where

$$J = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1-20}$$

denotes the *mean curvature* of the level surface passing through the point under consideration, with R_1 and R_2 being its principal radii of curvature. Eq. (1–19) is a nontrivial consequence of (1–14); its derivation can be found in (Heiskanen and Moritz, 1967, pp. 51–53). It will play a basic role in Wavre's theory of equilibrium figures.

Normal and anomalous gravity field. Since the actual gravity field is mathematically rather complicated, it is usually referred to a *normal gravity field* of a simple analytical nature. In general, the *normal gravity potential* U is chosen in such a way that the reference ellipsoid is an equipotential surface for U:

$$U(x, y, z) = U_0 = \text{const.} \quad , \tag{1-21}$$

in the same way as the geoid is an equipotential surface for the actual gravity potential W:

$$W(x, y, z) = W_0 = \text{const.} \quad ; \tag{1-22}$$

we may assume $U_0 = W_0$. The normal potential U will be considered in detail in Chapter 5; here we only mention Somigliana's closed formula for normal gravity γ on the ellipsoid:

$$\gamma = \frac{a\gamma_e \cos^2\phi + b\gamma_p \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \quad , \tag{1-23}$$

where a and b are shown in Fig. 1.1, γ_e and γ_p denote normal gravity at equator and pole, respectively, and ϕ indicates geographical latitude on the ellipsoid (sec. 1.4).

The difference

$$T = W - U \tag{1-24}$$

for the same point is called *anomalous potential*, or *disturbing potential.* Denoting by N the height of the geoid (1–22) above the reference ellipsoid (1–21), we have the famous formula

$$N = \frac{T}{\gamma} \quad , \tag{1-25}$$

also due to Bruns, which is as elementary as it is intriguing, besides being extremely useful.

1.3 Spherical Harmonics

In this section we shall collect some well-known but very important formulas for spherical harmonics for later reference; the notations follow (Heiskanen and Moritz, 1967), sections 1–8 through 1–15, 2–5, and 2–9.

Spherical coordinates r (radius vector), θ (polar distance), and λ (longitude) are related to rectangular coordinates x, y, z by

$$\begin{aligned} x &= r\sin\theta\cos\lambda \quad , \\ y &= r\sin\theta\sin\lambda \quad , \\ z &= r\cos\theta \quad ; \end{aligned} \tag{1–26}$$

see Fig. 1.3.

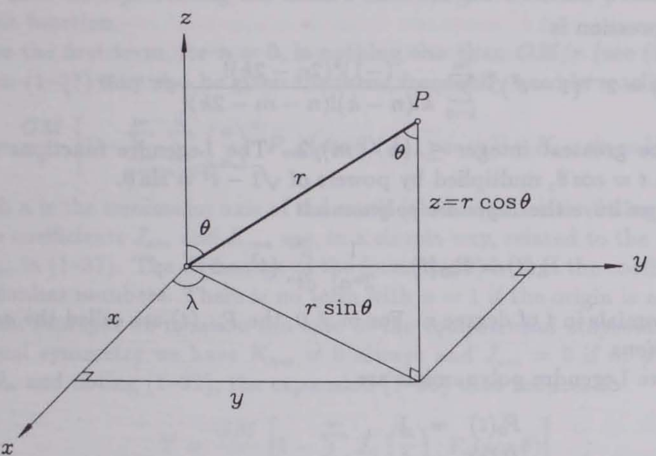

FIGURE 1.3: Spherical and rectangular coordinates

If we express Laplace's equation $\Delta V = 0$ in spherical coordinates and try to solve it by a product of three functions, each of which depends on only *one* spherical coordinate:

$$V = f(r)g(\theta)h(\lambda) \quad , \tag{1–27}$$

then the solutions are found to be

$$f(r) = r^n \quad \text{or} \quad f(r) = \frac{1}{r^{n+1}} \quad , \tag{1–28a}$$

$$g(\theta) = P_{nm}(\cos\theta) \quad , \tag{1–28b}$$

$$h(\lambda) = \cos m\lambda \quad \text{or} \quad h(\lambda) = \sin m\lambda \quad , \tag{1–28c}$$

where

$$\begin{aligned} n &= 0, 1, 2, 3, \ldots, \\ m &= 0, 1, \ldots, n \quad ; \end{aligned} \tag{1–29}$$

n is called the *degree*, and m the *order* of the functions under consideration. (There is a second solution for g which, however, will not be needed until much later, see sec. 5.1.)

Thus, the dependence on r and on λ is simple: $f(r)$ is a positive or negative power of r, and $h(\lambda)$ is a sine or cosine of multiples of λ.

