For the internal potential ($r < R$) we proceed in exactly the same way, substituting

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\psi) \quad , \tag{2-77}$$

instead of (2-72), into (2-71) and obtaining the internal potential $v = v_i$. Again, orthogonality eliminates all terms except $n = 2$, and (2-74) again applies. The result is

$$v_i = -\frac{8\pi}{15} G\rho f r^2 P_2(\cos\theta) \quad . \tag{2-78}$$

In order to apply (2-65), we must use for the inner spherical potential the expression (2-43), obtaining

$$V_i = \frac{4\pi}{3} G\rho \left[\frac{3}{2} R^2 - \frac{1}{2} r^2 - \frac{2}{5} f r^2 P_2(\cos\theta)\right] \tag{2-79}$$

as the formula for the internal potential of a homogeneous ellipsoid. In both formulas (2-76) and (2-79), terms of $O(f^2)$ are neglected. These two formulas will serve as a basis for computing the potential of a heterogeneous (stratified) ellipsoid.

2.4 Heterogeneous Ellipsoid

Homogeneous shell. As a preparation, consider a thin ellipsoidal shell (of infinitesimal thickness), bounded by two ellipsoids E_1 and E_2, within which the density ρ is constant.

In the same way as we have assigned, in Fig. 2.3, to an ellipsoid E its mean sphere S (of radius R, which defines R as mean radius for E), we can assign such spheres to E_1 and E_2; let q be the mean radius of the inner ellipsoid E_1 and $q + dq$ the mean radius of the outer ellipsoid E_2 (remember they are infinitesimally close to each other). Similarly let f denote the flattening of E_1 and $f + df$ that of E_2, and let f be a function of q,

$$f = f(q) \quad , \tag{2-80}$$

so that

$$df = \frac{df}{dq} dq \quad . \tag{2-81}$$

Then the equation of E_1 is, by (2-64),

$$r = q\left[1 - \frac{2}{3} f P_2(\cos\theta)\right] \tag{2-82}$$

and that of E_2,

$$r = (q + dq)\left[1 - \frac{2}{3}(f + df) P_2(\cos\theta)\right] \quad , \tag{2-83}$$

keeping in mind that f and df depend on q through (2-80) and (2-81).

Now comes the important step. In order to determine the potential of the shell, consider the homogeneous solid ellipsoid bounded by E_2, of constant density ρ, and

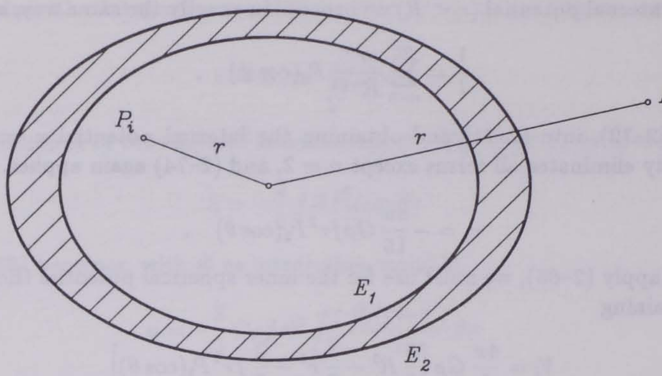

FIGURE 2.4: Ellipsoidal shell

remove the inner solid ellipsoid bounded by E_1; the result obviously is the ellipsoidal shell of Fig. 2.4. Thus the potential of the shell is the difference of the potentials of two homogeneous ellipsoids! This is why we have needed the theory of the homogeneous ellipsoid (sec. 2.3).

Thus, for the external potential, at some point P_e (Fig. 2.4), we have

$$dV_e = V_{shell} = V_e(E_2) - V_e(E_1) = \frac{dV_e}{dq}\,dq \quad , \tag{2–84}$$

since everything depends on q. Now V_e is given by (2–76), with R replaced by q, so that

$$dV_e = \frac{4\pi}{3}G\rho\frac{d}{dq}\left[\frac{q^3}{r} - \frac{2}{5}\frac{q^5}{r^3}fP_2(\cos\theta)\right]dq \quad . \tag{2–85}$$

Note that f, through (2–80), depends on q, but r, being the radius vector of P_e, is to be considered constant with respect to the differentiation. Thus, with

$$\frac{d}{dq}(q^5 f)dq = d(q^5 f) \quad ,$$

(2–85) becomes

$$dV_e = 4\pi G\rho\left[\frac{q^2}{r}\,dq - \frac{2}{15}\,\frac{P_2(\cos\theta)}{r^3}\,d(q^5 f)\right] \tag{2–86}$$

as the external potential of our thin ellipsoidal shell.

