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Phase transitions and structural properties of random graphs on surfaces

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CHAPTER 1

Introduction and main results

1.1. Introduction

1.1.1. **Random graphs.** Starting with their series of seminal papers [**40, 41, 42, 43**], Erdős and Rényi studied asymptotic stochastic properties of graphs chosen according to a certain probability distribution. Their approach laid the foundations for the classical theory of random graphs. The main questions considered by Erdős, Rényi, and many others are of the following type: consider the so-called *Erdős–Rényi random graph* $G(n, m)$, a graph chosen uniformly at random from the class $\mathcal{G}(n, m)$ of all graphs on vertex set $[n] := \{1, \ldots, n\}$ with $m = m(n)$ edges. What structural properties does $G(n, m)$ have *with high probability* (commonly abbreviated as whp), that is, with probability tending to one as n tends to infinity?

In general, such structural properties are heavily dependent on the parameters involved and small changes in even one parameter can result in tremendous differences in the results. In particular, changing the number m of edges in $G(n, m)$ minimally can result in very different behaviour. Such a change is called a *phase transition*. Since the initial papers of Erdős and Rényi, many such phase transitions have been observed, for instance in one of the most extensively studied properties of random graphs: the component structure.

Erdős and Rényi [**41**] proved that the *order* (which is the number of vertices) of the components of $G(n, m)$ undergoes such a phase transition when m is around $\frac{n}{2}$, i.e. when the average degree is around one. The result of Erdős and Rényi states that whp:

- if the average degree $\mu := 2\frac{m}{n}$ of $G(n, m)$ is smaller than one, then all components have at most logarithmic order;
- if $\mu = 1$, the largest component has order $n^{2/3}$;
- if $\mu > 1$, then there is a unique component of linear order, while the order of all other components is at most logarithmic.

This phenomenon became known as the *emergence of the giant component* and was considered by Erdős and Rényi to be 'one of the most striking facts concerning random graphs'.

While the result of Erdős and Rényi seems to indicate a 'double jump' in the order of the largest component from logarithmic to order $n^{2/3}$ to linear, Bollobás [**22**] proved that the phase transition is actually 'smooth' when we look more closely at the range of μ being around one, that is, when $s := m - \frac{n}{2}$ is sublinear. Bollobás' result, which was later improved by Łuczak [**75**], shows that the order of the largest component changes gradually, depending on whether s has order at most $n^{2/3}$ (known as the *critical regime*) or if s has larger order and $s > 0$ (the *supercritical regime*) or $s < 0$ (the *subcritical regime*). Subsequently, Aldous [**1**] further improved the result for the critical regime using multiplicative coalescent processes and inhomogeneous Brownian motion.

In the supercritical regime and in the regime $\mu > 1$, *central limit theorems* and *local limit theorems* provide stronger concentration results for the order and the size (which is, the number of edges) of the largest component. The methods used for these results range from counting techniques [**94, 101**] over Fourier analysis [**5**] to probabilistic methods such as Galton-Watson branching processes [**25**], two-round exposure [**4**], or random walks and martingales [**24**].

For the order of the largest component of $G(n,m)$, the critical behaviour is described by the results of Bollobás [**22**] and Łuczak [**75**] mentioned above. In order to formally state their results, we need to introduce some notation. A connected graph is called a *tree* if it has no cycles, *unicyclic* if it contains precisely one cycle, and *complex* (or *multicyclic*) otherwise. Given a graph G, we enumerate its components as $H_i = H_i(G)$, $i = 1, 2, \ldots$, in such a way that they are ordered from large to small, that is, the orders $|H_1|, |H_2|, \ldots$ of the components satisfy $|H_i| \geq |H_j|$ whenever $i < j$. We say that H_i is the *i-th-largest component* of G.

