XXVII.

Fragmente über die Grenzfälle der elliptischen Modulfunctionen.

I.

Additamentum ad $\S^{\text{um}}$ 40.

[Fundamenta nova theoriae functionum ellipticarum.]

Formulae in hoc $\S^{\circ}$ propositae in eo casu, ubi modulus ipsius q unitatem aequat, consideratione satis dignae videntur, quippe quae functiones unius variabilis pro quovis argumenti valore discontinuas praebeant.

Series quidem propositae magna ex parte pro modulo ipsius q unitati aequale non convergunt, sed integrando series convergentes inde derivari possunt; itaque primo integralia formularum 1—7 proponamus

$$(48)\quad \int_0 (\log k - \log 4\sqrt{q})\,\frac{dq}{q} = -4\log(1+q) + \frac{4}{4}\log(1+q^2) - \frac{4}{9}\log(1+q^3) + \frac{4}{16}\log(1+q^4) - \cdots$$

$$(49)\quad \int_0 -\log k'\,\frac{dq}{q} = 4\log\frac{1+q}{1-q} + \frac{4}{9}\log\frac{1+q^3}{1-q^3} + \frac{4}{25}\log\frac{1+q^5}{1-q^5} + \cdots$$

$$(50)\quad \int_0 \log\frac{2K}{\pi}\,\frac{dq}{q} = 4\log(1+q) + \frac{4}{9}\log(1+q^3) + \frac{4}{25}\log(1+q^5) + \cdots$$

$$(51)\quad \int_0 \left(\frac{2K}{\pi} - 1\right)\frac{dq}{q} = -4\log(1-q) + \frac{4}{3}\log(1-q^3) - \frac{4}{5}\log(1-q^5) + \cdots$$
$$= +2i\log\frac{1-qi}{1+qi} + \frac{2i}{2}\log\frac{1-q^2i}{1+q^2i} + \frac{2i}{3}\log\frac{1-q^3i}{1+q^3i} + \cdots$$

$$(52)\quad \int_0 \frac{2kK}{\pi}\,\frac{dq}{q} = 4\log\frac{1+\sqrt{q}}{1-\sqrt{q}} - \frac{4}{3}\log\frac{1+\sqrt{q^3}}{1-\sqrt{q^3}} + \frac{4}{5}\log\frac{1+\sqrt{q^5}}{1-\sqrt{q^5}} + \cdots$$
$$= 4i\log\frac{1-\sqrt{qi}}{1+\sqrt{qi}} + \frac{4i}{3}\log\frac{1-\sqrt{q^3i}}{1+\sqrt{q^3i}} + \frac{4i}{5}\log\frac{1-\sqrt{q^5i}}{1+\sqrt{q^5i}} + \cdots$$

$$(53)\quad \int_0 \left(\frac{2k'K}{\pi} - 1\right)\frac{dq}{q} = -4\log(1+q) + \frac{4}{3}\log(1+q^3) - \frac{4}{5}\log(1+q^5) + \cdots$$
$$= -2i\log\frac{1-qi}{1+qi} + \frac{2i}{2}\log\frac{1-q^2i}{1+q^2i} - \frac{2i}{3}\log\frac{1-q^3i}{1+q^3i} + \cdots$$

$$(54)\quad \int_0 \left(\frac{2\sqrt{k'}K}{\pi}-1\right)\frac{dq}{q} = -\tfrac{4}{2}\log(1+q^2)+\tfrac{4}{6}\log(1+q^6)$$

$$-\tfrac{4}{10}\log(1+q^{10})+\tfrac{4}{14}\log(1+q^{14})-\cdots$$

$$= -\frac{2i}{2}\log\frac{1-q^2i}{1+q^2i}+\frac{2i}{4}\log\frac{1-q^4i}{1+q^4i}$$

$$-\frac{2i}{6}\log\frac{1-q^6i}{1+q^6i}+\frac{2i}{8}\log\frac{1-q^8i}{1+q^8i}-\cdots$$

ubi logarithmos ita sumendos esse manifestum est, ut evanescant posito $q=0$.

Functiones eaedem ad dignitates ipsius q evolutae adhibitis Cl[i] Jacobi denotationibus hoc modo repraesentantur

$$(55)\quad \int_0 (\log k - \log 4\sqrt{q})\frac{dq}{q} = -4\sum\frac{\varphi(p)}{p^2}\Big(q^p-\frac{3q^{2p}}{4}-\frac{3}{16}q^{4p}$$

$$-\frac{3}{64}q^{8p}-\frac{3}{256}q^{16p}-\ldots\Big)$$

$$(56)\quad \int_0 -\log k'\frac{dq}{q} = 8\sum\frac{\varphi(p)}{p^2}q^p$$

$$(57)\quad \int_0 \log\frac{2K}{\pi}\frac{dq}{q} = 4\sum\frac{\varphi(p)}{p^2}\Big(q^p-\frac{1}{2}q^{2p}-\frac{1}{4}q^{4p}-\frac{1}{8}q^{8p}-\frac{1}{16}q^{16p}-\ldots\Big)$$

