which obviously is closely related to (1–46), ψ being the spherical distance between the points (θ, λ) and (θ', λ'):

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda) \quad . \tag{1–50}$$

A simple consequence of (1–49) is obtained by taking $f(\theta, \lambda) = Y_k(\theta, \lambda)$ with $k \neq n$:

$$\iint_\sigma Y_k(\theta', \lambda') P_n(\cos\psi) d\sigma = 0 \quad , \tag{1–51}$$

another important expression of orthogonality ($d\sigma = \sin\theta' d\theta' d\lambda'$ here).

Reciprocal distance. We finally mention the simple but fundamental spherical-harmonic development of $1/l$ occurring in equations such as (1–1) and (1–5). Consider two points P and P' in space, having spherical coordinates

$$P(r, \theta, \lambda) \quad \text{and} \quad P'(r', \theta', \lambda') \quad .$$

By applying the cosine theorem to the plane triangle OPP', O being the origin $r = 0$, we find for the spatial distance $l = PP'$:

$$l = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} \quad , \tag{1–52}$$

where ψ, the angle between the radius vectors $r = OP$ and $r' = OP'$, is again given by (1–50). The reciprocal distance may now be expanded into the series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{1–53}$$

which converges (uniformly in ψ) for $r' < r$ since

$$|P_n(\cos\psi)| \leq 1 \quad ;$$

it diverges for $r' > r$.

1.4 Elements of Ellipsoidal Geometry

For convenience and later reference we collect here some well-known (cf. Bomford, 1962, pp. 494–497; Heitz, 1988, pp. 99–105) and easily derivable formulas from ellipsoidal geometry.

Besides the semimajor axis a and semiminor axis b of the meridian ellipse (Fig. 1.1) we have already met the flattening

$$f = \frac{a - b}{a} \tag{1–54}$$

and the (first) excentricity e defined by

$$e^2 = \frac{a^2 - b^2}{a^2} \quad , \tag{1–55}$$

which are related by

$$e^2 = 2f - f^2 \quad . \tag{1–56}$$

After the second excentricity e' defined by

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{e^2}{1 - e^2} \tag{1–57}$$

is used.

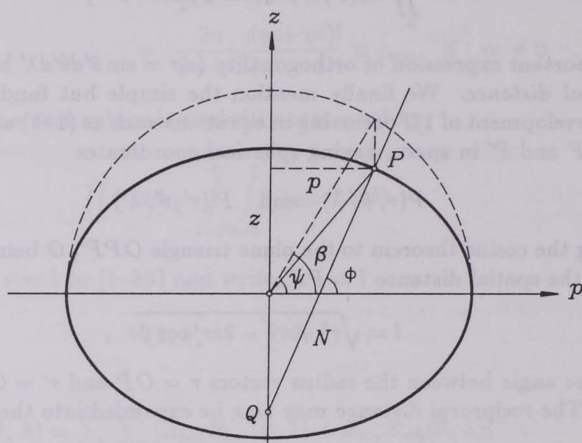

FIGURE 1.4: The meridian ellipse as parametrized by geographic latitude ϕ, geocentric latitude ψ or reduced latitude β

The meridian ellipse may be parametrized either by *geographic latitude* ϕ or by *reduced latitude* β or also by *geocentric latitude* ψ (Fig. 1.4). In coordinates p, z, we thus have

$$\begin{aligned} p &= a\cos\beta \quad , \\ z &= b\sin\beta \quad , \end{aligned} \tag{1–58}$$

or

$$\begin{aligned} p &= N\cos\phi \quad , \\ z &= \frac{b^2}{a^2}N\sin\phi \quad , \end{aligned} \tag{1–59}$$

or

$$\begin{aligned} p &= r\cos\psi \quad , \\ z &= r\sin\psi \quad , \end{aligned} \tag{1–60}$$

where r is determined so that the equation of the ellipse

$$\frac{p^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{1-61}$$

is satisfied, whence

$$r = \left(\frac{\cos^2 \psi}{a^2} + \frac{\sin^2 \psi}{b^2} \right)^{-\frac{1}{2}} \tag{1-62}$$

In (1–59), $N = PQ$ (Fig. 1.4) is given by

$$N = \frac{c}{(1 + e'^2 \cos^2 \phi)^{\frac{1}{2}}} \quad , \tag{1-63}$$

where

$$c = \frac{a^2}{b} \tag{1-64}$$

denotes the radius of curvature at the pole. Eqs. (1–58) satisfy (1–61) identically, and so do (1–59) with (1–63), and (1–60) with (1–62).

The radius of curvature of the meridian ellipse is

$$M = \frac{c}{(1 + e'^2 \cos^2 \phi)^{\frac{3}{2}}} \quad . \tag{1-65}$$

The geographic latitude ϕ, the geocentric latitude ψ and the reduced latitude β are related by

$$\tan \psi = \frac{b}{a} \tan \beta = \frac{b^2}{a^2} \tan \phi \quad , \tag{1-66}$$

which immediately follows from (1–58) to (1–60).

