2.7 Moments of Inertia

The moment of inertia of a body around an axis is given by the well-known formula

$$J = \iiint p^2 dm \quad , \tag{2–137}$$

where p denotes the distance of the mass element dm from the axis under consideration.

For the polar moment of inertia $J = C$, around the z-axis (mean axis of rotation) we thus have with $dm = \rho dv$:

$$C = \iiint (x^2 + y^2)\rho dv \quad , \tag{2–138}$$

since $p^2 = x^2 + y^2$ in this case.

Neglecting the flattening, we integrate over the sphere $r = R$, with volume element

$$dv = r^2 \sin\theta dr d\theta d\lambda \tag{2–139}$$

in spherical coordinates, with $x^2 + y^2 = r^2 \sin^2\theta$,

$$C = \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} r^4 \sin^3\theta \rho(r) dr d\theta d\lambda \quad . \tag{2–140}$$

This is the product of three integrals:

$$\int_0^{2\pi} d\lambda = 2\pi \quad ,$$

$$\int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3} \quad ,$$

$$\int_0^{R} \rho(r) r^4 dr \quad ,$$

whence

$$C = \frac{8\pi}{3} \int_0^{R} \rho(r) r^4 dr \quad . \tag{2–141}$$

This formula is nice but not very practical since it requires the knowledge of $\rho(r)$.

The essential feature of Radau's approximation (2–135) is that it permits us to transform (2–141) into a form that is independent of an explicit density law $\rho(r)$. By (2–128) we have

$$\rho = D + \frac{1}{3} qD' \, , \qquad D' = dD/dq \, , \tag{2–142}$$

so that (2–141) becomes, on replacing r by the mean radius q (the spherical configuration is the mean configuration for the ellipsoidal stratification!)

$$C = \frac{8\pi}{9} \int_0^R (3Dq^4 + D'q^5)dq \quad . \tag{2–143}$$

Integration by parts gives, for the infinite integral,

$$\int D'q^5 dq = \int \frac{dD}{dq} q^5 dq = Dq^5 - 5 \int Dq^4 dq \quad , \tag{2–144}$$

and for the definite integral

$$\int_0^R D'q^5 dq = \rho_m R^5 - 5 \int_0^R Dq^4 dq \quad , \tag{2–145}$$

where the earth's mean density is expressed by (2–116):

$$\rho_m = \frac{M}{4\pi R^3/3} \quad . \tag{2–146}$$

Thus (2–143) becomes

$$C = \frac{2}{3} MR^2 - \frac{16\pi}{9} \int_0^R Dq^4 dq \quad . \tag{2–147}$$

Now comes the crucial point: the integral can be evaluated by Radau's formula (2–135)! This is the reason why we have introduced, apparently out of the blue sky, the function (2–125). In fact, the integration of (2–135) gives

$$\int_0^R Dq^4 dq = \frac{1}{5} \rho_m R^5 \sqrt{1 + \eta_S} \quad , \tag{2–148}$$

considering that for $q = R$ we have $D = \rho_m$ and $\eta = \eta_S$ as given by (2–146) and (2–136). In view of (2–148), eq. (2–147) thus becomes

$$C = \frac{2}{3} MR^2 \left(1 - \frac{2}{5} \sqrt{1 + \eta_S}\right) \quad . \tag{2–149}$$

This equation is independent of the density law $\rho(q)$ and uses only the known surface value η_S! To be sure, it is based on the following presuppositions and approximations:

1. The earth is in hydrostatic equilibrium.
2. Second-order terms $O(f^2)$ can be neglected.
3. Radau's function $\psi(\eta) = 1$, see eq. (2–133).

Now the dynamical ellipticity (1–85)

$$H = \frac{C - A}{C} \tag{2–150}$$

is very accurately known from astronomical precession. From the theory of the external field we have (2–17),

$$J_2 = \frac{C - A}{Ma^2} \doteq \frac{C - A}{MR^2} = \frac{2}{3}f - \frac{1}{3}m \quad , \tag{2–151}$$

disregarding $O(f^2)$ as usual, whose numerical value is given by (1–77). Thus

$$\frac{J_2}{H} = \frac{(C - A)/MR^2}{(C - A)/C} = \frac{C}{MR^2} = \frac{2}{3}\left(1 - \frac{2}{5}\sqrt{1 + \eta_S}\right) \tag{2–152}$$

or, with (2–136),

$$\frac{J_2}{H} = \frac{2}{3}\left(1 - \frac{2}{5}\sqrt{\frac{5m}{2f} - 1}\right) \quad . \tag{2–153}$$

Substituting the numerical values (1–77), (1–79), (1–83) and (1–85) we get an inconsistency which, when confirmed by a more precise (second-order) theory, would show that the earth is not in hydrostatic equlibrium, cf. sec. 1.1.

Substituting (2–151) we get the relation

$$f - \frac{m}{2} = H\left(1 - \frac{2}{5}\sqrt{\frac{5m}{2f} - 1}\right) \quad , \tag{2–154}$$

which can be solved for f and permits the determination of f from H without knowing J_2 but assuming hydrostatic equilibrium.

Since f can now be determined from J_2 much more directly, without needing the hypothesis of hydrostatic equilibrium, this possibility is of historic interest only.

It remains of fundamental geophysical importance, however, whether (2–153), or rather a more accurate version, holds for the real earth or not. This will be considered later (sec. 4.2.5).

Mathematically speaking, an equation such as (2–153) is an (approximate) *first integral* of Clairaut's equation (2–114). The complete solution of this equation would, of course, be a representation of f as a function of q: $f = f(q)$, $0 \le q \le R$. Nevertheless it is extremely surprising that Radau could get as far as (2–153) without needing the density law $\rho = \rho(q)$.