or

$$g = \frac{4\pi G}{r^2} \int_0^r r^2 \rho(r) dr \quad ; \tag{2-57}$$

now we may, without danger of confusion, write r instead of r' in the integrand, a convenient and customary though somewhat questionable simplification since, after the integral sign, r denotes the integration variable, whereas as the upper limit of integration and before the integral sign, r denotes the radius vector of P at which V and g are considered (Fig. 2.2).

The physical interpretation of (2–57) is very clear. The part of the earth's mass which is enclosed by the surface S_P is

$$M_P = \int_{r'=0}^{r} \iint_{\sigma} \rho(r') r'^2 dr' d\sigma = 4\pi \int_0^r \rho r^2 dr \tag{2-58}$$

by (2–52), so that (2–57) may be written

$$g = \frac{GM_P}{r^2} , \tag{2-59}$$

in agreement with (2–33) and (2–37). This is the attraction of the "core" within S_P, whereas the attraction of the outer shell is zero, by (2–36). This is quite analogous to the homogeneous case (2–37).

Using this analogy, it is also extremely convenient and useful to introduce the *mean density* D within the sphere S_P by

$$D = \frac{3}{4\pi r^3} M_P \quad , \tag{2-60}$$

in agreement with (2–38), which is the fictitious constant density producing the same attraction (2–59) on and outside S_P as the real density distribution $\rho(r)$ inside S_P. By (2–58) we have

$$D = \frac{3}{r^3} \int_0^r \rho r^2 dr = D(r) \tag{2-61}$$

(D is constant within S_P but, depending on S_P, it depends on r!). Finally, (2–58), (2–59), and (2–61) give

$$g(r) = \frac{4\pi G}{3} r D(r) \quad ; \tag{2-62}$$

a useful formula which is the analogue of (2–39) for a heterogeneous, spherically symmetric stratification.

2.3 Homogeneous Ellipsoid: First-Order Theory

Since the earth is not homogeneous, the theory of a homogeneous ellipsoid only plays an auxiliary and preparatory role, although an important one.

Consider a homogeneous ellipsoid of revolution, of density

$$\rho = \text{const.} \tag{2-63}$$

By first-order theory we mean, as usual, that only terms linear in f are considered, $O(f^2)$ being neglected. To this approximation, its surface is given by (2–6),

$$r = R\left[1 - \frac{2}{3} f P_2(\cos\theta)\right] \quad . \tag{2-64}$$

This equation may be interpreted geometrically as in Fig. 2.3: the ellipsoid consists of

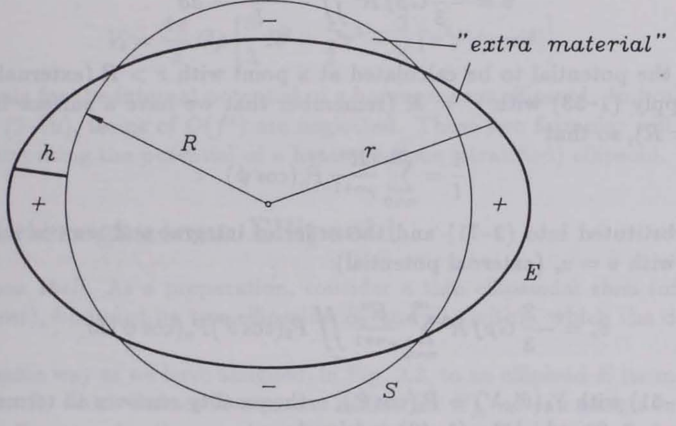

FIGURE 2.3: Ellipsoid and mean sphere

a "basic sphere" of radius R and "extra material" (plus or minus). Thus its potential is given by

$$V = V_{sphere} + v \quad . \tag{2-65}$$

Here v denotes the potential due to the "extra material", which to our approximation may be considered compressed into a surface layer on the sphere, of surface density

$$\mu = \rho h \quad , \tag{2-66}$$

where ρ is the volume density and h the thickness of the layer (Fig. 2.3). The potential of this layer is given by (1–5):

$$v = G \iint_S \frac{\mu}{l}\, dS = G\rho \iint_S \frac{h}{l}\, dS \quad , \tag{2-67}$$

in view of (2–63). Putting

$$dS = R^2 d\sigma \quad , \tag{2-68}$$

we may replace the integration over the sphere S by an integration over the unit sphere σ:

$$v = G\rho R^2 \iint_\sigma \frac{h}{l}\, d\sigma \quad . \tag{2–69}$$

The deviation h of the ellipsoid from the sphere (Fig. 2.3) is, by (2–64),

$$h = -\frac{2}{3} fRP_2(\cos\theta) \quad , \tag{2–70}$$

so that (2–69) becomes, with θ' as integration variable,

$$v = -\frac{2}{3} G\rho f R^3 \iint_\sigma \frac{P_2(\cos\theta')}{l}\, d\sigma \quad . \tag{2–71}$$

