2.5 Hydrostatic Equilibrium: Clairaut's Equation

The equation of motion for an ideal fluid is very simple:

$$\rho \ddot{\mathbf{x}} = \rho \mathbf{g} - \text{grad}p \quad . \tag{2–95}$$

$\ddot{\mathbf{x}} = d^2\mathbf{x}/dt^2$ is the acceleration of a fluid particle (the second derivative of the position vector $\mathbf{x}$ with respect to time t), ρ is the density, $\mathbf{g}$ is the force (per unit mass) acting on the particle, which in our case is gravity, and p denotes the pressure. This equation can be found in any textbook on theoretical physics in general and on hydrodynamics in particular; cf. also (Moritz and Mueller, 1987, p. 204).

For *hydrostatic equilibrium* there is no motion, hence $\ddot{\mathbf{x}} = 0$, and generally (1–8) holds. Thus (2–95) reduces to

$$0 = \rho \text{grad}W - \text{grad}p \quad . \tag{2–96}$$

Now we form the inner product with the vector $d\mathbf{x} = [dx, dy, dz]$, obtaining, e.g.,

$$\text{grad}W \cdot d\mathbf{x} = W_x dx + W_y dy + W_z dz = dW \quad , \tag{2–97}$$

which is nothing else than the total differential of the potential W. Thus (2–96) is equivalent to

$$dW = \rho^{-1} dp \quad . \tag{2–98}$$

Hence $dp = 0$ implies $dW = 0$ and vice versa: *the surfaces of constant potential* ($dW = 0$) *coincide with the surfaces of constant pressure* ($dp = 0$). Hence

$$p = p(W) \tag{2–99}$$

is a function of W only, and so is, by (2–98)

$$\rho = \frac{dp}{dW} = \rho(W) \quad . \tag{2–100}$$

Thus also *the surfaces of constant density coincide with the surfaces of constant potential.* For hydrostatic equilibrium, the surfaces of constant density ρ are also surfaces of constant potential W and surfaces of constant pressure p; we shall call them simply *equisurfaces*. Furthermore, ρ is assumed not to decrease towards the center.

Internal potential of an equilibrium ellipsoid. To get the gravity potential W, we must add the centrifugal potential Φ to the gravitational potential V given by (2–93). For Φ we have the expression (2–8), which holds for the interior as well as for the surface,

$$\Phi = \frac{1}{3}\omega^2 r^2 \left[1 - P_2(\cos\theta)\right] = \frac{1}{3}\omega^2 q^2 \left[1 - P_2(\cos\theta)\right] + O(f^2) \quad . \tag{2–101}$$

In (2–93) we express r by the ellipsoidal equation (2–82) and neglect $O(f^2)$. Thus we only need

$$\frac{1}{r} = \frac{1}{q}\left[1 + \frac{2}{3} f P_2(\cos\theta)\right] \tag{2–102}$$

in the first term on the right-hand side of (2–93); in the other small terms we may simply replace r by q as we did in (2–101). The result is

$$\begin{aligned} W &= V + \Phi = 4\pi G \left[\frac{1}{q} \int_0^q \rho q'^2 dq' + \int_q^R \rho q' dq' + \frac{\omega^2 q^2}{12\pi G} \right] + \\ &+ 4\pi G P_2(\cos\theta) \left[\frac{2}{3} \frac{f}{q} \int_0^q \rho q'^2 dq' - \frac{2}{15 q^3} \int_0^q \rho d(f q'^5) - \right. \\ & \left. - \frac{2}{15} q^2 \int_q^R \rho \frac{df}{dq'} dq' - \frac{\omega^2 q^2}{12\pi G} \right] = W(q, \theta) \quad . \end{aligned} \tag{2–103}$$

Now the ellipsoidal surfaces $q =$ const. are by definition (eq. (2–88)) also surfaces of constant density ρ and, as we have just seen, for hydrostatic equilibrium also surfaces of constant potential and hence *equisurfaces* as defined above. Therefore, for $q =$ const. there must also be $W =$ const., *which is only possible if* (2–103) *does not depend on* θ! Thus the bracket which is a factor of $P_2(\cos\theta)$ must be zero, and there remains

$$W = 4\pi G \left[\frac{1}{q} \int_0^q \rho q'^2 dq' + \int_q^R \rho q' dq' + \frac{\omega^2 q^2}{12\pi G} \right] \quad , \tag{2–104}$$

which clearly has the desired form $W = W(q)$.