$$(58)\quad \int_0 \left(\frac{2K}{\pi}-1\right)\frac{dq}{q} = 4\sum\frac{\psi(n)q^{2^l(4m-1)^2n}}{2^l(4m-1)^2n}$$

$$(59)\quad \int_0 \frac{2kK}{\pi}\frac{dq}{q} = 8\sum\frac{\psi(n)q^{\frac{(4m-1)^2n}{2}}}{(4m-1)^2n}$$

$$(60)\quad \int_0 \left(\frac{2k'K}{\pi}-1\right)\frac{dq}{q} = -4\sum\frac{\psi(n)q^{(4m-1)^2n}}{(4m-1)^2n}$$

$$+4\sum\frac{\psi(n)q^{2^{l+1}(4m-1)^2n}}{2^{l+1}(4m-1)^2n}$$

$$(61)\quad \int_0 \left(\frac{2\sqrt{k'}K}{\pi}-1\right)\frac{dq}{q} = -4\sum\frac{\psi(n)q^{2(4m-1)^2n}}{2(4m-1)^2n}$$

$$+4\sum\frac{\psi(n)q^{2^{l+2}(4m-1)^2n}}{2^{l+2}(4m-1)^2n}.$$

Accuratiori functionum propositarum disquisitioni tanquam lemma antemittimus theorema sequens generale.

Si series

$$a_0+a_1+a_2+\cdots$$

eo quo scripsimus ordine summata summam habet convergentem, functio ipsius r hac serie

$$a_0 + a_1 r + a_2 r^2 + \cdots$$

expressa, convergente r versus limitem 1, convergit versus valorem eundem.

Hinc facile deducitur

Si functio $f(q)$ complexae quantitatis q pro modulis ipsius q unitate minoribus exhibeatur per seriem

$$a_0 + a_1 q + a_2 q^2 + \cdots$$

hanc seriem pro valore q_0 cujus modulus sit unitas, si habeat summam, exprimere valorem eum, quem functio $f(q)$ nanciscatur convergente q versus q_0 ita, ut modulus tantum mutetur, i. e. secundum notam repraesentationem geometricam, appropinquante puncto, per quod quantitas q repraesentatur, in linea ad limitem spatii, pro quo functio est data, normali.

Quamobrem hos tantum valores functionum propositarum hic respicimus, etiamsi evolutiones 48—54 latius pateant.

Sit brevitatis gratia (x) aut absolute minima quantitatum a quantitate x numero integro distantium, aut, si x ex numero integro et fractione $\frac{1}{2}$ composita est, $= 0$, porro $E(x)$ numerus integer maximus non major quam x: obtinemus e 48, attribuendo ipsi q valorem $q_0 = e^{xi}$

$$(62)\quad \int_0^{e^{xi}} (\log k - \log 4\sqrt{q})\frac{dq}{q}$$

$$= -2\log 4 \cos\frac{x^2}{2} + \frac{2}{4}\log 4\cos\frac{2x^2}{2} - \frac{2}{9}\log 4 \cos\frac{3x^2}{2}$$

$$+ \frac{2}{16}\log 4 \cos\frac{4x^2}{2} - \cdots$$

$$-4\pi i\left(\frac{x}{2\pi}\right) + \frac{4\pi i}{4}\left(\frac{2x}{2\pi}\right) - \frac{4\pi i}{9}\left(\frac{3x}{2\pi}\right) + \frac{4\pi i}{16}\left(\frac{4x}{2\pi}\right) - \cdots$$

$$= 2\sum \frac{(-1)^n \log 4 \cos\frac{nx^2}{2}}{nn} \qquad \left[+ 4\pi i \sum \frac{(-1)^n}{nn}\left(\frac{nx}{2\pi}\right)\right].$$

Pars imaginaria hujus seriei convergit, quicunque est valor ipsius x, pars realis, si $\frac{x}{2\pi}$ est numerus surdus, non convergit, sin minus, denotando literis m, n numeros integros inter se primos, et ponendo $\frac{x}{2\pi} = \frac{m}{n}$ ita exhiberi potest

1^0 si n est impar, aequalis fit,

$$\frac{\pi^2}{n^2}\sum_{1,n-1}^{s}\frac{(-1)^s\cos\frac{\pi s}{n}}{\sin\frac{\pi s^2}{n}}\log 4\cos\frac{sm\pi^2}{n}-\frac{\pi^2}{6n^2}\log 4.$$

2^0 si n est par, designante p numerum imparem

$$=\frac{\pi^2}{n^2}\sum_{1,\frac{n}{2}-1}^{s}\frac{2(-1)^s\log 4\cos\frac{sm\pi^2}{n}}{\sin\pi\frac{s^2}{n}}+\frac{\pi^2}{3n^2}\log 4$$

$$+\frac{2\pi^2}{n^2}(-1)^{\frac{n}{2}}\left(\log\frac{q_0-q}{q_0+q}+\log n+\frac{8}{\pi^2}\sum\frac{\log p}{p^2}\right)$$

quae formula manifesto ita est intelligenda, functionem propositam, subtracta functione

$$\frac{2\pi^2}{n^2}(-1)^{\frac{n}{2}}\log\frac{q_0-q}{q_0+q},$$

si convergat q modo supra stabilito versus limitem q_0, convergere versus limitem finitum, ejusque valorem assignat.

