a modern Geodetic Reference System (Marussi et al., 1974).

The combination of satellite and terrestrial data has provided us with global models for the external anomalous gravitational potential (cf. Rapp, 1986) of considerable accuracy and resolution which call for an interpretation in terms of mass anomalies in the earth's mantle. This forces geodesists (though somewhat reluctantly) to go to the difficult and treacherous field of gravimetric inverse problems.

The principal problem in applying Molodensky's theory to mountainous areas is that this theory requires gravity or other data to be given *continuously* on the earth's surface. Real measurements, however, are made at *discrete* points only, and thus we have to interpolate in between. An indispensable tool for interpolation in mountain areas and for other geodetic purposes is *isostatic reduction* which, after having played a fundamental role in the first half of the present century, has somewhat been relegated to the background afterwards. Now isostasy witnesses a revival and it seems appropriate to reconsider it, especially since modern computers also permit the use of more sophisticated and more realistic models.

The aim of these historical remarks has only been to motivate a book of the present type. Anyone interested in geodetic history as such will find ample material in books from (Todhunter, 1873) to (Levallois, 1988).

1.2 Elements of Gravitation and Gravity

A basic background of elementary physical geodesy, corresponding perhaps to the first three chapters of (Heiskanen and Moritz, 1967), will facilitate reading this book. In order to make it as self-contained as possible, however, we shall here collect some elementary facts on potential theory and physical geodesy. Proofs and more details can be found, e.g., in (Sigl, 1985) and (Heiskanen and Moritz, 1967). Advanced aspects such as treated in (Moritz, 1980) will not be needed (with a few exceptions, cf. secs. 4.1.5, 7.5, and 8.3.2).

First we introduce a fundamental earth-fixed rectangular coordinate system xyz defined in the usual way: the origin is at the earth's center of mass (the *geocenter*), the z-axis coincides with the mean axis of rotation, the x-axis lies in the mean Greenwich meridian plane and is normal to the z-axis; the y-axis is normal to the xz-plane and directed so that the xyz system is right-handed; the xy-plane is thus the (mean) equatorial plane.

One uses a mean axis of rotation and a mean Greenwich meridian plane in order to get a definition independent of time, in view of very small and more or less periodic changes in the instantaneous rotation axis and of deformations of the earth's body (the interested reader might consult (Moritz and Mueller, 1987)).

The *gravitational potential* of the earth may be expressed by the *Newtonian integral*

$$V(P) = V(x, y, z) = G \iiint_v \frac{dm}{l} = G \iiint_v \frac{\rho}{l}\, dv \quad , \tag{1–1}$$

where (Fig. 1.2) $P(x, y, z)$ denotes the point at which V is calculated, Q is the

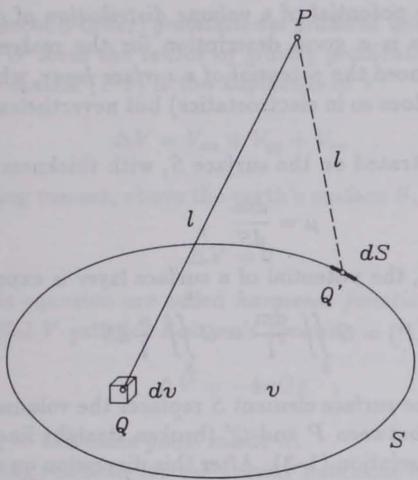

FIGURE 1.2: Illustrating eqs. (1–1) and (1–5)

point, variable within the earth's body, which forms the center of the mass element dm or the volume element dv, l is the distance between P and Q (solid straight line), and $\rho = \rho(Q) = dm/dv$ is the mass density at Q. G is the Newtonian gravitational constant

$$G = 6.673 \times 10^{-11}\,\mathrm{m^3 s^{-2} kg^{-1}} \quad . \tag{1–2}$$

The integral is to be extended over the whole earth's body v, which includes the solid and liquid parts. The (very small) effect of the atmosphere is usually disregarded; if necessary, it can be taken into account by corrections, which have the relative order of 10^{-6}. The same treatment may be applied to temporal variations of V (due to earth tides, etc.) which have the order of 10^{-7}. Atmospheric effects and temporal variations will consistently be disregarded here.

Even so, the representation (1–1) has only theoretical importance because its practical use would require the knowledge of the detailed density distribution within the earth, which obviously is not known.

For large distances

$$r = \sqrt{x^2 + y^2 + z^2} \quad ,$$

(1–1) may be expressed as

$$V = \frac{GM}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty \quad , \tag{1–3}$$

M denoting the total mass of the body and $O(1/r^2)$ (read $O(\epsilon)$ as "term(s) of order ϵ") symbolizing a term that, for $r \to \infty$, tends to zero as $1/r^2$. The physical sense of this equation is that, at large distances and approximately, any body acts gravitationally as a point mass.

Eq. (1–1) represents the potential of a *volume distribution* of density ρ, assumed piecewise continuous. This is a good description for the real earth. For certain computations we shall also need the *potential of a surface layer*, which does not occur naturally in gravitation (it does so in electrostatics) but nevertheless serves as a useful mathematical fiction.

The mass is now concentrated on the surface S, with thickness zero and density

$$\mu = \frac{dm}{dS} \quad , \tag{1–4}$$

so that, in analogy to (1–1), the potential of a surface layer is expressed by

$$V_S(P) = G \iint\limits_S \frac{dm}{l} = G \iint\limits_S \frac{\mu}{l}\, dS \quad , \tag{1–5}$$

where now $\mu = \mu(Q')$ and the surface element S replaces the volume element dv; now, of course, l is the distance between P and Q' (broken straight line in Fig. 1.2). For $r \to \infty$, V_S also satisfies the relation (1–3). After this digression on surface potentials, we return to the case of the real earth, eq. (1–1).