The functions $P_{nm}(\cos\theta)$ are less elementary. They are called Legendre functions and defined by (we put $\cos\theta = t$):

$$P_{nm}(t) = \frac{1}{2^n n!}(1-t^2)^{\frac{m}{2}} \frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n \quad . \tag{1-30}$$

An explicit expression is

$$P_{nm}(t) = 2^{-n}(1-t^2)^{\frac{m}{2}} \sum_{k=0}^{n_0} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-m-2k)!} t^{n-m-2k} \quad , \tag{1-31}$$

where n_0 is the greatest integer $\leq (n-m)/2$. The Legendre functions are thus polynomials in $t = \cos\theta$, multiplied by powers of $\sqrt{1-t^2} = \sin\theta$.

For $m = 0$ we have the *Legendre polynomials*

$$P_n(t) = P_{n0}(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n \quad ; \tag{1-32}$$

they are polynomials in t of degree n. For $m \neq 0$, the $P_{nm}(t)$ are called the *associated Legendre functions.*

The first five Legendre polynomials are

$$\begin{aligned} P_0(t) &= 1 \quad , \\ P_1(t) &= t \quad , \\ P_2(t) &= \frac{3}{2}t^2 - \frac{1}{2} \quad , \\ P_3(t) &= \frac{5}{2}t^3 - \frac{3}{2}t \quad , \\ P_4(t) &= \frac{35}{8}t^4 - \frac{15}{4}t^2 + \frac{3}{8} \quad . \end{aligned} \tag{1-33}$$

The products of functions (1–28b) and (1–28c),

$$\begin{aligned} R_{nm}(\theta,\lambda) &= P_{nm}(\cos\theta)\cos m\lambda \quad , \\ S_{nm}(\theta,\lambda) &= P_{nm}(\cos\theta)\sin m\lambda \quad , \end{aligned} \tag{1-34}$$

are *Legendre surface harmonics*, and the products of (1–28a, b, c),

$$r^n R_{nm}(\theta,\lambda) \quad , \qquad r^n S_{nm}(\theta,\lambda) \quad , \tag{1-35a}$$

$$r^{-(n+1)} R_{nm}(\theta,\lambda) \quad , \qquad r^{-(n+1)} S_{nm}(\theta,\lambda) \quad , \tag{1-35b}$$

are the corresponding *solid spherical harmonics* ($m = 0$: *zonal*, $0 < m \leq n$: *tesseral* ($m = n$: *sectorial*)). The functions (1–35), as well as their (finite or convergent infinite) linear combinations, are harmonic.

In particular, the series

$$V(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[A_{nm} \frac{R_{nm}(\theta, \lambda)}{r^{n+1}} + B_{nm} \frac{S_{nm}(\theta, \lambda)}{r^{n+1}} \right] \tag{1-36}$$

or, equivalently,

$$V(r, \theta, \lambda) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} P_{nm}(\cos\theta)(A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) \quad , \tag{1-37}$$

may be used for representing the earth's external gravitational potential, which is a harmonic function.

Since the first term, for $n = 0$, is nothing else than GM/r (see (1–3)), the series (1–36) or (1–37) may also be given the form, frequently used in satellite applications:

$$V = \frac{GM}{r} \left[1 - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r} \right)^{n} P_{nm}(\cos\theta)(J_{nm} \cos m\lambda + K_{nm} \sin m\lambda) \right] \quad , \tag{1-38}$$

in which a is the semimajor axis of the earth (that is, of a best-fitting earth ellipsoid) and the coefficients J_{nm} and K_{nm} are, in a simple way, related to the coefficients A_{nm} and B_{nm} in (1–37). The advantage of the form (1–38) is that the coefficients are small dimensionless numbers. There is no term with $n = 1$ if the origin is at the geocenter.

As an example we mention the case of the *equipotential ellipsoid*. In view of the rotational symmetry we have $K_{nm} = 0$ always and $J_{nm} = 0$ if $m \neq 0$. On putting $J_{n0} = J_n$ and noting (1–32), the expansion (1–38) thus reduces to

$$V = \frac{GM}{r} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{a}{r} \right)^{n} P_n(\cos\theta) \right] \quad , \tag{1-39}$$

and the coefficients are given by

$$\begin{aligned} J_{2\nu} &= (-1)^{\nu+1} \frac{3e^{2\nu}}{(2\nu+1)(2\nu+3)} \left(1 - \nu + 5\nu \frac{C-A}{Ma^2e^2} \right) \quad , \\ J_{2\nu+1} &= 0 \quad , \end{aligned} \tag{1-40}$$

where $e^2 = (a^2 - b^2)/a^2$, M denotes the mass of the (normal) earth, A (equatorial) and C (polar) are its principal moments of inertia, and $\nu = 1, 2, 3, \ldots$ (Heiskanen and Moritz, 1967, p. 73).