Perinde obtinetur

$$(63)\quad \int_0^{e^{xi}}-\log k'\frac{dq}{q}=-2\log\operatorname{tg}\frac{x^2}{2}-\frac{2}{9}\operatorname{tg}\frac{3x^2}{2}-\frac{2}{25}\log\operatorname{tg}\frac{5x^2}{2}-\cdots$$

$$+4\pi i\left(\left(\frac{x}{2\pi}\right)-\left(\frac{x}{2\pi}+\frac{1}{2}\right)\right)+\frac{4\pi i}{9}\left(\left(\frac{3x}{2\pi}\right)-\left(\frac{3x}{2\pi}+\frac{1}{2}\right)\right)$$

$$+\frac{4\pi i}{25}\left(\left(\frac{5x}{2\pi}\right)-\left(\frac{5x}{2\pi}+\frac{1}{2}\right)\right)+\cdots$$

$$=-\sum_{-\infty,\infty}\frac{\log\operatorname{tg}\frac{px^2}{2}}{p^2}+\left[4\pi i\sum_{1,\infty}\frac{1}{p^2}\left(\left(\frac{px}{2\pi}\right)-\left(\frac{px}{2\pi}+\frac{1}{2}\right)\right)\right]$$

$$(64)\quad \int_0^{e^{xi}}\log\frac{2K}{\pi}\frac{dq}{q}=2\log 4\cos\frac{x^2}{2}+\frac{2}{9}\log 4\cos\frac{3x^2}{2}+\cdots$$

$$+4\pi i\left(\frac{x}{2\pi}\right)+\frac{4\pi i}{9}\left(\frac{3x}{2\pi}\right)+\frac{4\pi i}{25}\left(\frac{5x}{2\pi}\right)+\cdots$$

$$=\sum_{-\infty,\infty}\frac{\log 4\cos\frac{px^2}{2}}{p^2}\quad\left[+4\pi i\sum_{1,\infty}\frac{1}{p^2}\left(\frac{px}{2\pi}\right)\right]$$

$$(65)\quad \int_0^{e^{xi}}\left(\frac{2K}{\pi}-1\right)\frac{dq}{q}=-2\log 4\sin\frac{x^2}{2}+\frac{2}{3}\log 4\sin\frac{3x^2}{2}$$

$$-\frac{2}{5}\log 4\sin\frac{5x^2}{2}+\cdots$$

$$- 4\pi i\left(\frac{x}{2\pi}+\frac{1}{2}\right)+\frac{4\pi i}{3}\left(\frac{3x}{2\pi}+\frac{1}{2}\right)-\cdots$$

$$= i\log\operatorname{tg}\left(\frac{2x+\pi}{4}\right)^2+\frac{i}{2}\log\operatorname{tg}\left(\frac{4x+\pi}{4}\right)^2+\frac{i}{3}\log\operatorname{tg}\left(\frac{6x+\pi}{4}\right)^2+\cdots$$

$$+2\pi\left(\left(\frac{x}{2\pi}+\frac{1}{4}\right)-\left(\frac{x}{2\pi}+\frac{3}{4}\right)\right)+\frac{2\pi}{2}\left(\left(\frac{2x}{2\pi}+\frac{1}{4}\right)-\left(\frac{2x}{2\pi}+\frac{3}{4}\right)\right)$$

$$+\frac{2\pi}{3}\left(\left(\frac{3x}{2\pi}+\frac{1}{4}\right)-\left(\frac{3x}{2\pi}+\frac{3}{4}\right)\right)+\cdots$$

$$(66)\quad \int_0^{e^{xi}}\frac{2kK}{\pi}\frac{dq}{q}=-2\log\operatorname{tg}\frac{x^2}{4}+\frac{2}{3}\log\operatorname{tg}\frac{3x^2}{4}-\frac{2}{5}\log\operatorname{tg}\frac{5x^2}{4}+\cdots$$

$$+4\pi i\left(\left(\frac{x}{4\pi}\right)-\left(\frac{x}{4\pi}+\frac{1}{2}\right)\right)-\frac{4\pi i}{3}\left(\left(\frac{3x}{4\pi}\right)-\left(\frac{3x}{4\pi}+\frac{1}{2}\right)\right)+\cdots$$