The *gravity potential* W is the sum of V and the potential of the centrifugal force,

$$\Phi = \frac{1}{2}\,\omega^2(x^2 + y^2) \quad , \tag{1–6}$$

so that

$$W(x,\, y,\, z) = V(x,\, y,\, z) + \frac{1}{2}\,\omega^2(x^2 + y^2) \quad , \tag{1–7}$$

ω being the angular velocity of the earth's rotation (which is considered constant).

The field of potential V is called the *gravitational field*; the field of potential W is the *gravity field*.

The *gravity vector* $\mathbf{g}$ is the gradient of W:

$$\mathbf{g} = \operatorname{grad} W = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} \quad ; \tag{1–8}$$

its components are the partial derivatives of W with respect to x, y, z. It is the resultant of the gravitational force grad$V$ and the centrifugal force grad$\Phi = [\omega^2 x,\, \omega^2 y,\, 0]$ (we do not notationally distinguish between row and column vectors and use boldface only for "geometric" vectors such as for the gravity vector $\mathbf{g}$ and the coordinate vector $\mathbf{x}$).

The potentials V and W, as well as the gravity vector $\mathbf{g}$, are continuous throughout the whole three-dimensional space. This no longer holds for second derivatives; cf. eqs. (1–13) and (1–14) below, which show a discontinuity at S.

The second-order partial derivatives of V form a symmetric matrix

$$\begin{bmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{bmatrix} \quad , \tag{1–9}$$

which is called the (second-order) *gravitational gradient tensor*. Similarly, the second-order derivatives of W form the *tensor of gravity gradients*.

The trace of the matrix (1–9) is the *Laplacian* of V:

$$\Delta V = V_{xx} + V_{yy} + V_{zz} \quad . \tag{1-10}$$

Outside the attracting masses, above the earth's surface S, V satisfies *Laplace's equation*

$$\Delta V = 0 \quad . \tag{1-11}$$

The solutions of this equation are called *harmonic functions*. In the earth's interior, inside S, the potential V satisfies *Poisson's equation*

$$\Delta V = -4\pi G\rho \quad , \tag{1-12}$$

ΔV and ρ referring to the same point inside S.

The corresponding relations for the gravity potential W are, in view of (1–6),

$$\Delta W = 2\omega^2 \quad \text{outside} \quad S \quad , \tag{1-13}$$

$$\Delta W = -4\pi G\rho + 2\omega^2 \quad \text{inside} \quad S \quad . \tag{1-14}$$

The magnitude, or norm, of the gravity vector $\mathbf{g}$ is *gravity* g

$$g = ||\mathbf{g}|| \quad ; \tag{1-15}$$

the direction of $\mathbf{g}$, expressed by the unit vector

$$\mathbf{n} = -g^{-1}\mathbf{g} \quad , \tag{1-16}$$

is the direction of the vertical, or plumb line; we have chosen the minus sign so that $\mathbf{n}$ points upwards.

As we have already mentioned in sec. 1.1, the surfaces

$$W(x, y, z) = \text{const.} \tag{1-17}$$

are the *level surfaces*, or *equipotential surfaces*. It is easily shown that the plumb line vector $\mathbf{n}$ is everywhere perpendicular to these surfaces.

Denoting by $\partial/\partial n$ the derivative along this vector $\mathbf{n}$, we readily get from (1–8) and (1–15)

$$g = -\frac{\partial W}{\partial n} \quad , \tag{1-18}$$

the minus sign being in agreement with (1–16).

For the derivative $\partial g/\partial n$ we have *Bruns' formula*

$$\frac{\partial g}{\partial n} = -2gJ + 4\pi G\rho - 2\omega^2 \quad , \tag{1-19}$$

where

$$J = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1–20}$$

denotes the *mean curvature* of the level surface passing through the point under consideration, with R_1 and R_2 being its principal radii of curvature. Eq. (1–19) is a nontrivial consequence of (1–14); its derivation can be found in (Heiskanen and Moritz, 1967, pp. 51–53). It will play a basic role in Wavre's theory of equilibrium figures.

Normal and anomalous gravity field. Since the actual gravity field is mathematically rather complicated, it is usually referred to a *normal gravity field* of a simple analytical nature. In general, the *normal gravity potential* U is chosen in such a way that the reference ellipsoid is an equipotential surface for U:

$$U(x,\, y,\, z) = U_0 = \text{const.} \quad , \tag{1–21}$$

in the same way as the geoid is an equipotential surface for the actual gravity potential W:

$$W(x,\, y,\, z) = W_0 = \text{const.} \quad ; \tag{1–22}$$

we may assume $U_0 = W_0$. The normal potential U will be considered in detail in Chapter 5; here we only mention Somigliana's closed formula for normal gravity γ on the ellipsoid:

$$\gamma = \frac{a\gamma_e \cos^2\phi + b\gamma_p \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \quad , \tag{1–23}$$

where a and b are shown in Fig. 1.1, γ_e and γ_p denote normal gravity at equator and pole, respectively, and ϕ indicates geographical latitude on the ellipsoid (sec. 1.4).

The difference

$$T = W - U \tag{1–24}$$

for the same point is called *anomalous potential*, or *disturbing potential.* Denoting by N the height of the geoid (1–22) above the reference ellipsoid (1–21), we have the famous formula

$$N = \frac{T}{\gamma} \quad , \tag{1–25}$$

also due to Bruns, which is as elementary as it is intriguing, besides being extremely useful.

1.3 Spherical Harmonics

In this section we shall collect some well-known but very important formulas for spherical harmonics for later reference; the notations follow (Heiskanen and Moritz, 1967), sections 1–8 through 1–15, 2–5, and 2–9.