Assuming the potential to be calculated at a point with $r > R$ (external potential), we may apply (1–53) with $r' = R$ (remember that we have a surface layer on the sphere $r = R$), so that

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} P_n(\cos\psi) \quad . \tag{2–72}$$

This is substituted into (2–71) and the order of integral and sum is interchanged, obtaining with $v = v_e$ (external potential):

$$v_e = -\frac{2}{3} G\rho f R^3 \sum_{n=0}^{\infty} \frac{R^n}{r^{n+1}} \iint_\sigma P_2(\cos\theta') P_n(\cos\psi) d\sigma \quad . \tag{2–73}$$

Now by (1–51) with $Y_k(\theta', \lambda') = P_2(\cos\theta')$, orthogonality removes all terms except the one with $n = 2$, for which by (1–49) we simply get

$$\iint_\sigma P_2(\cos\theta') P_2(\cos\psi) d\sigma = \frac{4\pi}{5} P_2(\cos\theta) \quad . \tag{2–74}$$

Thus (2–73) reduces to

$$v_e = -\frac{8\pi}{15} G\rho f \frac{R^5}{r^3} P_2(\cos\theta) \quad . \tag{2–75}$$

This is added to the potential of a homogeneous sphere as given by (2–31) with

$$M = \frac{4\pi}{3} \rho R^3$$

by (2–38), in agreement with (2–65). The result is

$$V_e = \frac{4\pi}{3} G\rho \left[\frac{R^3}{r} - \frac{2}{5} \frac{R^5}{r^3} f P_2(\cos\theta) \right] \quad . \tag{2–76}$$

This is the desired formula for the external potential of a homogeneous ellipsoid.

For the internal potential ($r < R$) we proceed in exactly the same way, substituting

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{R^{n+1}} P_n(\cos\psi) \quad , \tag{2–77}$$

instead of (2–72), into (2–71) and obtaining the internal potential $v = v_i$. Again, orthogonality eliminates all terms except $n = 2$, and (2–74) again applies. The result is

$$v_i = -\frac{8\pi}{15} G\rho f r^2 P_2(\cos\theta) \quad . \tag{2–78}$$

In order to apply (2–65), we must use for the inner spherical potential the expression (2–43), obtaining

$$V_i = \frac{4\pi}{3} G\rho \left[\frac{3}{2} R^2 - \frac{1}{2} r^2 - \frac{2}{5} f r^2 P_2(\cos\theta)\right] \tag{2–79}$$

as the formula for the internal potential of a homogeneous ellipsoid. In both formulas (2–76) and (2–79), terms of $O(f^2)$ are neglected. These two formulas will serve as a basis for computing the potential of a heterogeneous (stratified) ellipsoid.

2.4 Heterogeneous Ellipsoid

Homogeneous shell. As a preparation, consider a thin ellipsoidal shell (of infinitesimal thickness), bounded by two ellipsoids E_1 and E_2, within which the density ρ is constant.

In the same way as we have assigned, in Fig. 2.3, to an ellipsoid E its mean sphere S (of radius R, which defines R as mean radius for E), we can assign such spheres to E_1 and E_2; let q be the mean radius of the inner ellipsoid E_1 and $q + dq$ the mean radius of the outer ellipsoid E_2 (remember they are infinitesimally close to each other). Similarly let f denote the flattening of E_1 and $f + df$ that of E_2, and let f be a function of q,

$$f = f(q) \quad , \tag{2–80}$$

so that

$$df = \frac{df}{dq} dq \quad . \tag{2–81}$$

Then the equation of E_1 is, by (2–64),

$$r = q\left[1 - \frac{2}{3} f P_2(\cos\theta)\right] \tag{2–82}$$

and that of E_2,

$$r = (q + dq)\left[1 - \frac{2}{3}(f + df) P_2(\cos\theta)\right] \quad , \tag{2–83}$$

keeping in mind that f and df depend on q through (2–80) and (2–81).

Now comes the important step. In order to determine the potential of the shell, consider the homogeneous solid ellipsoid bounded by E_2, of constant density ρ, and