$$=2i\log\operatorname{tg}\left(\frac{x+\pi}{4}\right)^2+\frac{2i}{3}\log\operatorname{tg}\left(\frac{3x+\pi}{4}\right)^2$$

$$+\frac{2i}{5}\log\operatorname{tg}\left(\frac{5x+\pi}{4}\right)^2+\cdots$$

$$+4\pi\left(\left(\frac{x}{4\pi}+\frac{1}{4}\right)-\left(\frac{x}{4\pi}+\frac{3}{4}\right)\right)+\frac{4\pi}{3}\left(\left(\frac{3x}{4\pi}+\frac{1}{4}\right)-\left(\frac{3x}{4\pi}+\frac{3}{4}\right)\right)+\cdots$$

$$(67)\quad \int_0^{e^{xi}}\left(\frac{2k'K}{\pi}-1\right)\frac{dq}{q}=-2\log 4\cos\frac{x^2}{2}+\frac{2}{3}\log 4\cos\frac{3x^2}{2}$$

$$-\frac{2}{5}\log 4\cos\frac{5x^2}{2}+\cdots$$

$$-4\pi i\left(\frac{x}{2\pi}\right)+\frac{4\pi i}{3}\left(\frac{3x}{2\pi}\right)-\frac{4\pi i}{5}\left(\frac{5x}{2\pi}\right)+\cdots.$$

$$=-i\log\operatorname{tg}\left(\frac{2x+\pi}{4}\right)^2+\frac{i}{2}\log\operatorname{tg}\left(\frac{4x+\pi}{4}\right)^2$$

$$-\frac{i}{3}\log\operatorname{tg}\left(\frac{6x+\pi}{4}\right)^2+\cdots$$

$$-2\pi\left(\left(\frac{x}{2\pi}+\frac{1}{4}\right)-\left(\frac{x}{2\pi}+\frac{3}{4}\right)\right)$$

$$+\frac{2\pi}{2}\left(\left(\frac{2x}{2\pi}+\frac{1}{4}\right)-\left(\frac{2x}{2\pi}+\frac{3}{4}\right)\right)-\cdots.$$

$$(68)\quad \int_0^{e^{xi}}\left(\frac{2\sqrt{k'}K}{\pi}-1\right)\frac{dq}{q}=-\log 4\cos x^2+\frac{1}{3}\log 4\cos 3x^2$$

$$-\frac{1}{5}\log 4\cos 5x^2+\cdots$$

$$-2\pi i\left(\frac{x}{\pi}\right)+\frac{2\pi i}{3}\left(\frac{3x}{\pi}\right)-\frac{2\pi i}{5}\left(\frac{5x}{\pi}\right)+\cdots$$

$$=-\frac{i}{2}\log\operatorname{tg}\left(x+\frac{\pi}{4}\right)^2+\frac{i}{4}\log\operatorname{tg}\left(2x+\frac{\pi}{4}\right)^2$$

$$-\frac{i}{6}\log\operatorname{tg}\left(3x+\frac{\pi}{4}\right)^2+\cdots$$

$$-\pi\left(\left(\frac{x}{\pi}+\frac{1}{4}\right)-\left(\frac{x}{\pi}+\frac{3}{4}\right)\right)+\frac{\pi}{2}\left(\left(\frac{2x}{\pi}+\frac{1}{4}\right)-\left(\frac{3x}{\pi}+\frac{3}{4}\right)\right)$$
$$-\frac{\pi}{3}\left(\left(\frac{3x}{\pi}+\frac{1}{4}\right)-\left(\frac{3x}{\pi}+\frac{3}{4}\right)\right)+\cdots$$

Posito $x=\frac{m}{n}2\pi$ fit pars imaginaria formulae 65

1^0 si n est numerus par

$$=\sum_{0,\infty}^{s}-4\pi i\sum_{1,n-1}^{p}\frac{(-1)^{\frac{p-1}{2}}}{p+ns}\left(\frac{pm}{n}+\frac{1}{2}\right)(-1)^{\frac{ns}{2}}$$

2^0 si n est numerus impar

$$=\sum_{0,\infty}^{s}-4\pi i\sum_{1,2n-1}^{p}(-1)^{\frac{p-1}{2}}\frac{1}{p+2ns}\left(\frac{pm}{n}+\frac{1}{2}\right)(-1)^{s}$$

quam patet habere valorem finitum, nisi n est $\equiv 0$ mod 4.

