximately, is an ellipsoid). Denote by $\zeta = QP$ the distance of a "new" equisurface from the corresponding "old" one. The deviation ζ satisfies an integral equation which can be found as follows (Fig. 3.1).

Denote the "normal" gravity potential of the spherical configuration by U and that of the actual spheroidal configuration by W. The potential U is purely gravitational (nonrotating!), whereas W includes the centrifugal force.

The effect of the configuration change, spherical to spheroidal, consists of three parts:

1. The volume element dv, containing the density $\rho' = \rho(q')$, is moved from Q' to P'. Thus ρ' is now at P', whereas the new density at Q' is

$$\rho' - \frac{\partial \rho'}{\partial q'} \zeta' \quad .$$

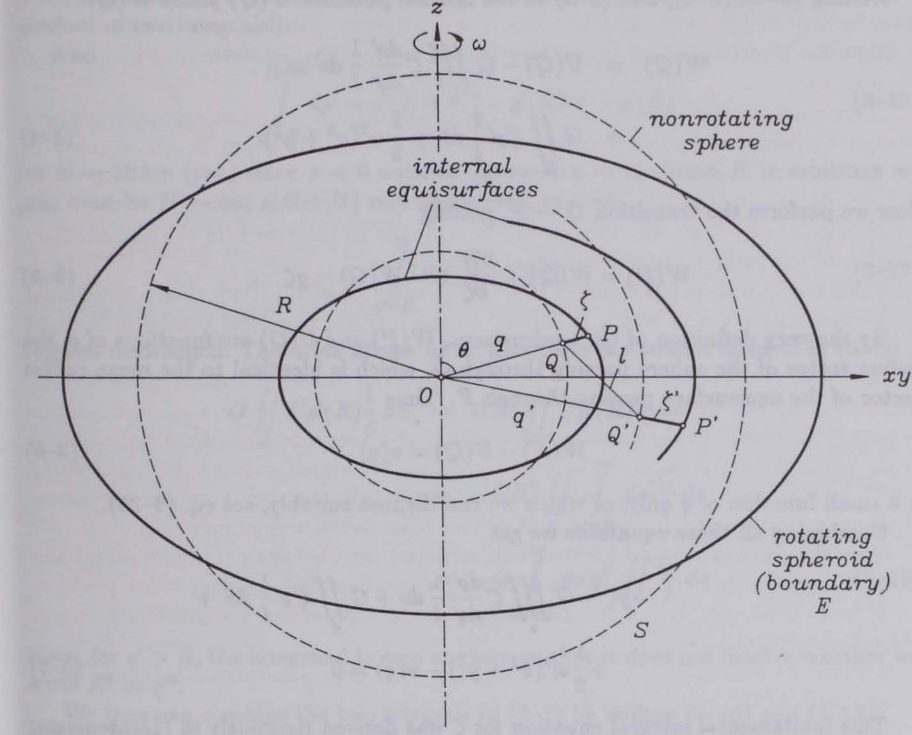

FIGURE 3.1: Rotation deforms a sphere into a spheroid

Thus the total effect of the change at the potential at point Q is

$$-G\iiint_v \zeta' \frac{\partial \rho'}{\partial q'}\frac{1}{l}\,dv \quad . \tag{3–1}$$

The meaning of $l = QQ'$, $q' = OQ'$ and ζ' is seen from Fig. 3.1, G denoting the gravitational constant and v the volume of S.

2. The effect of the "bulge" (positive if E is above S, negative otherwise) can be considered as a surface layer on the sphere S, with surface density $\rho'\zeta'$ (the integration variable is denoted by a prime also if the integration point is on S). This gives the contribution

$$G\iint_S \zeta'\rho'\frac{1}{l}\,dS \quad . \tag{3–2}$$

3. The centrifugal potential

$$\frac{1}{2}\omega^2(x^2+y^2) \quad . \tag{3–3}$$

Adding (3–1), (3–2), and (3–3) to the normal potential $U(Q)$ yields $W(Q)$:

$$W(Q) = U(Q) - G\iiint_v \zeta' \frac{d\rho'}{dq'}\frac{1}{l}\,dv + \\ + G\iint_S \zeta'\rho'\frac{1}{l}\,dS + \frac{1}{2}\omega^2(x^2+y^2) \quad . \tag{3–4}$$