Similarly (2–79) gives

$$dV_i = \frac{4\pi}{3}G\rho\frac{d}{dq}\left[\frac{3}{2}q^2 - \frac{1}{2}r^2 - \frac{2}{5}fr^2P_2(\cos\theta)\right]dq$$

or

$$dV_i = 4\pi G\rho\left[q\,dq - \frac{2}{15}r^2P_2(\cos\theta)df\right] \tag{2–87}$$

for the internal potential of the shell at a point P_i (Fig. 2.4) of radius vector r, keeping (2–81) in mind.

Heterogeneous ellipsoid. These formulas are used to build up the expressions for the potential of a heterogeneous ellipsoid. The heterogeneous ellipsoid (our ellipsoids are always rotationally symmetric) is bounded by the ellipsoidal surface with $q = R$. It has an ellipsoidal stratification, and let the set of ellipsoidal surfaces of constant density be labeled by the parameter q (the mean radius) just defined, so that the density

$$\rho = \rho(q) \tag{2–88}$$

is also a function of q (only), where

$$0 \leq q \leq R \quad . \tag{2–89}$$

Then any surface of constant density has an equation of form (2–82); since, by definition, q is constant along such a surface, ρ is also constant on it by (2–88) as it should. (The fact that q is constant on a surface of constant density, of course, does not imply that the surface is a sphere!) Note that q is the r of sec. 2.3; and already earlier (sec. 1.5) we have used r for the mean radius; we write q only when there is a danger of confusion.

The density ρ may even be considered constant between two infinitesimally close surfaces (2–82) and (2–83), or between E_1 and E_2. Thus the heterogeneous ellipsoid is built up from infinitely many infinitesimally thin homogeneous shells of Fig. 2.4, which means integration with respect to q from 0 to R. As a matter of fact, ρ varies from shell to shell according to (2–88).

External potential. The integration of (2–86) thus gives

$$\begin{aligned} V_e &= V_e(r,\theta) = \int_{q=0}^{R} dV_e = \\ &= \frac{4\pi G}{r}\int_0^R \rho q^2 dq - \frac{8\pi G}{15}\frac{P_2(\cos\theta)}{r^3}\int_{q=0}^{R} \rho d(fq^5) \quad . \end{aligned} \tag{2–90}$$

In the first term on the right-hand side we have

$$4\pi \int_0^R \rho q^2 dq = M \quad , \tag{2–91}$$

the total mass enclosed by the sphere $q = R$, as (2–58) shows. Thus

$$V_e = \frac{GM}{r} - \frac{8\pi G}{15}\frac{P_2(\cos\theta)}{r^3}\int_{q=0}^{R} \rho d(fq^5) \quad . \tag{2–92}$$

The integral being a constant, this expression very clearly is of form (1–39) for the expansion of a rotationally symmetric potential into a series of spherical harmonics, truncated at $n = 2$. This also shows that M is the total mass enclosed by the ellipsoid, which thus is seen to be equal to the mass of the auxiliary mean sphere of radius R.

This is quite normal since any ellipsoid E and its associated mean sphere S (of radius q) enclose the same volume by the very definition (2–82), in view of (2–53) for $n = 2$: the mean deviation between E and S is zero. This holds for any ellipsoid of constant density, $q < R$, as well as for the boundary ellipsoid $q = R$, which we are considering in (2–92).