Convergentia summae

$$a_0+a_1+a_2+a_3\ldots\ldots$$

postulat, ut data quantitate quamvis parva ε assignari possit terminus a_n, a quo summa usque ad terminum quemvis a_m extensa nanciscatur valorem absolutum ipso ε minorem. Iam posito brevitatis gratia

$$\varepsilon_{n+1}=a_{n+1}$$
$$\varepsilon_{n+2}=a_{n+1}+a_{n+2}$$
$$\varepsilon_{n+3}=a_{n+1}+a_{n+2}+a_{n+3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

functio

$$f(r)=a_0+a_1r+a_2r^2+\cdots$$

facile sub hac forma exhibetur

$$=a_0+a_1r+a_2r^2+\cdots+a_nr^n+\varepsilon_{n+1}r^{n+1}+(\varepsilon_{n+2}-\varepsilon_{n+1})r^{n+2}$$
$$+(\varepsilon_{n+3}-\varepsilon_{n+2})r^{n+3}$$
$$=a_0+a_1r+a_2r^2+\cdots+a_nr^n+\varepsilon_{n+1}(r^{n+1}-r^{n+2})$$
$$+\varepsilon_{n+2}(r^{n+2}-r^{n+3})+\cdots$$

Unde patet convergente r versus limitem 1 functionem $f(r)$ tandem quavis quantitate minus a valore seriei

$$a_0+a_1+a_2\ldots\ldots$$

distare. Summa terminorum altioris gradus quam n, quum sint ε_{n+1}, ε_{n+2}, .. ex hyp. omnes omisso signo $<\varepsilon$, differentiaeque $r^{n+1}-r^{n+2}$.. omnes positivae, manifesto evadit quantitate absoluta

$$<\varepsilon(r^{n+1}-r^{n+2})+\varepsilon(r^{n+2}-r^{n+3})\ldots$$
$$<\varepsilon r^{n+1}$$

summa autem terminorum non altioris gradus quam n est functio algebraica ipsius r, quam constat appropinquando r unitati summae

$$a_0 + a_1 + a_2 + \cdots + a_n$$

quantumvis appropinquari posse; unde patet appropinquando r unitati differentiam functionis $f(r)$ a valore seriei

$$a_0 + a_1 + \cdots$$

infra quantitatem quamvis datam descendere.

Ex hoc theoremate, quod Cl° Abel tribuendum esse Cl^us Dirichlet modo (1852 Sept. 14) quum antecedentia jam essent scripta monuit, facile deducitur . . . .

II.

$$\log k = \log 4\sqrt{q} + \sum (-1)^n \frac{4}{n} \frac{q^n}{1+q^n}, \qquad q = e^{xi}.$$

1) $x = \frac{2m}{n}\pi$, n ungerade.

$$\log k = i\left(\frac{x}{2} + \sum (-1)^s \frac{2}{s} \operatorname{tg} s \frac{x}{2}\right)$$

$$= i\left(\frac{x}{2} + \sum_{0,\infty}^{t} \sum_{1,2n}^{s} (-1)^s \frac{2}{2nt+s} \operatorname{tg} \frac{sm}{n}\pi\right)$$

$$= i\frac{x}{2} + 2i\int_0^1 \sum_{1,2n}^{s} (-1)^s \operatorname{tg} \frac{sm}{n}\pi \frac{x^{s-1}dx}{1-x^{2n}}$$

$$= i\frac{x}{2} + 2\int_0^1 \sum_{1,2n}^{s} (-1)^s \frac{\alpha^{2sm}-1}{\alpha^{2sm}+1} \frac{1}{2n} \sum_{1,2n}^{t} \frac{\alpha^{-ts}\alpha^t dx}{1-\alpha^t x}, \quad \alpha = e^{\frac{2\pi i}{2n}}$$

$$= i\frac{x}{2} + \frac{1}{2n}\int_0^1 \sum_{1,2n}^{t} \frac{\alpha^t dx}{1-\alpha^t x} 2 \sum_{1,n-1}^{\sigma} \sum_{1,2n}^{s} (-1)^{s+\sigma-1} \alpha^{s(2m\sigma-t)}$$

$$\frac{1}{1+r\alpha^{2sm}} = \sum \frac{(-1)^\sigma \alpha^{2s\sigma m} r^\sigma}{1-r^{2n}} = -\frac{1}{2n} \sum_{0,2n-1} (-1)^\sigma \sigma \alpha^{2s\sigma m}$$

$$= \frac{1}{2} \sum_{0,n-1} (-1)^\sigma \alpha^{2s\sigma m}$$

$$= i\frac{x}{2} + 2\sum_{1,n-1} \log(1-\alpha^{n+2m\sigma})(-1)^\sigma$$

$$= i\frac{x}{2} + \sum_{1,n-1} \log \alpha^{2m\sigma}(-1)^\sigma$$

$$= i\frac{x}{2} + 2\pi i\left(\sum_{1,n-1} \frac{2m\sigma}{2n}(-1)^\sigma - \sum_{1,n-1} (-1)^\sigma E\left(\frac{2m\sigma}{2n} + \frac{1}{2}\right)\right)$$

2) $x = \frac{m}{n}\pi$, m, n ungerade.