The results of Bollobás and Łuczak can now be described as follows (for all order notation in the following, see Definition 2.1.6). If m is smaller than $\frac{n}{2}$ and satisfies $\frac{n}{2} - m = \omega(n^{2/3})$, then whp all components of $G(n,m)$ have order $o(n^{2/3})$. Once $|\frac{n}{2} - m| = O(n^{2/3})$, several components of order $\Theta_p(n^{2/3})$ appear simultaneously. Finally, if m becomes even larger, i.e. $m > \frac{n}{2}$ and $m - \frac{n}{2} = \omega(n^{2/3})$, then the largest component H_1 whp has order $\omega(n^{2/3})$, while every other component has order $o(n^{2/3})$ whp. If we view this development as a process (increasing m one at a time), this means that all components of order $\Theta_p(n^{2/3})$ that appeared when $|m - \frac{n}{2}| = O(n^{2/3})$ later merge into a single component which is then the unique component of order $\omega(n^{2/3})$. This component is usually referred to as the *giant component.*

THEOREM 1.1.1 ([**22, 75**]). *Let* $m = \left(1 + \lambda n^{-1/3}\right)\frac{n}{2}$*, where* $\lambda = o(n^{1/3})$*, and let* $H_i = H_i(G)$*,* $i = 1, 2, \ldots$*, be the i-th-largest component of* $G = G(n,m)$*.*

(i) If $\lambda \to -\infty$*, then for every* $i \in \mathbb{N} \setminus \{0\}$ *whp* H_i *is a tree and has order*

$$(2 + o(1)) \log\left(|\lambda|^3\right) \frac{n^{2/3}}{\lambda^2}.$$

(ii) If $\lambda \to c$ *for a constant* $c \in \mathbb{R}$*, then the probability that* G *has complex components is bounded away both from 0 and 1. For every* $i \in \mathbb{N} \setminus \{0\}$ *the order of* H_i *is*

$$\Theta_p\left(n^{2/3}\right).$$

Furthermore, the probability that H_i *is complex is bounded away from 0.*

(iii) If $\lambda \to \infty$*, then whp the largest component* H_1 *of* G *is complex and has order*

$$(2 + o(1))\, \lambda\, n^{2/3}.$$

For $i \geq 2$*, whp* H_i *is a tree of order* $o(n^{2/3})$*.*

Even more exact structural properties of $G(n,m)$ in the critical regime have been shown by Łuczak, Pittel, and Wierman [**79**]. For an overview on further properties, see e.g. [**23, 64**].

Since the pioneering work of Erdős and Rényi, various random graph models have been introduced and studied, for example the binomial random graph model $G(n,p)$, where each edge is present with probability p. In this thesis, we focus on another very interesting model, that is, on random *planar* graphs or, more generally, random graphs that are embeddable on a fixed two-dimensional orientable surface.

1.1.2. **Random graphs on surfaces.** Frieze [**70**] was arguably the first to ask about properties of *random* planar graphs. Analogously to the Erdős–Rényi random graph $G(n,m)$, we denote by $G_g(n,m)$ a graph chosen uniformly at random among all graphs from the class $\mathcal{G}_g(n,m)$ of graphs embeddable on the orientable surface of genus g with vertex set $[n]$ and m edges. In particular, a random planar graph is denoted by $G_0(n,m)$. An important part of studying random embeddable graphs is enumerating them. Determining the numbers of graphs embeddable on surfaces and *maps*, which are graphs *embedded* on surfaces, have been one of the main objectives of enumerative combinatorics for the last 60 years. Compared to that, the use of enumerative results for random embeddable graphs is comparably recent.

Starting from the enumeration of *planar* maps by Tutte [**104**], various types of maps on the sphere were counted. Planar *cubic* maps were counted by Gao and Wormald [**54**]. Additionally, Tutte's methods were generalised to enumerate maps on surfaces of higher genus, in particular by Bender and Canfield [**7, 8**] and Bender and Wormald [**13**].

An important subclass of maps are *triangulations*, that is, maps where each face is a triangle. Brown [**29**] determined the number of triangulations of a disc, and Tutte enumerated planar triangulations [**103**]. Triangulations on other surfaces have since been considered as well. Gao enumerated 2-connected triangulations on the projective plane [**50**] as well as connected [**51**], 2-connected [**52**] and 3-connected [**53**] triangulations on surfaces of arbitrary genus.

