This equation could also have been derived as a first-order approximation to Somigliana's formula (1–23); similarly there is a rigorous, though less simple, equivalent of Clairaut's formula (2–26) for the level ellipsoid; cf. (Heiskanen and Moritz, 1967, secs. 2-8 and 2-10) and eq. (5–69) later in sec. 5.2.

If we had a uniform coverage of the earth by gravity measurements (unfortunately we don't), then we could try to fit a formula of type (2–29) (to a higher approximation) to these measurements, obtaining f^*. Then the flattening f could be derived by (2–26) from

$$f = -f^* + \frac{5}{2} m \quad . \tag{2–30}$$

This is a complete gravimetric analogue to (2–18): it permits to determine the flattening f from gravity flattening f^*, whereas (2–18) allows the computation of f from the satellite-determined J_2.

2.2 Internal Field of a Stratified Sphere

First-order ellipsoidal formulas, as we have seen and will see, are basically spherical formulas with corrections on the order of the flattening f. In this sense, the sphere serves as a reference for the ellipsoid, and it will be useful to study the gravitational field of a stratified sphere, such as shown by Fig. 1.5.

The *external* gravitational field of any spherically symmetric distribution is given simply by

$$V = \frac{GM}{r} \quad . \tag{2–31}$$

It is formally equal to the potential of a mass point, regardless of the inner structure of the body as long as it is spherically symmetric. This is seen immediately on writing the general spherical-harmonic expansion (1–36), with (1–47), in Laplace's form

$$V = \sum_{n=0}^{\infty} \frac{Y_n(\theta, \lambda)}{r^{n+1}} = \frac{Y_0}{r} + \sum_{n=1}^{\infty} \frac{Y_n(\theta, \lambda)}{r^{n+1}} \quad . \tag{2–32}$$

Of the Laplacian harmonics $Y_n(\theta, \lambda)$, only Y_0 is constant; cf. (1–33). In the case of spherical symmetry, all functions $Y_n(\theta, \lambda)$ must be missing except the constant Y_0 which, by (1–3), is seen to be equal to GM; this proves (2–31).

Gravity outside the sphere is then simply

$$g = -\frac{\partial V}{\partial r} = -\frac{dV}{dr} = \frac{GM}{r^2} \quad . \tag{2–33}$$

Note that if we consider the sphere as a zero-degree approximation to the ellipsoid, it must be nonrotating since $\omega^2 = O(f)$ by (2–10), so that $f = 0$ implies $\omega = 0$. Thus, to this primitive approximation, $W = V$, and gravity coincides with gravitational attraction. The spherical symmetry of (2–33) is obvious. Eqs. (2–31) and (2–33) are valid down to the surface of the sphere.

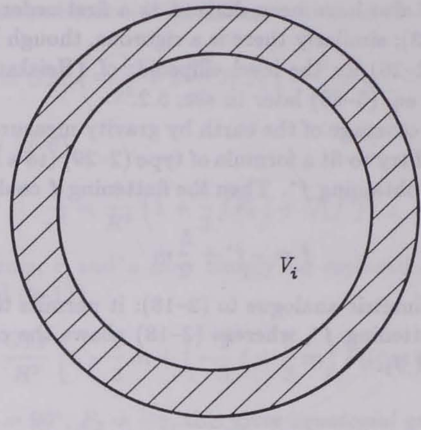

FIGURE 2.1: A spherical shell

The *internal* potential is more complicated. First we consider the potential in the interior of a hollow spherical shell (Fig. 2.1). It is easily seen to be constant:

$$V_i = C = \text{const.} \tag{2–34}$$

In fact, the potential V_i is a harmonic function, satisfying Laplace's equation $\Delta V = 0$, in the interior of the shell, and must therefore admit a spherical-harmonic expansion

$$V_i = \sum_{n=0}^{\infty} r^n Y_n(\theta, \lambda) = Y_0 + \sum_{n=1}^{\infty} r^n Y_n(\theta, \lambda) \quad , \tag{2–35}$$

analogous to (2–32), but with the outer harmonics (1–35b) replaced by the inner harmonics (1–35a). Repeating the previous argument considering spherical symmetry, only the term Y_0 can survive in (2–35), and setting $Y_0 = C$ proves (2–34).

It is clear that the structure of the shell has no influence as long as it is spherically symmetric: it may be homogeneous or layered (stratified).

Since the potential is identically constant inside the shell, the force vanishes there:

$$\mathbf{g} = \text{grad} V_i = 0 \tag{2–36}$$

inside the shell.