$$\log k = -\frac{q+q_0}{q-q_0}\frac{3}{2n^2}\sum_{1,\infty}\frac{1}{s^2} - \frac{1}{n}\log\frac{1+q^n}{1-q^n} \qquad \alpha = e^{\frac{2\pi i}{4n}}$$

$$+\frac{x}{2}i + 2\int_0^1 \sum_{1,4n-1}^{s} (-1)^s \frac{x^{s-1}dx}{1-x^{4n}}\frac{\alpha^{2ms}-1}{\alpha^{2ms}+1}$$

$$= A + \frac{x}{2}i +$$

$$2\int_0^1 \sum_{1,4n}^{t} \frac{\alpha^t dx}{1-\alpha^t x}\frac{1}{4n} \cdot -\frac{1}{2n}\sum_{1,4n-1}^{s}\sum_{0,2n-1}^{\sigma}(-1)^{s+\sigma}\sigma\alpha^{2s\sigma m}(\alpha^{2ms}-1)\alpha^{-st}$$

$$= A + \frac{x}{2}i + 2.2\pi i \sum_{1,n-1}^{s}\frac{s}{n}(-1)^s\left(\frac{ms-n}{2n} - E\left(\frac{ms}{2n}\right)\right), m\mu \equiv 1 \text{ mod. } 2n.$$

$$= A + \pi i\left(\frac{m-\mu}{2} + \frac{\mu}{2n} + 2\sum_{1,n-1}E\left(\frac{\mu s}{2n}\right)(-1)^s - 2\sum_{1,n-1}E\left(\frac{ms}{2n}\right)(-1)^s\right)$$

---

3) $x = \frac{m}{2n}\pi$, m ungerade.

$$\log k = \frac{q+q_0}{q-q_0}\frac{3}{8n^2}\sum\frac{1}{s^2} + \frac{1}{2n}\log\left(\frac{1+q^{2n}}{1-q^{2n}}\right)$$

$$+\frac{x}{2}i + i\sum^{t}\sum_{1,8n-1}^{s}(-1)^s\frac{2}{8nt+s}\operatorname{tg} s\frac{m}{4n}\pi$$

$$= A + \frac{x}{2}i + 2\int_0^1\sum_{1,8n-1}^{s}\frac{x^{s-1}dx}{1-x^{8n}}\frac{\alpha^{2ms}-1}{\alpha^{2ms}+1}(-1)^s \qquad \alpha = e^{\frac{2\pi i}{8n}}$$

$$= A + \frac{x}{2}i +$$

$$2\int_0^1\sum_{1,8n}^{t}\frac{\alpha^t dx}{1-\alpha^t x}\frac{1}{8n} \cdot -\frac{1}{4n}\sum_{1,8n-1}^{s}\sum_{0,4n-1}^{\sigma}(-1)^{s+\sigma}\sigma\alpha^{2s\sigma m}(\alpha^{2ms}-1)\alpha^{-st}$$

$$t \equiv 2rm + 4n \text{ mod. } 8n$$

$$= A + \frac{x}{2}i + 2\sum_{1,4n-1}^{r}\log(1-\alpha^{4n+2rm})\frac{1}{8n}\cdot$$

$$\cdot\frac{1}{4n}\left(8n\left((-1)^{r-1}(r-1) - (-1)^r r\right) + 8n(-1)^r(4n-1)\right)$$

$$= A + \frac{x}{2}i + 2\sum_{-2n+1,2n-1}^{s}\log(1-\alpha^{2sm})\frac{-s}{2n}(-1)^s$$

$$= A + \frac{x}{2} i - 4 \sum_{0,2n-1}^{s} \log(-\alpha^{2sm}) \frac{s}{4n} (-1)^s$$

$$= A + \frac{x}{2} i - 4 \sum_{0,2n-1}^{s} \left(\frac{sm}{4n} + \frac{1}{2}\right) \left(\frac{s}{4n}\right) (-1)^s 2\pi i$$

(x) = absolut kleinster Rest von x.

$$-\log k' = 8 \sum \frac{1}{t} \frac{q^t}{1-q^{2t}} = 4i \sum \frac{1}{t \sin tx}, \quad q = e^{xi}.$$

1) $x = \frac{m}{2n}\pi$, m ungerade.

$$-\log k' = 4i \sum_{0,\infty}^{t} \sum_{1,4n-1}^{s} \frac{1}{4nt+s} \frac{1}{\sin \frac{sm\pi}{2n}}$$

$$= 8 \int_0^1 \sum^{s} \frac{x^{s-1} dx}{1-x^{4n}} \frac{\alpha^{sm}}{1-\alpha^{2ms}} \qquad \alpha = e^{\frac{2\pi i}{4n}}$$

$$= 8 \int_0^1 \sum_{1,4n}^{t} \frac{\alpha^t dx}{1-\alpha^t x} \frac{1}{4n} \cdot - \frac{1}{2n} \sum_{1,4n-1}^{s} \sum_{0,2n-1}^{\sigma} \sigma \alpha^{ms(2\sigma+1)} \alpha^{-ts}$$