In addition, planar *graphs* were also studied, although they are arguably harder to enumerate. A first breakthrough result was achieved by McDiarmid, Steger, and Welsh proving the existence of a growth constant [**82**] for the class $\mathcal{G}_0(n)$ of planar graphs with vertex set $[n]$. That is, they proved that the limit

$$\lim_{n\to\infty}\left(\frac{|\mathcal{G}_0(n)|}{n!}\right)^{1/n}$$

exists and is finite. Giménez and Noy [**59**] then calculated this growth constant and proved various local and global limit laws, for example for the number of planar graphs with a given number of vertices and edges, obtaining the number of graphs in $\mathcal{G}_0(n,m)$ when $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu \in (2,6)$. These results were generalised to arbitrary non-negative genus g by Chapuy, Fusy, Giménez, Mohar and Noy [**30**] and independently by Bender and Gao [**9**].

An interesting subclass of planar graphs is the class of *cubic* planar graphs, which were counted by Bodirsky, Kang, Löffler and McDiarmid [**20**]. Cubic planar graphs occur as substructures of sparse planar graphs and were thus one of the essential ingredients in the study of sparse random planar graphs by Kang and Łuczak [**66**]. One main result of this thesis is to determine the number of cubic graphs embeddable on a surface of genus g (see Theorem 1.3.1).

Maps and embeddable graphs also have various applications in other fields. For example in algebra and geometry (see e.g. [**74**] for an overview) and statistical physics [**27, 65, 72**]. In some of these applications (e.g. [**72**]) *phase transitions* play a crucial role, thus it is of interest to consider *random* graphs embeddable on surfaces.

Returning to *random* embeddable graphs, Kang and Łuczak [**66**] proved that the random planar graph $G_0(n,m)$ features a similar phase transition to $G(n,m)$, that is, the giant component emerges at $m = \frac{n}{2} + O(n^{2/3})$.

THEOREM 1.1.2 ([**66**]). *Let $m = \left(1 + \lambda n^{-1/3}\right)\frac{n}{2}$, where $\lambda = \lambda(n) = o(n^{1/3})$, and let $H_i = H_i(G)$, $i = 1, 2, \dots$, be the i-th-largest component of $G = G_0(n,m)$. For every $i \in \mathbb{N} \setminus \{0\}$ whp*

$$|H_i| = \begin{cases} (2+o(1))\log\left(|\lambda|^3\right)\frac{n^{2/3}}{\lambda^2} & \text{if } \lambda \to -\infty, \\ \Theta\left(n^{2/3}\right) & \text{if } \lambda \to c \in \mathbb{R}, \\ (1+o(1))\,\lambda\, n^{2/3} & \text{if } \lambda \to \infty \text{ and } i = 1, \\ \Theta(n^{2/3}) & \text{if } \lambda \to \infty \text{ and } i \geq 2. \end{cases}$$

The main difference to the Erdős–Rényi random graph lies in the case $\lambda \to \infty$. In this regime, the largest component of $G_0(n,m)$ is roughly half as large as the largest component of $G(n,m)$. In contrast, the order of the second largest component (or more generally, of the i-th-largest component for every fixed $i \geq 2$) is much larger in $G_0(n,m)$ than in $G(n,m)$.

This behaviour, however, is not the most surprising feature of random planar graphs. Indeed, Kang and Łuczak [**66**] discovered that there is a second phase transition at $m = n + O(n^{3/5})$, which occurs when the giant component covers almost all vertices. Such a behaviour is not observed for Erdős–Rényi random graphs, where the number of vertices outside the giant component is linear in n as long as m is linear.

THEOREM 1.1.3 ([**66**]). *Let* $m = \left(2 + \zeta\, n^{-2/5}\right)\frac{n}{2}$, *where* $\zeta = \zeta(n) = o(n^{2/5})$. *Then whp the largest component* H_1 *of* $G_0(n,m)$ *is complex and*

$$n - |H_1| = \begin{cases} (1+o(1))|\zeta| n^{3/5} & \text{if } \zeta \to -\infty, \\ \Theta\left(n^{3/5}\right) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta\left(\zeta^{-3/2}\, n^{3/5}\right) & \text{if } \zeta \to \infty \text{ and } \zeta = o(n^{1/15}). \end{cases}$$

Given that this second phase transition has only been observed for random planar graphs, the fundamental question that is raised by Theorem 1.1.3 is whether this is an intrinsic phenomenon of planar graphs or whether this phenomenon can be observed elsewhere as well.

