The comparison with (1–74) shows that, to first order in f,

$$\phi - \psi = e^2 \cos\phi \sin\phi = 2f \cos\phi \sin\phi \quad , \tag{1–75}$$

neglecting f^2 in (1–56). To the same accuracy we may replace ψ by ϕ in (1–68), obtaining

$$\phi - \psi = 2f \cos\theta \sin\theta \quad , \tag{1–76}$$

accurate to order f.

1.5 Earth Models and Parameters

The Geodetic Reference System 1980. This system (GRS 1980) defines basic parameters for a globally best-fitting earth ellipsoid. It was adopted at the XVII General Assembly of the International Union of Geodesy and Geophysics (IUGG) in Canberra, December 1979, and still is the official reference system of the International Association of Geodesy (IAG) and is likely to remain so for the next years or even decades, since no significant changes have occurred or are to be expected in the near future.

The equipotential ellipsoid (1–21) and its external gravitational field are completely defined by four independent constants. The GRS 1980 takes

$$\begin{aligned} a &= 6378\,137\,\mathrm{m} \quad , \\ GM &= 3986\,005 \times 10^8 \mathrm{m^3 s^{-2}} \quad , \\ J_2 &= 108\,263 \times 10^{-8} \quad , \\ \omega &= 7292\,115 \times 10^{-11} \mathrm{s^{-1}} \end{aligned} \tag{1–77}$$

as the defining constants. The meanings of a (semimajor axis or equatorial radius, cf. Fig. 1.1), GM (geocentric gravitational constant including the atmosphere, cf. eq. (1–3)) and ω (angular velocity, cf. eq. (1–7)) are clear. Note that the product GM, rather than the earth's mass M, is given since the gravitational constant (1–2) is known only to 4 significant digits, whereas GM has 7 significant digits.

The "dynamical form factor" or zonal harmonic coefficient of degree 2 is defined by

$$J_2 = \frac{C - A}{Ma^2} \quad ; \tag{1–78}$$

it is dimensionless and, together with A (mean equatorial moment of inertia) and C (polar moment of inertia), has already been encountered in sec. 1.3; cf. (1–40).

Using the theory of the equipotential ellipsoid (cf. (Heiskanen and Moritz, 1967, pp. 64–79) and Chapter 5 of the present book), all other constants can be derived, e.g., flattening f, excentricity e, equatorial gravity γ_e, and ellipsoidal potential U_0:

$$f = 0.003\,352\,810\,681 = 1/298.257\,2221 \quad , \tag{1–79}$$

$$e^2 = 0.006\,694\,380\,023 \quad , \tag{1–80}$$

$$\gamma_e = 9.780\,326\,7715\,\mathrm{ms^{-2}} \quad , \tag{1–81}$$

$$U_0 = 6263\,686.0850 \times 10\,\mathrm{m^2 s^{-2}} \tag{1–82}$$

(digits above 7 are computationally consistent rather than significant!).

We shall also need the dimensionless auxiliary quantities

$$m = \frac{\omega^2 a^2 b}{GM} = 0.003\,449\,786 \quad , \tag{1–83}$$

$$f^* = \frac{\gamma_p - \gamma_e}{\gamma_e} = 0.005\,302\,440 \quad ; \tag{1–84}$$

the latter is called *gravity flattening*. More about the Geodetic Reference System can be found in (Moritz, 1984).

Other global constants. An extremely important constant that does not follow from the GRS 1980 but can be determined by the observation of astronomical precession is the *dynamical ellipticity*

$$H = \frac{C - A}{C} = 0.003\,273\,9935 \pm 7 \times 10^{-8} \quad , \tag{1–85}$$

quoted from (Landolt–Börnstein, 1984, p. 19).

Finally we give some useful rounded values:

$$R = 6371\,\mathrm{km} \quad , \tag{1–86}$$

$$\bar{g} = 9.81\,\mathrm{m\,s^{-2}} \tag{1–87}$$

for the mean radius of the earth and for mean gravity on its surface.

Models for the earth's interior. A schematic picture of the earth's interior is given in Fig. 1.5. We may roughly distinguish between crust (of thickness varying around 30 km), mantle and core (liquid outer and probably solid inner core), with

$$R_c = 3480\,\mathrm{km} \tag{1–88}$$

and

$$R_{ic} = 1220\,\mathrm{km} \tag{1–89}$$

as rather commonly accepted approximate values for their mean radii.

Standard sources for the structure of the earth's interior are (Bullen, 1975) and (Melchior, 1986), as well as (Landolt–Börnstein, 1984) for numerical values.

