Chapter 1

Background Information

1.1 Conceptual and Historical Background

There are basically three different possible definitions of "figure of the earth":

A) The solid and liquid earth bounded by the *physical earth's surface*, or *topographic surface*, which is the surface which we see, on which we stand, walk, drive, and, occasionally, swim. It is highly irregular, even after some obvious smoothing which is always necessary to make it a smooth surface amenable to mathematical treatment, and also after some averaging with respect to time since this surface undergoes temporal variations (on the order of decimeters or more) because of tidal effects, etc.

B) The (part of the earth bounded by the) *geoid*, which is a level surface coinciding (somewhat loosely speaking) with the free surface of the oceans together with its continuation under the continents. It is the geoid above which "heights above sea level" are measured. A level surface is everywhere horizontal, that is, perpendicular to the direction of the plumb line. Level surfaces are surfaces of constant gravity potential W (sec. 1.2), W =const., and the geoid is one of them, $W = W_0$, denoting the constant geoid potential by W_0. Again we are disregarding temporal (tidal) variations. Whereas the physical earth's surface, in its picturesque variety and beauty, is very irregular, the geoid is smoother and subject to a mathematical equation, $W = W_0$; however, even the gravity potential W is far from being a simple mathematical function. Therefore, the geoid is referred to a more regular, "normal", surface which approximates the geoid while being more regular in a mathematical or physical sense. Thus we arrive at the concept of a

C) *Normal earth*, or *reference earth*, or *earth model*. Mathematically the simplest model is an *ellipsoid* of revolution, which therefore is practically almost exclusively used. Physically the best reference for describing the small, more or less elastic, temporal variations (free and forced oscillations such as earth tides), is a hydrostatic *equilibrium figure*. Figures of hydrostatic equilibrium for the earth are very close to ellipsoids, but do not exactly coincide with an ellipsoid as we shall have ample opportunity to see in this book. By the way, we are frequently not distinguishing between a figure and the surface bounding it; this is customary and should not cause any confusion.

Classical mechanics, as created by the genius of Isaac Newton (1642–1727) and by many brilliant successors, provided the base for a scientific study of the earth's figure. Around 1735, the Paris Academy of Sciences decided to send two grade measuring expeditions to Peru (Bouguer, La Condamine, Godin) and to Lapland (Maupertuis, Clairaut) to determine experimentally the flattening of the earth's ellipsoid. In 1743, Clairaut (1713–1765) published his famous differential equation for the flattening of the earth as a function of depth, based on hydrostatic equilibrium, which has retained its fundamental validity to the present day. The well-known definition of the flattening f is illustrated by Fig. 1.1.

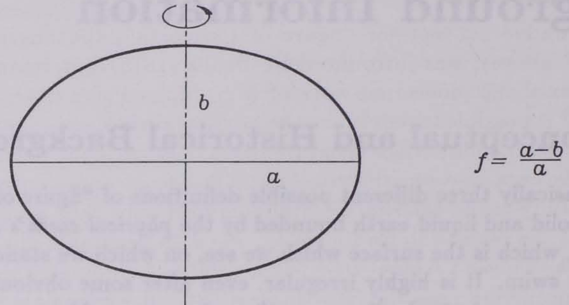

FIGURE 1.1: The earth ellipsoid as an ellipsoid of revolution with semimajor axis a, semiminor axis b, and flattening f

With the measuring accuracies available at that time, it was not necessary to distinguish between an ellipsoid and a figure of equilibrium, or even the geoid; the classical term "figure of the earth" was completely obvious to this approximation and was understood to denote some kind of normal figure.

It was only Carl Friedrich Gauss (1777–1855) who recognized the necessity for a precise definition of the geoid as a level surface. He called it "geometrical earth surface" (1828); the present term "geoid" was introduced by J.B. Listing in 1872. It is well known that the geoid deviates from a suitable mean ellipsoid by less than 100 meters.

A terrestrial equilibrium figure differs from an ellipsoid even much less, by a few meters, but even this is well within the accuracy limits of contemporary measurements.

In the 19th century, however, such petty differences had no practical significance whatsoever, and even the geoid was a theoretical concept rather than a practical reality. The main emphasis was on the determination of a global terrestrial ellipsoid (semimajor axis a and flattening f), and the theory of hydrostatic equilibrium was considered one of the best, if not the best, means for determining f. This situation lasted essentially until the advent of artificial satellites in 1957.

In 1909, J.F. Hayford obtained by a combination of triangulation and astronomical measurements with isostatic reduction (cf. Chapter 8) the following parameters

$$a = 6378\,388\,\mathrm{m}\,, \qquad f = 1/297 \quad .$$

In 1924, these parameters were adopted by the International Association of Geodesy as the International Ellipsoid; they were considered the best available for another twenty years.

Hayford's geometrical value of the flattening was confirmed by a very precise hydrostatic computation of Bullard (1948), who obtained 1/297.34, and thus a value of f around 1/297 was considered almost definitive until the advent of artificial satellites.