For m even larger, i.e. $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu \in (2,6)$, Giménez and Noy [**59**] showed, among several other results, that whp $G_0(n,m)$ has a component that covers all but a constant number of vertices. Observe that Theorem 1.1.3 leaves a gap of order $\Theta(n^{1/3})$ to the 'dense' regime considered by Giménez and Noy. Subsequently, Chapuy, Fusy, Giménez, Mohar, and Noy [**30**] and independently Bender and Gao [**9**] proved analogous results in the dense regime for $G_g(n,m)$.

In this thesis, we derive results analogous to Theorems 1.1.2 and 1.1.3 for general (constant) genus g. We shall prove that $G_g(n,m)$ features two phase transitions similar to the planar case, improving the results of Kang and Łuczak [**66**] at various places. In particular, we reduce the gap of $\Theta(n^{1/3})$ between Theorem 1.1.3 and results of Giménez and Noy [**59**], Chapuy, Fusy, Giménez, Mohar, and Noy [**30**], and Bender and Gao [**9**].

1.1.3. **Cubic graphs.** In the study of component structures of random graphs, cubic graphs emerge at various points as important substructures. Enumerating various classes of graphs and multigraphs has been a main topic of enumerative combinatorics for a long time [**21, 59, 61, 84, 85, 96, 97**]. In particular, the problem of enumerating all cubic graphs was successfully solved in 1978 by Bender and Canfield [**6**] and it was shown that for the class $\mathcal{S}(2n)$ of cubic graphs with $2n$ vertices and $3n$ edges we have

$$|\mathcal{S}(2n)| = \frac{(e^{-2}+o(1))(6n)!}{288^n(3n!)} = \left(\frac{1}{e^2\sqrt{2\pi}} + o(1)\right)\left(\frac{3}{2e}\right)^n (2n)! n^{n-1/2}. \tag{1.1}$$

Furthermore, they showed that the probability of a cubic multigraph being simple is positive. As we will see in Section 2.2.2, the number of weighted cubic multigraphs is also an important tool for enumerating general graphs and determining their structure.

Introducing the additional condition that the graph is also planar makes the enumeration more complicated. Results for planar cubic graphs were first presented by Bodirsky, Kang, Löffler, and McDiarmid [**20**], proving that

$$|\mathcal{S}_0(2n)| = (c_0 + o(1)) n^{-7/2} \gamma_S^n (2n)!. \tag{1.2}$$

Later, Noy, Requilé, and Rué [**88**] extended these results to the class $\mathcal{V}_0(2n)$ of cubic planar multigraphs, showing a similar formula with a different constant $\gamma_V > \gamma_S$. In particular, this proves that cubic planar multigraphs are with high probability not simple, in contrast to the general case.

1.2. Two phase transitions

The first group of main results of this thesis concerns the order and structure of the components of $G_g(n,m)$ for $(\frac{1}{2} - o(1))n \le m \le (1+o(1))n$. We prove the existence of two phase transitions, the first being the emergence of the giant component and the second being that the number of vertices outside the giant component changes from linear to sublinear. We thus extend the corresponding results for random planar graphs [**66**]. Additionally, we improve the error bounds of the planar case and show more precise bounds regarding the order and size of the components in a slightly larger regime. Furthermore, we prove that for $g \ge 1$ the giant component is the unique non-planar component and is not embeddable on a surface of smaller genus. This section is split into three different parts, each dealing with a different regime of $m = m(n)$.

Section 1.2.1 deals with the case $m = (\frac{1}{2} + o(1))n$, i.e. the analogue of Theorem 1.1.1 for random embeddable graphs $G_g(n,m)$. The special case $g = 0$ was already shown by Kang and Łuczak (see Theorem 1.1.2). We show that for embeddable graphs there are various differences in the behaviour to $G(n,m)$, most notably that the giant component is only half as large and that, in order to compensate for this difference, the second largest component (and even the k-th-largest component for any fixed $k \ge 2$) is of larger order than in $G(n,m)$. Additionally, we show that for positive genus g, the giant component is not only complex, but is also the unique non-planar component, which clearly distinguishes the case of positive genus from the planar case.