For the flattening of the core-mantle boundary, considered an ellipsoid of revolution of semiaxes a_c and b_c, we have the recent determination (Denis and Ibrahim, 1981, p. 189).

$$f_c = \frac{a_c - b_c}{a_c} \doteq 0.00256 \doteq \frac{1}{390} \quad , \tag{1–90}$$

which is in good agreement with (Bullen, 1975, p. 58).

A representative picture of our ideas regarding the distribution of density, pressure, and gravity inside the earth is presented by Fig. 1.6 following (Bullen, 1975, pp. 360–362). Today the most widely used earth model is PREM (Preliminary Reference Earth Model) due to Dziewonski and Anderson; see (Landolt–Börnstein, 1984, pp. 85–96). Its density distribution is shown in Fig. 1.7. In view of its importance, we

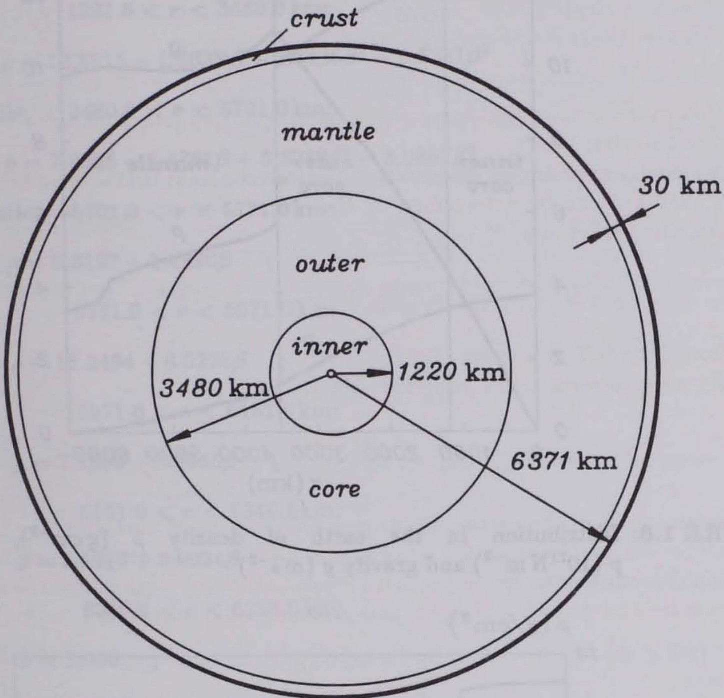

FIGURE 1.5: The earth's interior

shall also quote from (Landolt–Börnstein, 1984, p. 88) the following numerical values corresponding to a piecewise polynomial representation

$$\rho = a_0 + a_1\beta + a_2\beta^2 + a_3\beta^3 \quad , \tag{1--91}$$

where

$$\beta = \frac{r}{R} \tag{1--92}$$

is the normalized radius vector, increasing from 0 (geocenter) to 1 (earth's surface); it is clear that all models for the earth's interior here are spherical. The density ρ is in g/cm^3.

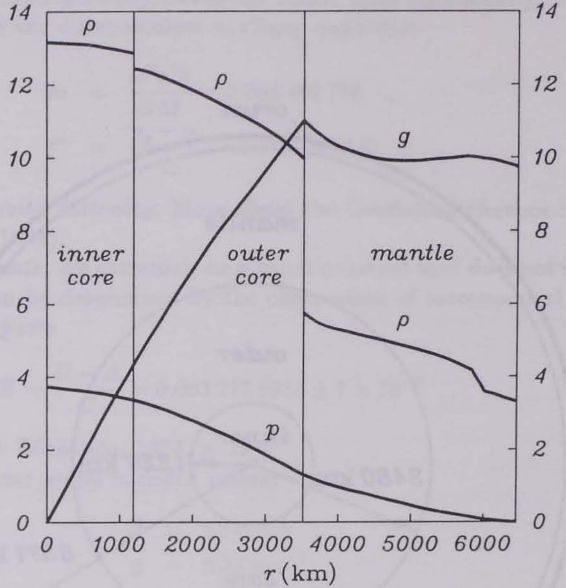

FIGURE 1.6: Distribution in the earth of density ρ ($\mathrm{g\,cm^{-3}}$), pressure p ($10^{11}\mathrm{N\,m^{-2}}$) and gravity g ($\mathrm{m\,s^{-2}}$)