Now we perform the transition $Q \to P$, getting

$$W(P) = W(Q) + \frac{\partial W}{\partial \zeta}\zeta = W(Q) - g\zeta \quad . \tag{3–5}$$

By the very definition of the equisurfaces, $W(P)$ and $U(Q)$ are functions of q, the radius vector of the sphere passing through Q, which is identical to the mean radius vector of the equisurface passing through P. Thus

$$W(P) - U(Q) = v(q) \tag{3–6}$$

is a small function of q only, of which we can dispose suitably, see eq. (3–25).

Combining all these equations we get

$$-g\zeta - G\iiint_v \zeta'\frac{d\rho'}{dq'}\frac{1}{l}\,dv + G\iint_S \zeta'\rho'\frac{1}{l}\,dS + \\ +\frac{1}{2}\omega^2(x^2+y^2) - v(q) = 0 \quad . \tag{3–7}$$

This fundamental integral equation for ζ was derived rigorously in (Lichtenstein, 1933, pp. 97-101). Note that the integrals are extended over the original spherically symmetric configuration.

Another form of (3–7). This is obtained by writing, cf. (2–68):

$$\iint_S dS = R^2 \iint_\sigma d\sigma \quad , \tag{3–8}$$

as well as

$$\iiint_v dv = \int_{q'=0}^{R} dq' q'^2 \iint_\sigma d\sigma \quad , \tag{3–9}$$

cf. (2–46) with $r' = q'$. Thus the first integral in (3–7) becomes

$$-G\iiint_v \zeta'\frac{d\rho'}{dq'}\frac{1}{l}\,dv = -G\int_{q'=0}^{R}\frac{d\rho'}{dq'}dq'q'^2\iint_\sigma \frac{\zeta'}{l}\,d\sigma \\ = -G\int_{q'=0}^{R} d\rho' q'^2 \iint_\sigma \frac{\zeta'}{l}\,d\sigma \tag{3–10}$$

(note that also ζ'/l depends on q' so that (3–10) is an iterated integral rather than a product of two integrals!).

Also

$$\int_{q'=R}^{R+H} d\rho' = \rho'(R+H) - \rho'(R) = -\rho'(R) \tag{3–11}$$

for $H = 10\,\mathrm{km}$ (say) since $\rho = 0$ outside the earth $q = R$. Since H is arbitrary we may even let $H \to \infty$, $\rho(R+H)$ remaining zero, so that

$$\int_{q'=R}^{\infty} d\rho' = -\rho'(R) \tag{3–12}$$

remains unchanged. This trick allows us to transform the second integral in (3–7):

$$\begin{aligned} G \iint_S \zeta'\rho'(R)\frac{1}{l}\,dS &= GR^2 \iint_\sigma \zeta'\rho'(R)\frac{1}{l}\,d\sigma \\ &= -GR^2 \iint_\sigma \int_{q'=R}^{\infty} d\rho'\frac{\zeta'}{l}\,d\sigma \\ &= -G \int_{q'=R}^{\infty} d\rho' q'^2 \iint_\sigma \frac{\zeta'}{l}\,d\sigma \end{aligned} \tag{3–13}$$

since, for $q' > R$, the integrand is zero anyway, so that it does not matter whether we write R^2 or q'^2.

We thus can combine the two integrals in (3–7) by adding (3–10) and (3–13):

$$\int_{q'=0}^{R} + \int_{q'=R}^{\infty} = \int_{q'=0}^{\infty} \tag{3–14}$$

to get a somewhat simpler form for (3–7):

$$g\zeta + G\int_{q'=0}^{\infty} d\rho' q'^2 \iint_\sigma \frac{\zeta'}{l}d\sigma - \frac{1}{2}\omega^2(x^2+y^2) + v(q) = 0 \quad . \tag{3–15}$$

Mathematicians call integrals like

$$\int_{x=a}^{b} g(x)df(x) \tag{3–16}$$

a *Stjeltjes integral*, but nonmathematicians might disregard this fact (in the exercise at the end of sec. 2.5 we had another Stjeltjes integral!). The physicist will probably be satisfied with the present heuristic derivation of (3–15); the mathematical reader is invited to make the argument more rigorous.

