# Chapter 2

# The Equilibrium Figure of the Earth: Basic Theory

## 2.1 External Ellipsoidal Field to First-Order Approximation

Let us first consider the ellipsoid of revolution as a level surface; this is a good approximation to the earth as we have seen in Chapter 1. In view of the smallness of the flattening $f(\doteq 0.003)$, we shall in this chapter disregard $f^2$ and other higher powers of the flattening. This is the *first-order theory* also considered by Clairaut (1743). For present accuracies, a second-order theory, accurate up to $f^2$, is required. This will be done in Chapter 4. The first-order theory, however, is much simpler and very beautiful and instructive and will, therefore, be treated first.

*Equation of the ellipsoid.* To first order, (1–73) reduces to

$$r = a(1 - f\cos^2\theta) \quad . \tag{2–1}$$

It will be useful to introduce spherical harmonics. By eq. (1–33), the Legendre polynomial $P_2$ is given by

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} \quad , \tag{2–2}$$

so that (2–1) may be transformed into

$$r = a\left[1 - \frac{1}{3}f - \frac{2}{3}f\,P_2(\cos\theta)\right] \quad . \tag{2–3}$$

The mean earth radius $R$, cf. (1–86), is the average of $r$ over the unit sphere:

$$R = \frac{1}{4\pi}\iint_\sigma r\,d\sigma = a\left(1 - \frac{1}{3}f\right) \tag{2–4}$$

since the integral over $P_2$ is zero:

$$\iint_\sigma P_2\,d\sigma = \iint_\sigma P_0P_2\,d\sigma = 0 \tag{2–5}$$

by (1–33) and (1–41), in view of the orthogonality of spherical harmonics. Thus (2–3) becomes

$$r = R\left[1 - \frac{2}{3} f\, P_2(\cos\theta)\right] \quad ; \tag{2–6}$$

*note that we are consistently neglecting* $f^2$! This equation expresses, for the ellipsoid, the radius vector $r$ as a function of $\theta$ (and $\lambda$). The longitude $\lambda$ does not occur explicitly because our ellipsoid is a surface of revolution; for the same reason, (2–6) does not contain tesseral harmonics which depend explicitly on $\lambda$ (sec. 1.3).

The fact that only the *even* polynomial $P_2$ enters into (2–6), expresses equatorial symmetry (symmetry with respect to the equatorial plane), which would be distroyed by the odd polynomials $P_1, P_3, \ldots$; cf. (1–33).

*Gravity potential.* The gravitational potential may be expressed by the rotationally symmetric zonal expansion (1–39), retaining only $J_2$:

$$V = \frac{GM}{r}\left[1 - \frac{a^2}{r^2} J_2 P_2(\cos\theta)\right] \quad . \tag{2–7}$$

In fact, (1–77) shows that $J_2$ is of order $f$; $J_3$ is missing because of equatorial symmetry, and $J_4$ is already of order of $J_2^2$ or of $f^2$ and must therefore be neglected (for numerical values of $J_4$ cf. sec. 6.4 later in the book).

For the centrifugal potential we have by (1–6), (1–26) and (2–2):

$$\Phi = \frac{1}{2}\omega^2(x^2 + y^2) = \frac{1}{2}\omega^2 r^2 \sin^2\theta = \frac{1}{3}\omega^2 r^2 \left[1 - P_2(\cos\theta)\right] \quad . \tag{2–8}$$

The sum of (2–7) and (2–8) gives the gravitational potential $W$:

$$W = \frac{GM}{r}\left[1 - \frac{a^2}{r^2} J_2 P_2(\cos\theta)\right] + \frac{1}{3}\omega^2 r^2 \left[1 - P_2(\cos\theta)\right] \quad . \tag{2–9}$$