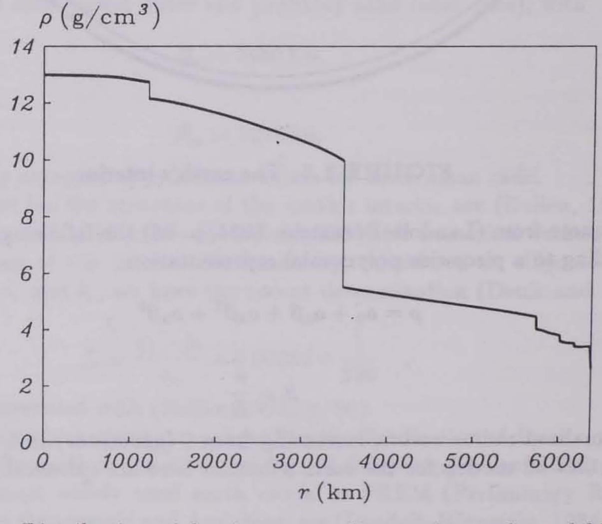

FIGURE 1.7: Distribution of density ρ according to the earth model PREM

Inner core, $0 < r < 1221.5\,\mathrm{km}$:

$$\rho = 13.0885 - 8.8381\,\beta^2 \quad ;$$

outer core, $1221.5 < r < 3480.0\,\mathrm{km}$:

$$\rho = 12.5815 - 1.2638\,\beta - 3.6426\beta^2 - 5.5281\beta^3 \quad ;$$

lower mantle, $3480.0 < r < 5701.0\,\mathrm{km}$:

$$\rho = 7.9565 - 6.4761\beta + 5.5283\beta^2 - 3.0807\beta^3 \quad ;$$

upper mantle, $5701.0 < r < 5771.0\,\mathrm{km}$: (1–93)

$$\rho = 5.3197 - 1.4836\beta \quad ;$$

$5771.0 < r < 5971.0\,\mathrm{km}$:

$$\rho = 11.2494 - 8.0298\beta \quad ;$$

$5971.0 < r < 6151.0\,\mathrm{km}$:

$$\rho = 7.1089 - 3.8045\beta \quad ;$$

$6151.0 < r < 6346.6\,\mathrm{km}$:

$$\rho = 2.6910 + 0.6924\beta \quad ;$$

crust, $6346.6 < r < 6356.0\,\mathrm{km}$:

$$\rho = 2.900 \quad ;$$

$6356.0 < r < 6368.0\,\mathrm{km}$:

$$\rho = 2.600 \quad ;$$

ocean, $6368.0 < r < 6371.0\,\mathrm{km}$:

$$\rho = 1.020 \quad .$$

Other earth models, e.g., are PEM, C2 and 1066A, B; for details and references see (Landolt–Börnstein, 1984, p. 96) and (Melchior, 1986, pp. 29 and 41).

Mass and mean density. Dividing GM from (1–77) by the gravitational constant (1–2) we get the *earth's mass*

$$M = 5.973 \times 10^{24}\mathrm{kg} \quad , \tag{1–94}$$

and dividing M by the earth's volume

$$v = \frac{4\pi}{3}\,a^2 b = 1.0832 \times 10^{21}\mathrm{m}^3 \quad , \tag{1–95}$$

we find the *earth's mean density*

$$\rho_m = 5.514\,\mathrm{g/cm^3} \quad . \tag{1-96}$$

Approximate density laws. Recent estimates for the earth's principal moments of inertia (Burša, 1982; Burša and Pěč, 1988, p. 130) are

$$\begin{aligned} A &= (8.01 \pm 0.01) \times 10^{37}\mathrm{kg\,m^2} \doteq B \quad , \\ C &= (8.04 \pm 0.01) \times 10^{37}\mathrm{kg\,m^2} \quad . \end{aligned} \tag{1-97}$$

The mean moment of inertia of the earth, considered as a sphere, thus is

$$\bar{C} = \frac{1}{3}(2A + C) = 8.02 \times 10^{37}\mathrm{kg\,m^2} \quad . \tag{1-98}$$