Solution of (3–15). Lichtenstein (1933, p. 22) has shown that equilibrium figures must be symmetric with respect to the equatorial plane (the xy plane in Fig. 3.1). If, in addition, we assume rotational symmetry, ζ must have the form

$$\zeta = \sum_{\nu=0}^{\infty} \zeta_{2\nu}(q) P_{2\nu}(\cos\theta) \quad , \tag{3–17}$$

containing only *even* zonal harmonics. (The existence of odd zonal harmonics in the geopotential is another indication of the earth's deviation from hydrostatic equilibrium!) The assumption of rotational symmetry is not necessary as the three-axial ellipsoids of Jacobi (cf. Chandrasekhar, 1969, pp. 101-103) show, but it is entirely natural: the Jacobi ellipsoids have a weird shape which is completely different from the earth and "earthlike" equilibrium figures.

Limiting ourselves to the first approximation, we thus have

$$\zeta = \zeta_0(q) + \zeta_2(q) P_2(\cos\theta) \tag{3–18}$$

and, of course,

$$\zeta' = \zeta_0(q') + \zeta_2(q') P_2(\cos\theta') \quad . \tag{3–19}$$

This is substituted into (3–15), together with

$$\frac{1}{l} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{q'^n}{q^{n+1}} P_n(\cos\psi) \,, & q' < q \quad , \\ \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{q'^{n+1}} P_n(\cos\psi) \,, & q' > q \quad . \end{cases} \tag{3–20}$$

This is an application of (2–48a, b) to the present case. As usual, we interchange integration and summation. Orthogonality then removes all terms except two; cf. (1–51). For the remaining terms we apply (2–52) and (2–74).

For $q' < q$ we then have

$$\iint_\sigma \frac{\zeta'}{l}\, d\sigma = 4\pi \frac{1}{q} \zeta_0(q') + \frac{4\pi}{5} \frac{q'^2}{q^3} P_2(\cos\theta) \zeta_2(q') \tag{3–21a}$$

and for $q' > q$,

$$\iint_\sigma \frac{\zeta'}{l}\, d\sigma = 4\pi \frac{1}{q'} \zeta_0(q') + \frac{4\pi}{5} \frac{q^2}{q'^3} P_2(\cos\theta) \zeta_2(q') \quad . \tag{3–21b}$$

This gives the inner integral in (3–15). In view of the difference between (3–21a, b), we must split up the outer integral as

$$\int_0^\infty = \int_0^q + \int_q^\infty \quad . \tag{3–22}$$

Finally we have by (2–8) to $O(f^2)$:

$$\frac{1}{2}\omega^2(x^2+y^2) = \frac{1}{3}\omega^2 q^2 - \frac{1}{3}\omega^2 q^2 P_2(\cos\theta) \quad . \tag{3–23}$$

All this is substituted into (3–15) with the result

$$\begin{aligned} & g\zeta_0 + g\zeta_2 P_2(\cos\theta) + 4\pi G\left[\frac{1}{q}\int_0^q \zeta_0' q'^2 d\rho' + \int_q^\infty \zeta_0' q' d\rho'\right] + \\ & +\frac{4\pi G}{5}\left[\frac{1}{q^3}\int_0^q \zeta_2' q'^4 d\rho' + q^2\int_q^\infty \frac{\zeta_2'}{q'}\,d\rho'\right] P_2(\cos\theta) - \\ & -\frac{1}{3}\omega^2 q^2 + \frac{1}{3}\omega^2 q^2 P_2(\cos\theta) + v(q) = 0 \quad , \end{aligned} \tag{3–24}$$

with the obvious notation $\zeta_k = \zeta_k(q)$, $\zeta_k' = \zeta_k(q')$, with $k = 0$ or 2.