Homogeneous sphere. The gravity (gravitational attraction) of a homogeneous sphere at an internal point P is found by a simple but very useful trick (this trick is one reason for treating the physically rather uninteresting homogeneous case here). Consider the sphere S_P passing through P (Fig. 2.2). Then *gravity* g_1 *due to the shell between* S *and* S_P, *is zero* by (2–36). The gravity g_2 due to the "core" bounded by S_P is then given by the "external" formula (2–33):

$$g_2 = \frac{GM_P}{r^2} \quad , \tag{2–37}$$

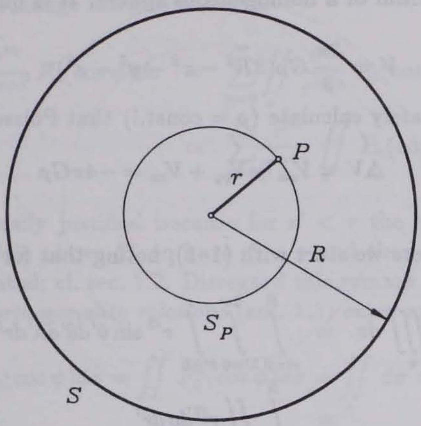

FIGURE 2.2: Computation point P inside the sphere

where

$$M_P = \frac{4\pi}{3} r^3 \rho \tag{2–38}$$

denotes the mass of the part enclosed by S_P; ρ is the constant density. ("Core" is meant in a figurative sense and has, of course, nothing to do with the actual earth's core!)

Thus

$$g_P = g_1 + g_2 = g_2 = \frac{4\pi G}{3} r\rho \quad , \tag{2–39}$$

by (2–36), (2–37), and (2–38).

In order to find the potential V, we integrate (2–33) in our case,

$$\frac{dV}{dr} = -g = -\frac{4\pi G}{3} \rho r \quad , \tag{2–40}$$

which gives

$$V = -\frac{2\pi G}{3} \rho r^2 + C_1 \quad . \tag{2–41}$$

The integration constant C_1 is determined such that, at the outer surface $r = R$, (2–41) must yield the same result as (2–31):

$$V(R) = -\frac{2\pi G}{3} \rho R^2 + C_1 = \frac{GM}{R} = \frac{4\pi G}{3} R^3 \rho \frac{1}{R} \quad , \tag{2–42}$$

whence $C_1 = 2\pi G \rho R^2$, and

$$V_i = V(r) = 2\pi G \rho \left(R^2 - \frac{1}{3} r^2 \right) \tag{2–43}$$

gives the internal potential of a homogeneous sphere. It is quadratic in x, y, z since $r^2 = x^2 + y^2 + z^2$:

$$V = \frac{2\pi}{3} G\rho(3R^2 - x^2 - y^2 - z^2) \quad , \tag{2–44}$$

from which we immediately calculate (ρ = const.!) that Poisson's equation

$$\Delta V = V_{xx} + V_{yy} + V_{zz} = -4\pi G\rho \tag{2–45}$$

is satisfied.

Stratified sphere. Here we start with (1–1), noting that for the sphere

$$\begin{aligned} \iiint\limits_{v} dv &= \int\limits_{r'=0}^{R} \int\limits_{\lambda'=0}^{2\pi} \int\limits_{\theta'=0}^{\pi} r'^2 \sin\theta' d\theta' d\lambda' dr' \\ &= \int\limits_{r'=0}^{R} \iint\limits_{\sigma} r'^2 d\sigma dr' \end{aligned} \tag{2–46}$$

by (1–43); the integration variables are denoted by r', θ', λ', reserving r, θ, λ for the interior point P at which V is to be computed. Thus (1–1) becomes

$$V = V(r, \theta, \lambda) = G \int\limits_{r'=0}^{R} r'^2 \rho(r') \iint\limits_{\sigma} \frac{d\sigma}{l} dr' \quad . \tag{2–47}$$

Note that ρ is a function only of the radius vector r' because of spherical symmetry.

Now we apply the basic series (1–53). Note that r' and r play a symmetric role in (1–52), but in (1–53), the larger one of the two must be in the denominator in order to obtain a convergent series. Thus within S_P (Fig. 2.2) there is $r' < r$ and we must use

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \qquad 0 < r' < r \quad , \tag{2–48a}$$

whereas in the shell between S_P and S there is $r' > r$ and we must use

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\psi) \quad , \qquad r < r' < R \quad . \tag{2–48b}$$

Thus (2–47) must be split up into two parts:

$$V = V_1 + V_2 \quad , \tag{2–49}$$

$$V_1 = G \int\limits_{r'=0}^{r} r'^2 \rho(r') \iint\limits_{\sigma} \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) d\sigma dr' \quad , \tag{2–50a}$$

$$V_2 = G \int\limits_{r'=r}^{R} r'^2 \rho(r') \iint\limits_{\sigma} \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos\psi) d\sigma dr' \quad . \tag{2–50b}$$