Section 1.2.2 focuses on the regime $m = (1+o(1))n$. In this regime, Kang and Łuczak [**66**] observed a *second phase transition* for $G_0(n,m)$. They showed that in contrast to $G(n,m)$, $G_0(n,m)$ admits yet another phase transition after which the number of vertices outside the giant component is sublinear. While it is known that in $G(n,m)$, as long as m is linear in n, the number of vertices outside the giant component will remain linear, this is not the case for planar graphs. For $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu \in (2,6)$, it was shown by Giménez and Noy [**59**] that only a constant number of vertices remain outside the giant component. This bound ($\mu = 2$) was shown by Kang and Łuczak to be the point of the phase transition for the number of vertices outside the giant component. The second main result of this thesis is that the second phase transition occurs for arbitrary constant genus g. Furthermore, we improve the error terms given in [**66**].

Section 1.2.3 closes the gap left by the previous two regimes and focuses on $G_g(n,m)$ when $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu \in (1,2)$. We show that the behaviour of $G_g(n,m)$ with respect to the order and the structure of the components coincides at both ends of the interval with the respective behaviour in the first two main results. Thus we prove a 'smooth' transition between the different phases.

1.2.1. **The first phase transition.** For this section, suppose that m is given as $m = (1+\lambda n^{-1/3})\frac{n}{2}$ with $\lambda = o(n^{1/3})$. For m in this regime various phase transitions in $G(n,m)$ have been observed, the most well-known is the emergence of the giant component (see Theorem 1.1.1). That is the order of the largest component changes from $o(n^{2/3})$ for $\lambda \to -\infty$ to $(2+o(1))\,\lambda n^{2/3}$ for $\lambda \to \infty$. Additionally, all other components are of smaller order. Moreover, the giant component H_1 is whp the unique *complex* component. In comparison, for planar graphs (see Theorem 1.1.2) H_1 is with positive probability not the unique complex component, but still whp by a factor of λ larger than the second-largest component. The first result of this thesis states that for constant genus these differences to $G(n,m)$ still occur.

THEOREM 1.2.1. *Let* $m = \left(1+\lambda n^{-1/3}\right)\frac{n}{2}$, *where* $\lambda = \lambda(n) = o(n^{1/3})$, *and denote by* $H_i = H_i(G)$, $i = 1, 2, \ldots$, *the i-th-largest component of* $G = G_g(n,m)$.

(i) If $\lambda \to -\infty$, *then for every* $i \geq 1$ *whp* H_i *is a tree of order*

$$(2+o(1))\log\left(|\lambda|^3\right)\frac{n^{2/3}}{\lambda^2}.$$

(ii) If $\lambda \to c$ *for a constant* $c \in \mathbb{R}$, *then the probability that* G *has complex components is bounded away both from 0 and 1. The i-th-largest component has order*

$$\Theta_p\left(n^{2/3}\right).$$

(iii) If $\lambda \to \infty$, *then whp* H_1 *is complex and has order*

$$\lambda n^{2/3} + O_p(n^{2/3}).$$

The rest $G \setminus H_1$ *of the graph has* $O_p(1)$ *complex components, each of which has order* $O_p(n^{2/3})$.

For $i \geq 2$, *we have* $|H_i| = \Theta_p(n^{2/3})$. *The probability that* G *has at least i complex components is bounded away both from 0 and 1.*

This theorem states that the order of the giant component in $G_g(n,m)$ develops in the same way as for $G_0(n,m)$. This also means that the differences to $G(n,m)$ regarding the order of the components still occurs for *any* fixed positive genus. Comparing to Theorem 1.1.1(iii), the order of H_1 is still only half as large, whereas the order of further components is larger. On the other hand, we also show distinctly more that in the planar case. Not only are we giving estimates on the *order* of the giant component, but we also determine the genus of the components.