Now we note that

$$J_2 = O(f) \quad , \qquad \omega^2 = O(f) \quad , \tag{2–10}$$

where, as we have already remarked, the symbol $O(f)$ reads "on the order of $f$", denoting quantities of order $f$. The first equation has been explained above; the second will be justified later; cf. eq. (2–14). Thus, in keeping with our approximation and neglecting $O(f^2)$, we can put $a^2/r^2 \doteq 1$ in (2–7) because it already is multiplied by $J_2 = O(f)$. For the same reason we may put $r^2 \doteq R^2$ in (2–8). Thus (2–9) becomes

$$W = \frac{GM}{r}(1 - J_2 P_2) + \frac{1}{3}\omega^2 R^2 (1 - P_2) \quad , \tag{2–11}$$

abbreviating

$$P_2(\cos\theta) = P_2 \quad . \tag{2–12}$$

By (2–6),

$$\frac{1}{r} = \frac{1}{R}\left(1 + \frac{2}{3} f\, P_2\right) + O(f^2)$$

(binomial series!). This is substituted into (2–11), the multiplications are carried out, and $O(f^2)$ is neglected. The result may be written

$$W = \frac{GM}{R}\left[1 + \frac{1}{3}m + \left(\frac{2}{3}f - J_2 - \frac{1}{3}m\right)P_2(\cos\theta)\right] \quad , \tag{2-13}$$

where

$$m = \frac{\omega^2 R^3}{GM} = \frac{\omega^2 a^2 b}{GM} = 0.00345 \tag{2-14}$$

by (1–83), which is indeed of order $f$ and thus justifies putting $\omega^2 = O(f)$ as in (2–10).

If our ellipsoid is to be a level surface, $W$ must be constant on it:

$$W = W_0 \quad , \tag{2-15}$$

so that the coefficient of $P_2(\cos\theta)$ in (2–13) must vanish. This gives

$$W = \frac{GM}{R}\left(1 + \frac{1}{3}m\right) = W_0 \tag{2-16}$$

and

$$\frac{2}{3}f - J_2 - \frac{1}{3}m = 0 \quad ,$$

which yields an extremely important relation between $f$ and $J_2$:

$$J_2 = \frac{2}{3}f - \frac{1}{3}m \tag{2-17}$$

or, inversely,

$$f = \frac{3}{2}J_2 + \frac{1}{2}m \quad . \tag{2-18}$$

This is not only a beautiful relation between geometrical ($f$) and physical ($J_2$, $m$) quantities, but is the key formula for the direct determination of the flattening $f$ from the satellite-determined coefficient $J_2$. Of course, practically a higher-order approximation is required, but nothing shows the essential structure of the problem more clearly than (2–18).

Finally we note that, using the ellipsoid as a model for the geoid, we simply have identified the actual potential $W$ with the normal potential $U$, in keeping with Clairaut's approximation; cf. sec. 1.2.

*Gravity.* The radial component of gravity $g$ is

$$-\frac{\partial W}{\partial r} \quad ,$$

the $\theta$-component is

$$\frac{1}{r}\frac{\partial W}{\partial \theta} \quad ,$$

so that

$$g = \sqrt{\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2} \doteq -\frac{\partial W}{\partial r} \tag{2-19}$$

since $(\partial W/\partial\theta)^2$ is of second order. The differentiation of (2–9) gives

$$\frac{\partial W}{\partial r} = GM\left(-\frac{1}{r^2} + 3\,\frac{a^2}{r^4}\,J_2\,P_2\right) + \frac{2}{3}\,\omega^2 r(1 - P_2) \quad . \tag{2–20}$$

Now we substitute, by (2–6),

$$\frac{1}{r^2} = \frac{1}{R^2}\left(1 + \frac{4}{3}\,fP_2\right) + O(f^2) \quad ; \tag{2–21}$$

in the other small terms, $r$ and $a$ may simply be replaced by $R$. This gives, also considering (2–14) and (2–17),

$$g = \frac{GM}{R^2}\left[1 - \frac{2}{3}\,m + \left(-\frac{2}{3}\,f + \frac{5}{3}\,m\right)P_2(\cos\theta)\right] \quad . \tag{2–22}$$

For the equator, $\theta = 90°$, $P_2 = -\frac{1}{2}$, this gives *equatorial gravity*

$$\gamma_e = \frac{GM}{R^2}\left(1 + \frac{1}{3}\,f - \frac{3}{2}\,m\right) \tag{2–23}$$