The sum of all terms multiplied by $P_2(\cos\theta)$ must vanish since (3–24) holds identically for all θ. Then also the sum of the remaining terms must be zero; taking

$$v(q) = \frac{1}{3}\omega^2 q^2 \quad , \tag{3–25}$$

$$\zeta_0(q) \equiv 0 \tag{3–26}$$

will achieve this. Thus $\zeta_0(q)$ identically vanishes. The sum of all terms multiplied by P_2 is

$$g\zeta + \frac{4\pi G}{5}\left[\frac{1}{q^3}\int_0^q \zeta' q'^4 d\rho' + q^2\int_q^\infty \frac{\zeta'}{q'}\,d\rho'\right] + \frac{1}{3}\omega^2 q^2 = 0 \quad . \tag{3–27}$$

Here we have omitted the subscript 2 in ζ_2. Now we apply partial integration:

$$\begin{aligned} \frac{1}{q^3}\int_0^q \zeta' q'^4 d\rho' &= \frac{1}{q^3}\left[\zeta' q'^4 \rho'\right]_0^q - \frac{1}{q^3}\int_0^q \rho' d(\zeta' q'^4) \\ &= \zeta q \rho - \frac{1}{q^3}\int_0^q \rho' d(\zeta' q'^4) \end{aligned} \tag{3–28}$$

since the term within brackets vanishes for $q' = 0$. Similarly

$$\begin{aligned} q^2\int_q^\infty \frac{\zeta'}{q'}\,d\rho' &= q^2\left[\frac{\zeta'}{q'}\rho'\right]_q^\infty - q^2\int_q^\infty \rho' d\left(\frac{\zeta'}{q'}\right) \\ &= -\zeta q \rho - q^2\int_q^\infty \rho' d\left(\frac{\zeta'}{q'}\right) \end{aligned} \tag{3–29}$$

since $\rho' = 0$ for $q' > R$ and also for $q' \to \infty$.

We further take from (2–57)

$$g(q) = \frac{4\pi G}{q^2} \int_0^q \rho' q'^2 dq' \tag{3-30}$$

and restore the subscript 2 to ζ. Then the comparison of (3–18) and (2–82), noting $r = OP = q + \zeta$ (Fig. 3.1) and $\zeta_0 = 0$, gives

$$\zeta = \zeta_2(q) = -\frac{2}{3} q f(q) \quad .$$

Thus (3–27) becomes, on omitting the prime on ρ so that $\rho = \rho(q')$ and similarly for f after the integral,

$$-\frac{2}{3}\frac{f}{q} \int_0^q \rho q'^2 dq' + \frac{2}{15}\frac{1}{q^3} \int_0^q \rho d(f q'^5) + \\ + \frac{2}{15} q^2 \int_q^R \rho \, df + \frac{\omega^2 q^2}{12\pi G} = 0 \ , \tag{3-31}$$

which is identical to (2–106) (up to a factor $15q^3/2$ which cancels), on noting, e.g.,

$$df = \frac{df}{dq'} dq' \quad .$$

Since Clairaut's equation (2–114), plus boundary condition (2–118), was a direct consequence of (2–106), it equally follows from (3–31).

This provides another method for deriving Clairaut's equation, which has the advantage of using an integral equation similar to the integral equations customary from Molodensky's approach to physical geodesy.

Therefore it is not surprising after all that even Molodensky (1988) occupied himself with the integral equation of Lichtenstein!

3.2 The Geometry of Equilibrium Surfaces

Clairaut's equation (2–114) for the basic geometric quantity, the flattening f, is a *homogeneous* differential equation.

Homogeneous differential equations (with right-hand side zero) with independent variable t, time, correspond to *free* motion, as opposed to forced motion. In the present case, the independent variable is the radius r rather than time, but the argument may indicate that the geometry of the equisurfaces for equilibrium figures seems to have a considerable autonomy.

This idea was thoroughly investigated in the fundamental book (Wavre, 1932). Since it is little known in the English-speaking scientific community, we shall outline Wavre's theory of stratification of equilibrium figures (which is *rigorous*).