This holds for the real earth. If the earth were *homogeneous*, then obviously $D = \rho$, so that Clairaut's equation reduces to

$$\frac{d^2 f}{dq^2} + \frac{6}{q}\frac{df}{dq} = 0 \quad , \tag{2–120}$$

which has the solution $df/dq = 0$ or $f = \text{const}$. Thus the equisurfaces of homogeneous ellipsoidal equilibrium figures are geometrically similar ellipsoids (all have the same flattening f). This was derived here as an approximation of first order in f, but may be shown to be rigorously valid. This is the case of the Maclaurin ellipsoid to be considered in sec. 5.4.

Finally we mention that, instead of solving the differential equation (2–114) with the appropriate boundary conditions, we could also try to solve the original equivalent integro-differential equations (2–105) or (2–111) iteratively. This approach may have numerical advantages (Denis, 1989), but from the conceptual and analytical point of view, which we are emphasizing throughout this book, the elegant and mathematically simple and transparent equation of Clairaut remains fundamental.

Our further considerations will, therefore, follow the classical approach, submitting Clairaut's equation to an ingenious transformation due to Radau.

Exercise. Wavre (1932, p. 96) gives the elegant integro-differential equation

$$\frac{d}{dq}(Df) = \frac{3}{q^6}\int\limits_{q'=0}^{q} f(q')q'^5 d\rho \quad ,$$

where for differentiable ρ

$$d\rho = \frac{d\rho(q')}{dq'}\,dq' \quad .$$

Show its equivalence to Clairaut's equation (2–114) by differentiation.

2.6 Radau's Transformation

Radau (1885) introduces the parameter

$$\eta = \frac{q}{f}\frac{df}{dq} = \frac{d\ln f}{d\ln q} \quad . \tag{2–121}$$

In terms of Radau's parameter we thus have

$$\frac{df}{dq} = \frac{\eta}{q} f \quad , \tag{2–122}$$

and differentiation gives

$$\begin{aligned} \frac{d^2 f}{dq^2} &= -\frac{\eta}{q^2} f + \frac{f}{q}\frac{d\eta}{dq} + \frac{\eta}{q}\frac{df}{dq} \\ &= \left(\frac{1}{q}\frac{d\eta}{dq} + \frac{\eta^2 - \eta}{q^2}\right) f \quad , \end{aligned} \tag{2–123}$$

where we have used (2–122). On substituting this into Clairaut's equation (2–114), cancelling the common factor f, and multiplying by q^2 we get *Radau's equation*

$$q\frac{d\eta}{dq} + \eta^2 - \eta - 6 + 6\frac{\rho}{D}(1+\eta) = 0 \quad . \tag{2–124}$$

In this way we have transformed the second-order linear differential equation (2–114) into the first-order non-linear differential equation (2–124).

As such, this is not very exciting; it even follows a standard mathematical procedure employed in such cases (equations of Riccati type). It will, however, be found to work surprisingly well, almost by miracle.

Consider the function (its choice will be motivated later)

$$F(q) = Dq^5\sqrt{1+\eta} \quad , \tag{2–125}$$

remembering that both the mean density D (in the volume enclosed by the equisurface q) and Radau's parameter η are functions of q. Its logarithmic derivative is (the prime denotes derivatives with respect to q)

$$\frac{F'(q)}{F(q)} = \frac{d\ln F}{dq} = \frac{D'}{D} + \frac{5}{q} + \frac{\eta'}{2(1+\eta)} \quad . \tag{2–126}$$

For $\eta' = d\eta/dq$ we get from (2–124):

$$\begin{aligned} q\eta' &= \eta - \eta^2 + 6 - 6(1+\eta) - 2q\frac{D'}{D}(1+\eta) \\ &= -\eta^2 - 5\eta - 2q\frac{D'}{D}(1+\eta) \quad . \end{aligned} \tag{2–127}$$

Here we have used

$$\frac{\rho}{D} = 1 + \frac{1}{3}q\frac{D'}{D} \quad , \tag{2–128}$$

which is an immediate consequence of (2–113). With (2–127), eq. (2–126) becomes

$$q\frac{F'(q)}{F(q)} = q\frac{D'}{D} + 5 - \frac{\eta^2+5\eta}{2(1+\eta)} - q\frac{D'}{D} \quad .$$

Thus D'/D cancels (the first, minor, miracle) and there remains

$$q\frac{F'(q)}{F(q)} = \frac{10+5\eta-\eta^2}{2(1+\eta)} \quad . \tag{2–129}$$

Remembering the definition of $F(q)$ by (2–125) we thus have

$$F'(q) = 5Dq^4\frac{1+\frac{1}{2}\eta-\frac{1}{10}\eta^2}{\sqrt{1+\eta}} \tag{2–130}$$

or

$$F'(q) = 5Dq^4\psi(\eta) \tag{2–131}$$

with

$$\psi(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}} \quad . \tag{2–132}$$

Now comes the second, major, miracle:

$$\psi(\eta) \overset{\text{almost}}{\equiv} 1 \tag{2–133}$$

for a broad range of values of η (Table 2.1). Thus the rigorous formula (2–131), written by (2–125) as

$$\frac{d}{dq}\left(Dq^5\sqrt{1+\eta}\right) = 5Dq^4\psi(\eta) \quad , \tag{2–134}$$

may be replaced by Radau's approximation

TABLE 2.1: The function $\psi(\eta)$

η	$\psi(\eta)$		
0	1.00000	$\longleftarrow$	earth's center
0.3	1.00072		
0.5	1.00021		
0.572	0.99959	$\longleftarrow$	earth's surface
0.6	0.99928		
1.0	0.98995		
2.0	0.92376		
3.0	0.80000		

$$Dq^4 = \frac{1}{5}\frac{d}{dq}\left(Dq^5\sqrt{1+\eta}\right) \quad . \tag{2–135}$$

In fact, at the earth's surface ($q = R$) we have by (2–121) and (2–118)

$$\eta = \frac{5}{2}\frac{m}{f} - 2 = 0.572 = \eta_S \quad , \tag{2–136}$$

and at the earth's center there is $\eta = 0$ by (2–121) since $f \neq 0$ for $q = 0$. So for $0 \le q \le R$, $\psi(\eta)$ will always be very close to 1, to an accuracy comparable with our first-order approximation.

Eq. (2–134), especially in Radau's approximation (2–135), will play a fundamental role in the next section.