At any rate, for two hundred years beginning with Clairaut (1743), the theory of hydrostatic equilibrium figures had been considered a central topic of geodesy, and, with a few exceptions, "figure of the earth" was almost synonymous with "equilibrium figure".

The alternative concept for "figure of the earth", the physical earth's surface (concept "A" above), was introduced by Bruns (1878) but remained rather in the background until the pioneering work of M.S. Molodensky since 1945. Molodensky showed that it is possible to determine the physical surface of the earth and its external gravitational field from surface measurements only, without needing a reduction to the geoid. He even rejected the concept of the geoid and wanted geodesy to limit itself to the study of the *external* gravitational field only, thus avoiding problems with the density of the masses above the geoid, which in general is unknown *a priori*.

Molodensky's ideas became extremely fruitful and dominated the thinking of physical geodesy until very recently; cf. (Moritz, 1980). It was mainly the conceptual and theoretical interest of Molodensky's approach which made it so attractive, although few geodesists were ready to follow him into abandoning the geoid altogether; cf. (Heiskanen and Moritz, 1967, sec. 8–13). In fact, the number of geoids produced during the last two or three decades is hardly countable ...

At any rate, present geodetic interest in equilibrium figures is rather low indeed. The last major textbook in physical geodesy that extensively considers them was (Ledersteger, 1969), whereas (Moritz, 1980) does not even mention them.

Besides the modern, Molodensky-oriented, approach to physical geodesy, there is another major reason for this. One of the first results of satellite geodesy, already around 1960, was a value of the earth's flattening (about 1/298.25), which was established beyond doubt and seems to be incompatible with hydrostatic equilibrium, which has been seen to lead to a value of about 1/297. Recent studies (cf. Denis, 1989), however, indicate that the last word may not have been spoken in this matter yet.

Now we seem again to witness a turn of the tide. The increasing interactions between geodesy and geophysics force geodesy, so to speak, to go below the earth's surface again. The theory of earth rotation and earth tides are considered, at least to a considerable extent, to belong to geodesy, and a hydrostatic earth model serves as a natural reference for the theory of a rotating and oscillating earth (cf. Melchior, 1983, sec. 6.2; Moritz and Mueller, 1987, sec. 4.2).

Today, finally, we know the earth ellipsoid to an accuracy of about a meter (Geodetic Reference System 1980, etc.), and in constructing, on this basis, a standard earth model also for the earth's interior, we have to consider *interior* level surfaces and mass distributions for an equipotential ellipsoid which underlies the definition of a modern Geodetic Reference System (Marussi et al., 1974).

The combination of satellite and terrestrial data has provided us with global models for the external anomalous gravitational potential (cf. Rapp, 1986) of considerable accuracy and resolution which call for an interpretation in terms of mass anomalies in the earth's mantle. This forces geodesists (though somewhat reluctantly) to go to the difficult and treacherous field of gravimetric inverse problems.

The principal problem in applying Molodensky's theory to mountainous areas is that this theory requires gravity or other data to be given *continuously* on the earth's surface. Real measurements, however, are made at *discrete* points only, and thus we have to interpolate in between. An indispensable tool for interpolation in mountain areas and for other geodetic purposes is *isostatic reduction* which, after having played a fundamental role in the first half of the present century, has somewhat been relegated to the background afterwards. Now isostasy witnesses a revival and it seems appropriate to reconsider it, especially since modern computers also permit the use of more sophisticated and more realistic models.

The aim of these historical remarks has only been to motivate a book of the present type. Anyone interested in geodetic history as such will find ample material in books from (Todhunter, 1873) to (Levallois, 1988).

1.2 Elements of Gravitation and Gravity

A basic background of elementary physical geodesy, corresponding perhaps to the first three chapters of (Heiskanen and Moritz, 1967), will facilitate reading this book. In order to make it as self-contained as possible, however, we shall here collect some elementary facts on potential theory and physical geodesy. Proofs and more details can be found, e.g., in (Sigl, 1985) and (Heiskanen and Moritz, 1967). Advanced aspects such as treated in (Moritz, 1980) will not be needed (with a few exceptions, cf. secs. 4.1.5, 7.5, and 8.3.2).

First we introduce a fundamental earth-fixed rectangular coordinate system xyz defined in the usual way: the origin is at the earth's center of mass (the *geocenter*), the z-axis coincides with the mean axis of rotation, the x-axis lies in the mean Greenwich meridian plane and is normal to the z-axis; the y-axis is normal to the xz-plane and directed so that the xyz system is right-handed; the xy-plane is thus the (mean) equatorial plane.

One uses a mean axis of rotation and a mean Greenwich meridian plane in order to get a definition independent of time, in view of very small and more or less periodic changes in the instantaneous rotation axis and of deformations of the earth's body (the interested reader might consult (Moritz and Mueller, 1987)).

The *gravitational potential* of the earth may be expressed by the *Newtonian integral*

$$V(P) = V(x, y, z) = G \iiint_v \frac{dm}{l} = G \iiint_v \frac{\rho}{l}\, dv \quad , \tag{1-1}$$

where (Fig. 1.2) $P(x, y, z)$ denotes the point at which V is calculated, Q is the