(we do not distinguish here between gravity $g$ and normal gravity $\gamma$!); for the pole, $\theta = 0°$, $P_2 = 1$, we have *polar gravity*

$$\gamma_p = \frac{GM}{R^2}\left(1 - \frac{2}{3}\,f + m\right) \quad , \tag{2–24}$$

so that for the *gravity flattening* (1–84) we get

$$f^* = \frac{\gamma_p - \gamma_e}{\gamma_e} = -f + \frac{5}{2}\,m \quad . \tag{2–25}$$

This gives another beautiful formula

$$f + f^* = \frac{5}{2}\,m \tag{2–26}$$

due to Clairaut, which relates the geometrical flattening $f$ and the gravity flattening $f^*$ in a surprisingly simple way. There is a physical interpretation also for the dimensionless quantity $m$: by (2–14) and (2–23) we have, disregarding $O(f^2)$,

$$m = \frac{\omega^2 R}{GM/R^2} = \frac{\omega^2 a}{\gamma_e} = \frac{\text{centrifugal force at equator}}{\text{gravity at equator}} \quad . \tag{2–27}$$

Then (2–22) may be transformed, using (2–2), to

$$g = \gamma_e\left[1 + \left(-f + \frac{5}{2}\,m\right)\cos^2\theta\right] \tag{2–28}$$

or, by (2–26),

$$g = \gamma_e(1 + f^*\cos^2\theta) \quad . \tag{2–29}$$

This equation could also have been derived as a first-order approximation to Somigliana's formula (1–23); similarly there is a rigorous, though less simple, equivalent of Clairaut's formula (2–26) for the level ellipsoid; cf. (Heiskanen and Moritz, 1967, secs. 2-8 and 2-10) and eq. (5–69) later in sec. 5.2.

If we had a uniform coverage of the earth by gravity measurements (unfortunately we don't), then we could try to fit a formula of type (2–29) (to a higher approximation) to these measurements, obtaining $f^*$. Then the flattening $f$ could be derived by (2–26) from

$$f = -f^* + \frac{5}{2}m \quad . \tag{2–30}$$

This is a complete gravimetric analogue to (2–18): it permits to determine the flattening $f$ from gravity flattening $f^*$, whereas (2–18) allows the computation of $f$ from the satellite-determined $J_2$.

## 2.2 Internal Field of a Stratified Sphere

First-order ellipsoidal formulas, as we have seen and will see, are basically spherical formulas with corrections on the order of the flattening $f$. In this sense, the sphere serves as a reference for the ellipsoid, and it will be useful to study the gravitational field of a stratified sphere, such as shown by Fig. 1.5.

The *external* gravitational field of any spherically symmetric distribution is given simply by

$$V = \frac{GM}{r} \quad . \tag{2–31}$$

It is formally equal to the potential of a mass point, regardless of the inner structure of the body as long as it is spherically symmetric. This is seen immediately on writing the general spherical-harmonic expansion (1–36), with (1–47), in Laplace's form

$$V = \sum_{n=0}^{\infty} \frac{Y_n(\theta,\, \lambda)}{r^{n+1}} = \frac{Y_0}{r} + \sum_{n=1}^{\infty} \frac{Y_n(\theta,\, \lambda)}{r^{n+1}} \quad . \tag{2–32}$$

Of the Laplacian harmonics $Y_n(\theta,\, \lambda)$, only $Y_0$ is constant; cf. (1–33). In the case of spherical symmetry, all functions $Y_n(\theta,\, \lambda)$ must be missing except the constant $Y_0$ which, by (1–3), is seen to be equal to $GM$; this proves (2–31).

Gravity outside the sphere is then simply

$$g = -\frac{\partial V}{\partial r} = -\frac{dV}{dr} = \frac{GM}{r^2} \quad . \tag{2–33}$$

Note that if we consider the sphere as a zero-degree approximation to the ellipsoid, it must be nonrotating since $\omega^2 = O(f)$ by (2–10), so that $f = 0$ implies $\omega = 0$. Thus, to this primitive approximation, $W = V$, and gravity coincides with gravitational attraction. The spherical symmetry of (2–33) is obvious. Eqs. (2–31) and (2–33) are valid down to the surface of the sphere.