THEOREM 1.2.2. *Let* $m = \left(1+\lambda n^{-1/3}\right)\frac{n}{2}$, *where* $\lambda = \lambda(n) = o(n^{1/3})$ *and* $\lambda \to \infty$. *Then, for* $g \geq 1$, *whp* H_1 *is not embeddable on* $\mathbb{S}_{g-1}$, *while all other components of* G *are planar.*

In some sense this theorem can be seen as a bridge between $G_0(n,m)$ and $G(n,m)$, as while the giant component is still not the unique complex component, it is at least the unique non-planar component and there is a way of identifying the giant component without comparing its order to that of other components. Such a method exists for $G(n,m)$ (the giant component is the unique complex component), but does not exist for planar graphs (with positive probability there are multiple complex components and it is not clear, a priori, which of them is the giant component without comparing orders).

We prove Theorems 1.2.1 and 1.2.2 in Chapter 3.

1.2.2. **The second phase transition.** When Kang and Łuczak first published their result on random planar graphs [**66**], they described a new phase transition not observed in the Erdős-Rényi random graph model $G(n,m)$. While in $G(n,m)$ the number of vertices outside the giant component remains linear in n as long as $m = \lfloor \mu\frac{n}{2} \rfloor$, $\mu \in \mathbb{R}$, this is not the case for planar graphs. In particular, it was shown by Giménez and Noy [**59**] that the number of vertices outside the giant component is $O(1)$ for $m = \left\lfloor \mu\frac{n}{2} \right\rfloor$ with $\mu > 2$. The bound $\mu > 2$ was due to their methods, as for smaller μ, their error terms grew too fast. Kang and Łuczak showed that this is not just a problem in the method, but $G_0(n,m)$ indeed behaves differently at $m = (1+o(1))n$. More precisely, the number of vertices outside the giant component undergoes a phase transition from there being $\Theta(n)$ vertices outside when $\mu < 2$ to sublinearly many vertices when $\mu = 2$ (see Theorem 1.1.3).

With the methods used, there still remained the gap for m between $m = n + o(n^{2/3})$ and $m = n + o(n)$ where no result was known. In addition to extending the result to arbitrary positive genus, we improve on this gap. We extend the results of Theorem 1.1.3 for the random graph $G_g(n,m)$ of arbitrary genus to all $m = n + o\big((\log n)^{-2/3} n\big)$. Additionally, we derive estimates for the i-th-largest component for any fixed $i \geq 2$.

THEOREM 1.2.3. *Let $m = \big(2 + \zeta\, n^{-2/5}\big)\frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$. Then whp the largest component $H_1 = H_1(G)$ of $G = G_g(n,m)$ is complex and*

$$n - |H_1| = \begin{cases} (1+o(1))|\zeta| n^{3/5} & \text{if } \zeta \to -\infty, \\ \Theta\big(n^{3/5}\big) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta\Big(\zeta^{-3/2}\, n^{3/5}\Big) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

For $i \geq 2$, we have

$$|H_i| = \begin{cases} \Theta_p\Big(|\zeta|^{2/3} n^{2/5}\Big) & \text{if } \zeta \to -\infty, \\ \Theta_p\big(n^{2/5}\big) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta_p\big(\zeta^{-1}\, n^{2/5}\big) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

Observe that these results are stronger than the results by Kang and Łuczak for the planar case (see Theorem 1.1.3). Not only hold our results for a wider range of ζ (up to $\zeta = o\big((\log n)^{-2/3} n^{2/5}\big)$ instead of $\zeta = o\big(n^{1/15}\big)$), but we also improve on the order of the i-th-largest component for all $i \geq 2$.

As in the planar case, corresponding results for $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu > 2$ were already shown by Bender and Gao [**9**] and by Chapuy, Fusy, Giménez, Mohar, and Noy [**30**]. In this case it is known that, as in the planar case, the number of vertices outside the giant component is $O(1)$. Also, as in the first phase transition, we can determine the genus of the components.