$$\frac{1}{1-r\alpha^{2ms}} = \sum_{0,2n-1}^{\sigma} \frac{r^\sigma \alpha^{2ms\sigma}}{1-r^{2n}}$$

$$\frac{1}{1-\alpha^{2ms}} = -\frac{1}{2n} \sum_{0,2n-1}^{\sigma} \sigma \alpha^{2ms\sigma} = \frac{1}{2} \sum_{0,n-1}^{\sigma} \alpha^{2ms\sigma}$$

$$= \sum_{0,n-1} [\log(1+\alpha^{m(2r+1)}) - \log(1+\alpha^{-m(2r+1)})]$$

$$= -\pi i \left((m-2)n - 4 \sum_{0,n-1}^{s} E\left(\frac{m(2s+1)}{4n}\right)\right)$$

---

2) $x = \frac{m\pi}{n}$, n ungerade. $\qquad \alpha = e^{\frac{2\pi i}{2n}}$

$$-\log k' = -\frac{q+q_0}{q-q_0} \frac{\pi^2}{4n^2} q_0^{-n} + 8 \int_0^1 \sum_{1,2n-1}^{s} \frac{x^{s-1} dx}{1-x^{2n}} \frac{\alpha^{ms}}{1-\alpha^{2ms}}$$

$$= A +$$

$$8 \int_0^1 \sum_{1,2n}^{t} \frac{\alpha^t dx}{1-\alpha^t x} \cdot - \frac{1}{2n} \sum_{1,2n-1}^{s} \sum_{0,n-1}^{\sigma} \left(\frac{\sigma - \left(\frac{n-1}{2}\right)}{n}\right) \alpha^{ms(2\sigma+1)} \alpha^{-ts}$$

1) $t \equiv m(2r+1)$ mod. $2n$

2) $t \equiv m(2r+1) + n$

$$= A + 8 \sum_{0,\, n-1} \log\left(1 - \alpha^{m(2r+1)}\right) \frac{1}{2n} \left(\frac{r - \frac{n-1}{2}}{n}\right) n$$

$$- 8 \sum \log\left(1 - \alpha^{m(2r+1)+n}\right) \frac{1}{2n} \left(\frac{r - \frac{n-1}{2}}{n}\right) n$$

$$= A + 8 \sum_{1,\, \frac{n-1}{2}} \frac{1}{2} \left(\frac{s}{n}\right) \left(\log(1-\alpha^{2ms+mn}) - \log(1-\alpha^{-2ms+mn})\right)$$

$$- 4 \sum \left(\frac{s}{n}\right) \left(\log(1-\alpha^{2ms+(m+1)n}) - \log(1-\alpha^{-2ms+(m+1)n})\right)$$

$$= A + 8\pi i \sum_{1,\, \frac{n-1}{2}} \left(\frac{s}{n}\right) \left(\left(\frac{2ms+(m+1)n}{2n}\right) - \left(\frac{2ms+mn}{2n}\right)\right)$$

$$= A + 4\pi i \sum \left(\frac{s}{n}\right) \left(\cdots\cdots\right)$$

$$= A + 4\pi i \sum \left(\frac{\mu s}{n}\right) \left(\left(\frac{2s+(m+1)n}{2n}\right) - \left(\frac{2s+mn}{2n}\right)\right),$$

$$m\mu \equiv 1 \text{ mod. } n$$

$$= A + 4\pi i (-1)^{m+1} \sum_{1,\, \frac{n-1}{2}} \left(\frac{\mu s}{n}\right)$$

$$= (-1)^{m+1} \left[\frac{\pi^2}{4n^2} \frac{q+q_0}{q-q_0} + \pi i \left(\frac{n^2-1}{2n} \mu - 4 \sum_{1,\, \frac{n-1}{2}} E\left(\frac{\mu s}{n} + \frac{1}{2}\right)\right)\right]$$

$$\log \frac{2K}{\pi} = 4 \sum \frac{q^t}{t(1+q^t)} = \log\left(\frac{q_0+q}{q_0-q}\right) + 4 \sum \frac{1}{t} \left(\frac{q^t}{1+q^t} - \frac{1}{2} \frac{q^t}{q_0^t}\right)$$

$$= \log \frac{q_0+q}{q_0-q} + 2i \sum \frac{1}{t} \operatorname{tg} t \frac{x}{2}$$

1) $x = \frac{2m}{n}\pi$, n ungerade.