From Chapter 2 we take the formula (2–141),

$$\bar{C} = \frac{8\pi}{3}\int_0^R \rho(r) r^4 dr \quad . \tag{1-99}$$

For a *homogeneous* earth ($\rho = \rho_m = $ const.) this becomes

$$C_0 = \frac{8\pi}{15}\rho_m R^5 \quad . \tag{1-100}$$

Defining the mean radius by

$$R = \sqrt[3]{a^2 b} \tag{1-101}$$

as the radius of the sphere of equal volume, we have in agreement with (1–94), (1–95), and (1–96)

$$M = \frac{4\pi}{3} R^3 \rho_m \quad , \tag{1-102}$$

so that (1–100) becomes

$$C_0 = \frac{2}{5} M R^2 = 9.70 \times 10^{37}\mathrm{kg\,m^2} \quad , \tag{1-103}$$

whence

$$\frac{\bar{C}}{C_0} = \frac{\bar{C}}{2MR^2/5} = 0.827 \quad . \tag{1-104}$$

Thus, globally speaking, any global density law must satisfy three basic conditions:

1. It must provide the correct total mass M (1–94) or, equivalently, the mean density ρ_m (1–96).

2. It must give the value $\bar{C}$ (1–98) for the mean moment of inertia or, equivalently, the ratio (1–104).

3. It must reproduce the density at the base of the continental layers, which may be taken as about 3.2 to 3.3 g/cm^3, e.g., the conventional density just below the Mohorovičić discontinuity much used in isostasy:

$$\rho_1 = 3.27\,\mathrm{g/cm^3} \quad . \tag{1–105}$$

Now the mass of the earth is given by the well-known formula, cf. also eq. (2–58) from Chapter 2:

$$M = 4\pi \int_0^R \rho(r) r^2 dr \quad . \tag{1–106}$$

Thus our three conditions say that any admissible density law $\rho(r)$ must satisfy (1–106), (1–99), and (1–105) for $r = R$ (neglecting the crust as regards density).

The simplest way to represent an overall smoothed density law is by means of a polynomial, e.g., with even powers only:

$$\rho(r) = a + b\left(\frac{r}{R}\right)^2 + c\left(\frac{r}{R}\right)^4 \quad . \tag{1–107}$$

In this case, the three conditions are

$$\begin{aligned} \frac{1}{3}a + \frac{1}{5}b + \frac{1}{7}c &= \frac{1}{3}\rho_m = 1.838\mathrm{g/cm^3} \quad , \\ \frac{1}{5}a + \frac{1}{7}b + \frac{1}{9}c &= \frac{0.827}{5}\rho_m = 0.912\mathrm{g/cm^3} \quad , \\ a + b + c &= \rho_1 \quad . \end{aligned} \tag{1–108}$$

Following this procedure, Bullard (1954, p. 89) has found the law (ρ in g/cm^3)

$$\rho = 12.19 - 16.71\left(\frac{r}{R}\right)^2 + 7.82\left(\frac{r}{R}\right)^4 \quad , \tag{1–109}$$

which represents "the general trend of possible density distributions" (his constants at the right-hand side of (1–108) are very slightly different than ours).

An older model is the *law of Roche*

$$\rho = a + b\left(\frac{r}{R}\right)^2 \quad , \tag{1–110}$$

which is no longer able to satisfy all three conditions (1–108). The model (Bullen, 1975, p. 73)

$$\rho = 10.53 - 8.35\left(\frac{r}{R}\right)^2 \quad (\mathrm{g/cm^3}) \quad , \tag{1–111}$$

which meets the first two conditions, gives the surface density $\rho_1 = 2.18\mathrm{g/cm^3}$ which is too low. Its importance, however, lies in the fact that it is the mathematically simplest two-parameter model and as such may still be useful. Fig. 1.8 illustrates these models.

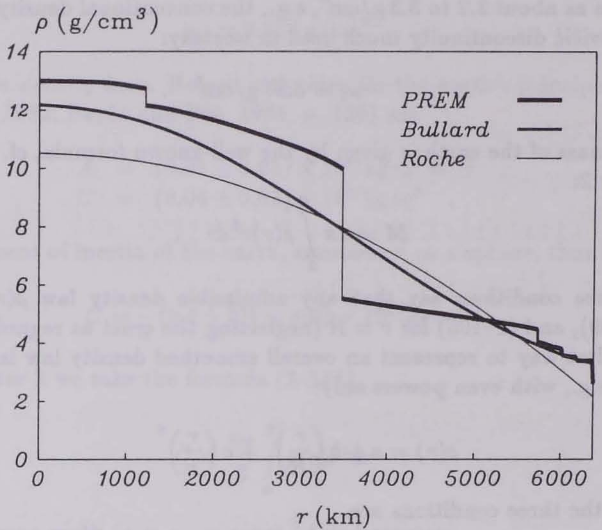

FIGURE 1.8: The PREM model and the approximate models by Bullard and Roche