Turning now to the ellipsoid as a surface of revolution, the rectangular coordinates xyz of an ellipsoidal point P are, of course, given by

$$\begin{aligned} x &= p \cos \lambda \quad , \\ y &= p \sin \lambda \quad , \\ z &= z \quad , \end{aligned} \tag{1-67}$$

λ denoting the geographic longitude as in (1–26). It is thus identical to λ in Fig. 1.3, whereas θ in this figure is the complement of geocentric latitude:

$$\theta = 90° - \psi \quad . \tag{1-68}$$

The two principal radii of curvature of our ellipsoid are M and N as given by (1–65) and (1–63). This is obvious for the "meridian radius of curvature" M but less so for the "normal radius of curvature" N. The simple geometrical interpretation of N as the "normal terminated by the minor axis" (Bomford, 1962, p. 497) as shown in Fig. 1.4, by the way, holds for *all* surfaces of revolution. This is a simple consequence of Meusnier's theorem well known from differential geometry; cf. (Baeschlin, 1948,

p. 24). (There is no danger whatsoever to confuse the normal radius of curvature with the geoidal height, so that the same letter N may be used for both.) The mean curvature of the ellipsoid, corresponding to (1–20), thus is

$$J = \frac{1}{2}\left(\frac{1}{M} + \frac{1}{N}\right) \quad . \tag{1–69}$$

Series expansions. First we shall derive a truncated series for r as a function of θ, which will play a basic role in Chapter 4. From (1–62) and (1–68) we get

$$\frac{1}{r^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} \quad . \tag{1–70}$$

By (1–54) we have

$$b = a(1-f) \quad , \tag{1–71}$$

whence

$$\frac{a^2}{r^2} = \sin^2\theta + \frac{\cos^2\theta}{(1-f)^2}$$

or

$$\begin{aligned} \frac{a^2}{r^2}(1-f)^2 &= (1-f)^2\sin^2\theta + 1 - \sin^2\theta \\ &= 1 - (2f - f^2)\sin^2\theta \quad . \end{aligned}$$

Hence

$$r = a(1-f)\left[1 - (2f - f^2)\sin^2\theta\right]^{-\frac{1}{2}} \quad . \tag{1–72}$$

This formula is still rigorous. Now we expand the last factor as a binomial series, disregarding f^3 and higher powers:

$$r = a(1-f)\left[1 + (f - \frac{1}{2}f^2)\sin^2\theta + \frac{3}{2}f^2\sin^4\theta \cdots\right] \quad .$$

Carrying out the multiplication by $(1-f)$ and making some elementary transformations we get the desired formula

$$r = a(1 - f\cos^2\theta - \frac{3}{8}f^2\sin^2 2\theta) \quad , \tag{1–73}$$

which is accurate up to f^2.

The second truncated series expansion we shall need is for $\phi - \psi$, the difference between geographic and geocentric latitude. By (1–66) and (1–55) we have

$$\tan\psi = (1 - e^2)\tan\phi \quad . \tag{1–74}$$

On the left-hand side we put $\psi = \phi - (\phi - \psi)$ and expand by Taylor's theorem:

$$\tan\psi = \tan\left[\phi - (\phi - \psi)\right] = \tan\phi - \frac{\phi - \psi}{\cos^2\phi} + \cdots$$

The comparison with (1–74) shows that, to first order in f,

$$\phi - \psi = e^2 \cos\phi \sin\phi = 2f \cos\phi \sin\phi \quad , \tag{1–75}$$

neglecting f^2 in (1–56). To the same accuracy we may replace ψ by ϕ in (1–68), obtaining

$$\phi - \psi = 2f \cos\theta \sin\theta \quad , \tag{1–76}$$

accurate to order f.

1.5 Earth Models and Parameters

The Geodetic Reference System 1980. This system (GRS 1980) defines basic parameters for a globally best-fitting earth ellipsoid. It was adopted at the XVII General Assembly of the International Union of Geodesy and Geophysics (IUGG) in Canberra, December 1979, and still is the official reference system of the International Association of Geodesy (IAG) and is likely to remain so for the next years or even decades, since no significant changes have occurred or are to be expected in the near future.

The equipotential ellipsoid (1–21) and its external gravitational field are completely defined by four independent constants. The GRS 1980 takes

$$\begin{aligned} a &= 6378\,137\,\mathrm{m} \quad , \\ GM &= 3986\,005 \times 10^8 \mathrm{m}^3\mathrm{s}^{-2} \quad , \\ J_2 &= 108\,263 \times 10^{-8} \quad , \\ \omega &= 7292\,115 \times 10^{-11}\mathrm{s}^{-1} \end{aligned} \tag{1–77}$$

as the defining constants. The meanings of a (semimajor axis or equatorial radius, cf. Fig. 1.1), GM (geocentric gravitational constant including the atmosphere, cf. eq. (1–3)) and ω (angular velocity, cf. eq. (1–7)) are clear. Note that the product GM, rather than the earth's mass M, is given since the gravitational constant (1–2) is known only to 4 significant digits, whereas GM has 7 significant digits.

The "dynamical form factor" or zonal harmonic coefficient of degree 2 is defined by

$$J_2 = \frac{C - A}{Ma^2} \quad ; \tag{1–78}$$

it is dimensionless and, together with A (mean equatorial moment of inertia) and C (polar moment of inertia), has already been encountered in sec. 1.3; cf. (1–40).

Using the theory of the equipotential ellipsoid (cf. (Heiskanen and Moritz, 1967, pp. 64–79) and Chapter 5 of the present book), all other constants can be derived, e.g., flattening f, excentricity e, equatorial gravity γ_e, and ellipsoidal potential U_0:

$$f = 0.003\,352\,810\,681 = 1/298.257\,2221 \quad , \tag{1–79}$$

$$e^2 = 0.006\,694\,380\,023 \quad , \tag{1–80}$$

$$\gamma_e = 9.780\,326\,7715\,\mathrm{ms}^{-2} \quad , \tag{1–81}$$

$$U_0 = 6263\,686.0850 \times 10\,\mathrm{m}^2\mathrm{s}^{-2} \tag{1–82}$$