Derivation of Clairaut's equation. As we have just seen, the factor $P_2(\cos\theta)$ in (2–103) must be zero. This gives, on multiplying by $(-15/2)$,

$$\begin{aligned} &-5 \frac{f}{q} \int_0^q \rho q'^2 dq' + \frac{1}{q^3} \int_0^q \rho \frac{d(f q'^5)}{dq'} dq' + \\ &+ q^2 \int_q^R \rho \frac{df}{dq'} dq' + \frac{5\omega^2}{8\pi G} q^2 = 0 \quad . \end{aligned} \tag{2–105}$$

This equation must hold identically for all $q \leq R$.

We shall now try to eliminate the integrals by differentiation. First we multiply by q^3:

$$-5 f q^2 \int_0^q \rho q'^2 dq' + \int_0^q \rho \frac{d(f q'^5)}{dq'} dq' + q^5 \int_q^R \rho \frac{df}{dq'} dq' + \frac{5\omega^2}{8\pi G} q^5 = 0 \quad . \tag{2–106}$$

Now we differentiate with respect to q, which is possible since (2–106) is an identity in q. For instance,

$$\frac{d}{dq} \int_0^q \rho \frac{d(f q'^5)}{dq'} dq' = \rho \frac{d(f q^5)}{dq} - 0 = \rho \frac{d(f q^5)}{dq} \tag{2–107}$$

since the integrand vanishes at $q = 0$, and

$$\frac{d}{dq}\int_q^R \rho \frac{df}{dq'}\,dq' = 0 - \rho\frac{df}{dq} = -\rho\frac{df}{dq} \tag{2-108}$$

since ρ, and hence the integrand, vanishes at the boundary ellipsoid. (More precisely, we could extend the integral (1–1) for the potential to a surface $\bar{S}$ enclosing the boundary surface S since $\rho = 0$ between S and $\bar{S}$; and hence we could extend the integral to $R + \epsilon$ instead of R, and at $R + \epsilon$, ρ is certainly zero.)

In this way, the differentiation of (2–106) gives

$$-5\left(q^2\frac{df}{dq} + 2qf\right)\int_0^q \rho q'^2 dq' - 5fq^4\rho + \rho\frac{d(fq^5)}{dq} -$$

$$-q^5\rho\frac{df}{dq} + 5q^4\int_q^R \rho\frac{df}{dq'}\,dq' + \frac{25\omega^2}{8\pi G}q^4 = 0 \quad .$$

It is clear that $\rho = \rho(q')$, $f = f(q')$ inside the integrals and $\rho = \rho(q)$, $f = f(q)$ outside. In view of

$$\rho\frac{d(fq^5)}{dq} = \rho\left(q^5\frac{df}{dq} + 5q^4 f\right) \quad ,$$

which cancels with two other terms, there remains simply

$$-5\left(q^2\frac{df}{dq} + 2qf\right)\int_0^q \rho q'^2 dq' + 5q^4\int_q^R \rho\frac{df}{dq'}\,dq' + \frac{25\omega^2}{8\pi G}q^4 = 0 \quad . \tag{2-109}$$

Now we introduce the mean density D by (2–61), with q instead of r:

$$D = D(q) = \frac{3}{q^3}\int_0^q \rho q'^2 dq' \quad , \tag{2-110}$$

so that (2–109) becomes, on multiplying by $(-3/(5q^4))$,

$$\left(q\frac{df}{dq} + 2f\right)D - 3\int_q^R \rho\frac{df}{dq'}dq' - \frac{15\omega^2}{8\pi G} = 0 \quad . \tag{2-111}$$

Now we may differentiate again with respect to q:

$$\left(q\frac{d^2f}{dq^2} + 3\frac{df}{dq}\right)D + \left(q\frac{df}{dq} + 2f\right)\frac{dD}{dq} + 3\rho\frac{df}{dq} = 0 \quad . \tag{2-112}$$

Differentiating (2–110) we find

$$\frac{dD}{dq} = -\frac{9}{q^4}\int_0^q \rho q'^2 dq' + \frac{3}{q^3}\cdot\rho q^2$$

or

$$\frac{dD}{dq} = -\frac{3}{q}(D - \rho) \quad . \tag{2–113}$$

This is substituted into (2–112), and after some easy algebra we get

$$\frac{d^2 f}{dq^2} + \frac{6}{q}\frac{\rho}{D}\frac{df}{dq} - \frac{6}{q^2}\left(1 - \frac{\rho}{D}\right) f = 0 \quad . \tag{2–114}$$

This is the famous *differential equation of Clairaut* (1743), which is basic for the whole theory of terrestrial equilibrium figures. What follows in this book on equilibrium figures, may be considered variations on this theme by Clairaut. Eq. (2–114) will be derived in various alternative (and very instructive) ways, and, being valid only to a first-order approximation, it will be corrected by terms of second order in the flattening.