In (2–50a) we can interchange integral and sum:

$$\iint_\sigma \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) d\sigma = \sum_{n=0}^{\infty} \iint_\sigma \frac{r'^n}{r^{n+1}} P_n(\cos\psi) d\sigma = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} \iint_\sigma P_n(\cos\psi) d\sigma \quad . \tag{2–51}$$

(This interchange is easily justified because for $r' < r$ the series is uniformly convergent: lift P very little above S_P to have $r' < r$ and then move it back to S_P by continuity of the potential; cf. sec. 1.2. Disregard this remark if you are not a mathematician.) Then the *orthogonality relations* (sec. 1.3) enter and give, by (1–42) with $n = 0$,

$$\iint_\sigma P_0(\cos\psi) d\sigma = \iint_\sigma P_0^2(\cos\psi) d\sigma = \iint_\sigma d\sigma = 4\pi \quad , \tag{2–52}$$

whereas, by (1–51) with $k = 0$, for $n > 0$

$$\iint_\sigma P_n(\cos\psi) d\sigma = 0 \tag{2–53}$$

since $P_0 = R_{00} = 1$ and $Y_0 =$ const.

Thus, in the sum (2–51), only the term $n = 0$ survives (there remains only $4\pi/r$), and (2–50a), as well as (2–50b) by analogy, reduce to

$$\begin{aligned} V_1 &= 4\pi G \frac{1}{r} \int_{r'=0}^{r} r'^2 \rho(r') dr' \quad , \\ V_2 &= 4\pi G \int_{r'=r}^{R} r' \rho(r') dr' \quad . \end{aligned} \tag{2–54}$$

Thus the *internal potential* V, by (2–49), finally becomes

$$V = 4\pi G \left[\frac{1}{r} \int_{r'=0}^{r} r'^2 \rho(r') dr' + \int_{r'=r}^{R} r' \rho(r') dr' \right] \quad . \tag{2–55}$$

As a check we put $\rho =$ const. and get (2–43).

Gravity inside the earth. In agreement with (2–33) we have

$$g = -\frac{dV}{dr} \quad . \tag{2–56}$$

Differentiating (2–55) according to the usual rules of calculus we get

$$g = -4\pi G \left[-\frac{1}{r^2} \int_0^r r'^2 \rho(r') dr' + \frac{1}{r} \cdot r^2 \rho(r) - r\rho(r) \right]$$

or

$$g = \frac{4\pi G}{r^2} \int_0^r r^2 \rho(r) dr \quad ; \tag{2–57}$$

now we may, without danger of confusion, write r instead of r' in the integrand, a convenient and customary though somewhat questionable simplification since, after the integral sign, r denotes the integration variable, whereas as the upper limit of integration and before the integral sign, r denotes the radius vector of P at which V and g are considered (Fig. 2.2).

The physical interpretation of (2–57) is very clear. The part of the earth's mass which is enclosed by the surface S_P is

$$M_P = \int_{r'=0}^{r} \iint_{\sigma} \rho(r') r'^2 dr' d\sigma = 4\pi \int_0^r \rho r^2 dr \tag{2–58}$$

by (2–52), so that (2–57) may be written

$$g = \frac{GM_P}{r^2} , \tag{2–59}$$

in agreement with (2–33) and (2–37). This is the attraction of the "core" within S_P, whereas the attraction of the outer shell is zero, by (2–36). This is quite analogous to the homogeneous case (2–37).

Using this analogy, it is also extremely convenient and useful to introduce the *mean density* D within the sphere S_P by

$$D = \frac{3}{4\pi r^3} M_P \quad , \tag{2–60}$$

in agreement with (2–38), which is the fictitious constant density producing the same attraction (2–59) on and outside S_P as the real density distribution $\rho(r)$ inside S_P. By (2–58) we have

$$D = \frac{3}{r^3} \int_0^r \rho r^2 dr = D(r) \tag{2–61}$$

(D is constant within S_P but, depending on S_P, it depends on r!). Finally, (2–58), (2–59), and (2–61) give

$$g(r) = \frac{4\pi G}{3} r D(r) \quad ; \tag{2–62}$$

a useful formula which is the analogue of (2–39) for a heterogeneous, spherically symmetric stratification.

2.3 Homogeneous Ellipsoid: First-Order Theory

Since the earth is not homogeneous, the theory of a homogeneous ellipsoid only plays an auxiliary and preparatory role, although an important one.