Internal potential. We shall use a similar artifice (Fig. 2.5) as for the sphere

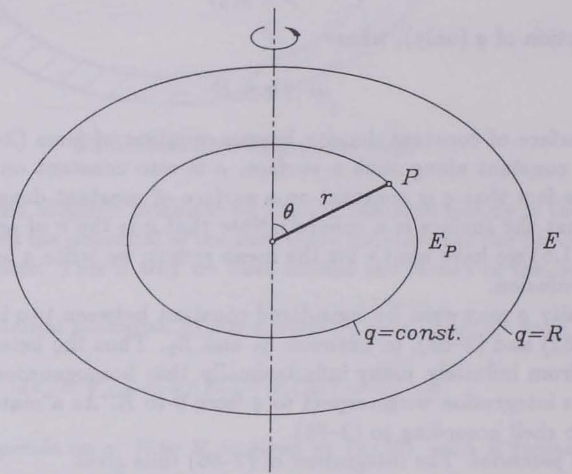

FIGURE 2.5: Illustrating the potential at an interior point P

(Fig. 2.2), considering the ellipsoid (= ellipsoidal surface) of constant density E_P passing through the interior point P at which the potential $V = V_i$ is to be computed. The ellipsoid E_P is characterized by its value q (the radius of the corresponding mean sphere); along E_P, the value of q is, of course, constant as we have already remarked. The equation of E_P is (2–82); r and θ are shown in Fig. 2.5.

Again we shall build up the potential by summing (integrating) the contributions of the infinitesimal shells bounded by ellipsoids of constant density as shown in Fig. 2.4. These contributions are given by (2–86) and (2–87). Since q has been reserved for E_P (Fig. 2.5), we shall denote the integration variable by q', similarly as we did for the sphere, cf. (2–47). For the interior of E_P, i.e. for $q' < q$, we take (2–86); for the shell between E_P and E, i.e. for $q < q' < R$, we take (2–87): P is external for the region inside E_P (being just on its external boundary E_P) and internal for the shell. Thus we get

$$
\begin{aligned}
V_i &= V(r, \theta) = \\
&= \frac{4\pi G}{r} \int_0^q \rho(q') q'^2 dq' - \frac{8\pi G}{15} \frac{P_2(\cos\theta)}{r^3} \int_0^q \rho d(f q'^5) + \\
&+ 4\pi G \int_q^R \rho(q') q' dq' - \frac{8\pi G}{15} r^2 P_2(\cos\theta) \int_q^R \rho \frac{df}{dq'} dq' \quad . \qquad (2\text{–}93)
\end{aligned}
$$

As a matter of fact,

$$\rho = \rho(q') \quad , \qquad f = f(q') \qquad (2\text{–}94)$$

in the second and fourth integral. As a check note that, for $q = R$, (2–93) reduces to (2–90) as it should; obviously the somewhat clumsy notation q' for q when used as integration variable, was not needed in (2–90).

Eq. (2–93) gives the potential at an interior point P if r is expressed in terms of q and θ by (2–82), cf. again Fig. 2.5. The derivation (which goes back to Laplace) is standard (cf. Jeffreys, 1976, sec. 4.03), but nevertheless the attentive reader may have noticed that we are not playing the game quite fair. In fact, working with a series such as (2–77), we must presuppose that P lies *inside the sphere* of radius R since $r < R$. However, we are using the result (2–78) *inside the ellipsoid* and, as Fig. 2.3 shows, a point may well lie inside the ellipsoid but outside the sphere.

The easy answer is that, having derived the basic expressions of sec. 2.3 by regarding the "extra material" of Fig. 2.3 as being compressed as a surface layer on the sphere, so that, for the present purposes, the ellipsoid can formally be identified with the sphere. This answer is not so bad as it looks, but it is not very convincing either. In the past, several mathematicians from Liapunov to Wavre have worried about this problem and tried to solve it. Later, in Chapter 4, we shall attempt to give a simple but quite rigorous argument; in the meantime the reader is asked to take (2–93) on belief.

A remark on infinitesimals. Physicists and other appliers of mathematics have always worked with differentials basically in the sense of Leibnitz, as "infinitely small quantities". This naive approach is eminently successful but has long been frowned upon by mathematicians, who demanded a rigorous limiting process in each instance, which is possible in principle but usually disproportionally laborious. Legitimately, we may interpret differential formulas such as (2–86) as approximations for small but finite dq, which become better and better the more dq decreases.

Only recently, "actual infinitesimals" or real infinitely small quantities, very much in the sense of Leibnitz, have been introduced in mathematics in a rigorous way. This is the subject of "nonstandard analysis" which nowadays enjoys a fair amount of popularity. A very readable introduction is, e.g., (Keisler, 1976).

What counts for the applier is that differentials and infinitesimals can indeed be used in a mathematically respectable fashion, whatever be their interpretation.