$$\alpha = e^{\frac{2\pi}{2n} i}, \quad \frac{1}{1 + r\alpha^{2sm}} = \sum_{0,\, n-1} \frac{(-1)^\sigma r^\sigma \alpha^{2s\sigma m}}{1+r^n}.$$

$$\log \frac{2K}{\pi} = \log \frac{q_0+q}{q_0-q} + 2 \sum \sum_{1,\, 2n-1}^{s} \frac{1}{2nt+s} \frac{\alpha^{2ms}-1}{\alpha^{2ms}+1}$$

$$= \log \frac{q_0+q}{q_0-q} + 2 \int_0^1 \sum_{1,\, 2n}^{t} \frac{\alpha^t dx}{1-\alpha^t x} \cdot - \frac{1}{2n} \sum^{s} \alpha^{-ts} \sum_{1,\, n-1}^{\sigma} (-1)^\sigma \alpha^{2s\sigma m}$$

$$= \log \frac{q_0+q}{q_0-q} + 2 \sum_{1,\, n-1} \log\left(1 - \alpha^{2rm}\right) (-1)^r \frac{1}{2n} n$$

$$- 2 \sum_{1,\, n-1} \log\left(1 - \alpha^{2rm+n}\right) (-1)^r \frac{1}{2n} n$$

$$= A + \tfrac{1}{2}\sum\left(\frac{rm}{n} + \tfrac{1}{2}\right)(-1)^r 2\pi i - \tfrac{1}{2}\sum\left(\frac{rm}{n}\right)(-1)^r 2\pi i$$

$$= \log\frac{q_0 + q}{q_0 - q} + 2\pi i \sum_{1,\frac{n-1}{2}}^{s}\left(\left(s\frac{2m}{n} + \tfrac{1}{2}\right) - \left(s\frac{2m}{n}\right)\right)$$

---

2) $x = \frac{m}{n}\pi$, n ungerade, m ungerade, $\alpha = e^{\frac{2\pi i}{4n}}$

$$\log\frac{2K}{\pi} = \frac{q + q_0}{q - q_0}\,\frac{\pi^2}{4n^2} + \log\frac{q_0 + q}{q_0 - q} + 2\sum\sum_{1,4n-1}^{s}\frac{1}{4nt + s}\,\frac{\alpha^{2ms} - 1}{\alpha^{2ms} + 1}$$

$$= A +$$

$$2\int_0^1 \sum_{1,4n}^{t}\frac{\alpha^t dx}{1 - \alpha^t x}\,\frac{1}{4n}\cdot - \frac{1}{2n}\sum_{1,4n-1}^{s}\sum_{0,2n-1}^{\sigma}(-1)^\sigma \sigma\alpha^{2s\sigma m}(\alpha^{2ms} - 1)\alpha^{-ts}$$

$$= A + 2\int_0^1 \sum_{1,4n}^{t}\frac{\alpha^t dx}{1 - \alpha^t x}\,\frac{1}{4n}\,2\sum_{1,4n-1}^{s}\sum_{1,2n-1}^{\sigma}(-1)^\sigma\left(\frac{\sigma - n}{2n}\right)\alpha^{2ms\sigma}\alpha^{-ts}$$

1) $t \equiv 2mr \mod. 4n$
2) $t \equiv 2mr + 2n$

$$= A - 2\sum_{1,2n-1}\log(1 - \alpha^{2mr})\,\frac{1}{4n}(-1)^r\left(\frac{r - n}{2n}\right)4n$$

$$+ 2\sum\log(1 - \alpha^{2mr + 2n})(-1)^r\left(\frac{r - n}{2n}\right)$$

$$= A - 2\pi i\sum_{1,2n-1}(-1)^r\left(\left(\frac{mr + n}{2n}\right) - \left(\frac{mr}{2n}\right)\right)\left(\frac{r - n}{2n}\right)$$

$$= A - 2\pi i\sum_{1,2n-1}(-1)^r\left(\left(\frac{r + n}{2n}\right) - \left(\frac{r}{2n}\right)\right)\left(\frac{\mu r - n}{2n}\right),$$

$$m\mu \equiv 1 \mod. 2n$$

$$= A + 2\pi i\sum_{1,n-1}(-1)^r\left(\frac{\mu r - n}{2n}\right)$$

---

3) $x = \frac{m}{2n}\pi$, m ungerade.

$$\log\frac{2K}{\pi} = \log\frac{q_0 + q}{q_0 - q} + 2\sum\sum_{1,4n-1}^{s}\frac{1}{4nt + s}\,\frac{\alpha^{ms} - 1}{\alpha^{ms} + 1} \qquad \alpha = e^{\frac{2\pi i}{4n}}$$

$$= A + 2\int_0^1 \sum_{1,2n}^{t}\frac{\alpha^t dx}{1 - \alpha^t x}\,\frac{1}{4n}\,2\sum_{1,4n-1}^{s}\sum_{1,4n-1}^{\sigma}(-1)^\sigma\left(\frac{\sigma - 2n}{4n}\right)\alpha^{ms\sigma}\alpha^{-ts}$$

$$= A + 2\pi i\sum_{1,2n-1}(-1)^r\left(\frac{\mu r - 2n}{4n}\right), \quad m\mu \equiv 1 \mod. 4n.$$