Orthogonality relations. The integral over the unit sphere of the product of any two *different* functions R_{nm} or S_{nm} is zero:

$$\left.\begin{aligned} \iint_\sigma R_{nm}(\theta, \lambda) R_{sr}(\theta, \lambda) d\sigma &= 0 \\ \iint_\sigma S_{nm}(\theta, \lambda) S_{sr}(\theta, \lambda) d\sigma &= 0 \end{aligned}\right\} \quad \text{if} \quad s \neq n \quad \text{or} \quad r \neq m \quad \text{or both,}$$

$$\iint_\sigma R_{nm}(\theta, \lambda) S_{sr}(\theta, \lambda) d\sigma = 0 \qquad \text{in any case} \quad . \tag{1-41}$$

The symbol σ denotes the unit sphere $r = 1$, and $d\sigma$ its surface element. For the product of two *equal* functions we have

$$\begin{aligned} \iint_\sigma [R_{n0}(\theta, \lambda)]^2 \, d\sigma &= \frac{4\pi}{2n+1} \equiv \kappa_{n0} \quad , \\ \iint_\sigma [R_{nm}(\theta, \lambda)]^2 \, d\sigma &= \iint_\sigma [S_{nm}(\theta, \lambda)]^2 \, d\sigma = \\ &= \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!} \equiv \kappa_{nm} \quad \text{if} \quad m \neq 0 \quad . \end{aligned} \tag{1–42}$$

The integral over the unit sphere is explicitly expressed by

$$\iint_\sigma (\cdot) d\sigma = \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} (\cdot) \sin\theta d\theta d\lambda \quad . \tag{1–43}$$

Let us now put $r = 1$ in (1–36) and write

$$V(1, \theta, \lambda) = f(\theta, \lambda) \quad , \tag{1–44}$$

so that

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [A_{nm} R_{nm}(\theta, \lambda) + B_{nm} S_{nm}(\theta, \lambda)] \quad . \tag{1–45}$$

We multiply $f(\theta, \lambda)$ by $R_{nm}(\theta, \lambda)$ or $S_{nm}(\theta, \lambda)$ and integrate over the unit sphere, taking into account (1–41) and (1–42). This determines the coefficients as

$$\begin{aligned} A_{nm} &= \frac{1}{\kappa_{nm}} \iint_\sigma f(\theta, \lambda) R_{nm}(\theta, \lambda) d\sigma \quad , \\ B_{nm} &= \frac{1}{\kappa_{nm}} \iint_\sigma f(\theta, \lambda) S_{nm}(\theta, \lambda) d\sigma \quad . \end{aligned} \tag{1–46}$$

Finally we introduce the *Laplace surface harmonics* $Y_n(\theta, \lambda)$ of $f(\theta, \lambda)$, defined by

$$Y_n(\theta, \lambda) = \sum_{m=0}^{n} [A_{nm} R_{nm}(\theta, \lambda) + B_{nm} S_{nm}(\theta, \lambda)] \quad , \tag{1–47}$$

and write

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} Y_n(\theta, \lambda) \quad . \tag{1–48}$$

Then the Laplace harmonic of degree n is given by the expression

$$Y_n(\theta, \lambda) = \frac{2n+1}{4\pi} \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} f(\theta', \lambda') P_n(\cos\psi) \sin\theta' d\theta' d\lambda' \quad , \tag{1–49}$$

which obviously is closely related to (1–46), ψ being the spherical distance between the points (θ, λ) and (θ', λ'):

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda) \quad . \tag{1–50}$$

A simple consequence of (1–49) is obtained by taking $f(\theta, \lambda) = Y_k(\theta, \lambda)$ with $k \neq n$:

$$\iint_\sigma Y_k(\theta', \lambda') P_n(\cos\psi) d\sigma = 0 \quad , \tag{1–51}$$

another important expression of orthogonality ($d\sigma = \sin\theta' d\theta' d\lambda'$ here).

Reciprocal distance. We finally mention the simple but fundamental spherical-harmonic development of $1/l$ occurring in equations such as (1–1) and (1–5). Consider two points P and P' in space, having spherical coordinates

$$P(r, \theta, \lambda) \quad \text{and} \quad P'(r', \theta', \lambda') \quad .$$

By applying the cosine theorem to the plane triangle OPP', O being the origin $r = 0$, we find for the spatial distance $l = PP'$:

$$l = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} \quad , \tag{1–52}$$

where ψ, the angle between the radius vectors $r = OP$ and $r' = OP'$, is again given by (1–50). The reciprocal distance may now be expanded into the series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{1–53}$$

which converges (uniformly in ψ) for $r' < r$ since

$$|P_n(\cos\psi)| \leq 1 \quad ;$$

it diverges for $r' > r$.

1.4 Elements of Ellipsoidal Geometry

For convenience and later reference we collect here some well-known (cf. Bomford, 1962, pp. 494–497; Heitz, 1988, pp. 99–105) and easily derivable formulas from ellipsoidal geometry.

Besides the semimajor axis a and semiminor axis b of the meridian ellipse (Fig. 1.1) we have already met the flattening

$$f = \frac{a - b}{a} \tag{1–54}$$

and the (first) excentricity e defined by

$$e^2 = \frac{a^2 - b^2}{a^2} \quad , \tag{1–55}$$