Eq. (2–114) thus is a homogeneous ordinary differential equation of second order for $f = f(q)$ $(0 \leq q \leq R)$ (it is clear that the expression "second order" is used here in a quite different sense than in the sentence before!). It must be completed by two initial or boundary conditions, e.g., by prescribing f and df/dq at the surface $q = R$.

In fact, df/dq can be computed as follows. Eq. (2–111) gives for $q = R$:

$$\left(R\frac{df}{dq} + 2f\right) D = \frac{15\omega^2}{8\pi G} \tag{2–115}$$

and with

$$D = \frac{M}{4\pi R^3/3} = \rho_m \quad , \tag{2–116}$$

which, since $q = R$, gives the mean density of the whole earth, we get

$$R\frac{df}{dq} + 2f = \frac{5}{2}\frac{\omega^2 R^3}{GM}$$

or, in view of (2–14),

$$R\frac{df}{dq} + 2f = \frac{5}{2}m \quad . \tag{2–117}$$

Thus

$$\frac{df}{dq} = \frac{1}{R}\left(\frac{5}{2}m - 2f\right) \tag{2–118}$$

at the earth's surface.

The numerical values (1–79) and (1–83) indicate that

$$\frac{df}{dq} > 0 \tag{2–119}$$

at the earth's surface, and it may be shown (cf. Jeffreys, 1976, p. 198; Wavre, 1932, p. 98) that (2–119) also holds in the earth's interior. Thus *the flattening of the equisurfaces decreases* with decreasing q, that is *with increasing depth*.

This holds for the real earth. If the earth were *homogeneous*, then obviously $D = \rho$, so that Clairaut's equation reduces to

$$\frac{d^2 f}{dq^2} + \frac{6}{q}\frac{df}{dq} = 0 \quad , \tag{2–120}$$

which has the solution $df/dq = 0$ or $f = \text{const}$. Thus the equisurfaces of homogeneous ellipsoidal equilibrium figures are geometrically similar ellipsoids (all have the same flattening f). This was derived here as an approximation of first order in f, but may be shown to be rigorously valid. This is the case of the Maclaurin ellipsoid to be considered in sec. 5.4.

Finally we mention that, instead of solving the differential equation (2–114) with the appropriate boundary conditions, we could also try to solve the original equivalent integro-differential equations (2–105) or (2–111) iteratively. This approach may have numerical advantages (Denis, 1989), but from the conceptual and analytical point of view, which we are emphasizing throughout this book, the elegant and mathematically simple and transparent equation of Clairaut remains fundamental.

Our further considerations will, therefore, follow the classical approach, submitting Clairaut's equation to an ingenious transformation due to Radau.

Exercise. Wavre (1932, p. 96) gives the elegant integro-differential equation

$$\frac{d}{dq}(Df) = \frac{3}{q^6}\int_{q'=0}^{q} f(q')q'^5 d\rho \quad ,$$

where for differentiable ρ

$$d\rho = \frac{d\rho(q')}{dq'}dq' \quad .$$

Show its equivalence to Clairaut's equation (2–114) by differentiation.

2.6 Radau's Transformation

Radau (1885) introduces the parameter

$$\eta = \frac{q}{f}\frac{df}{dq} = \frac{d\ln f}{d\ln q} \quad . \tag{2–121}$$

In terms of Radau's parameter we thus have

$$\frac{df}{dq} = \frac{\eta}{q}f \quad , \tag{2–122}$$

and differentiation gives

$$\begin{aligned}\frac{d^2 f}{dq^2} &= -\frac{\eta}{q^2}f + \frac{f}{q}\frac{d\eta}{dq} + \frac{\eta}{q}\frac{df}{dq} \\ &= \left(\frac{1}{q}\frac{d\eta}{dq} + \frac{\eta^2 - \eta}{q^2}\right)f \quad ,\end{aligned} \tag{2–123}$$