THEOREM 1.2.4. *Let $m = \big(2 + \zeta\, n^{-2/5}\big)\frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$ and if $\lambda \to \infty$ then $\zeta = o((\log n)^{-2/3} n^{2/5})$. Then, for $g \geq 1$, whp H_1 is not embeddable on $\mathbb{S}_{g-1}$, while all other components of G are planar.*

That is, as in the first phase transition, the genus of $G_g(n,m)$ is concentrated on the giant component and it is the unique non-planar component.

We prove Theorems 1.2.3 and 1.2.4 in Chapter 3.

1.2.3. **Between the phase transitions.** Considering the previous two sections, there are three regimes left when $m = \lfloor \mu \frac{n}{2} \rfloor$, i.e. $\mu < 1$, $1 < \mu < 2$, and $\mu > 2$.

For $\mu < 1$, it was already shown by Erdős and Rényi [**40**] that for such m, $G(n,m)$ is planar with high probability and thus all results for $G(n,m)$ also hold in this regime.

For $\mu > 2$, Chapuy, Fusy, Giménez, Mohar, and Noy [**30**] proved local and global limit laws showing that all but a constant number of vertices are in the giant component, that the giant component is not embeddable on $\mathbb{S}_{g-1}$, and that the probability that $G_g(n,m)$ is connected is bounded away from zero and one.

The case $1 < \mu < 2$ was shown for the planar case by Kang and Łuczak. We extend this to arbitrary genus g.

THEOREM 1.2.5. *Let $m = \lfloor \mu \frac{n}{2} \rfloor$, where $\mu = \mu(n)$ converges to a constant in $(1,2)$, and let $H_i = H_i(G)$, $i = 1, 2, \ldots$, be the i-th-largest component of $G = G_g(n,m)$. Then whp H_1 is complex and has order*

$$|H_1| = (\mu - 1)n + O_p\Big(n^{2/3}\Big).$$

For $i \geq 2$, we have $|H_i| = \Theta_p(n^{2/3})$.

Theorem 1.2.5 covers the ground between the supercritical regime of Theorem 1.2.1 and the subcritical regime of Theorem 1.2.3 in a 'smooth' manner in the sense that

- if we set $\mu = 1 + \lambda n^{-1/3}$ in Theorem 1.2.5, the order of H_1 coincides with the supercritical case of Theorem 1.2.1;
- for $\mu = 2 + \zeta n^{-2/5}$ in Theorem 1.2.5, we get the same value for $n - |H_1|$ as in the subcritical case of Theorem 1.2.3.

As in Theorems 1.2.2 and 1.2.4, we also show that the giant component uses the complete genus.

THEOREM 1.2.6. *Let $m = \lfloor \mu \frac{n}{2} \rfloor$. Then, for $g \geq 1$, whp H_1 is not embeddable on $\mathbb{S}_{g-1}$, while all other components of G are planar.*

Thus, we have determined the order of the largest component for all m (except for the small gap in the second supercritical phase). In Figure 1.1, we compare this order with the order of the largest component of $G(n, m)$.

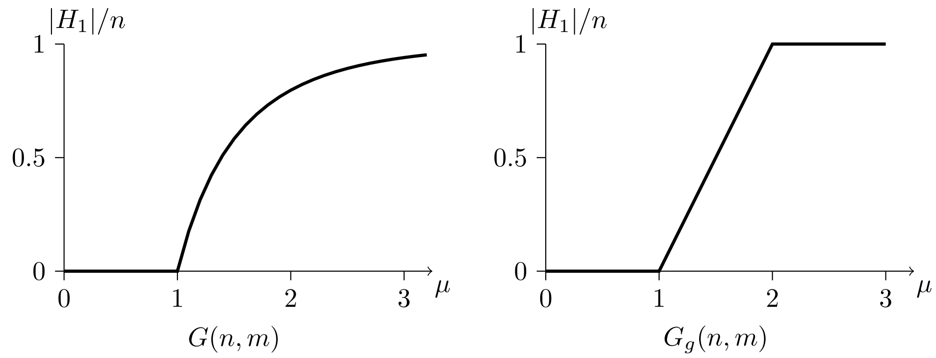

FIGURE 1.1. Rescaled order of the largest component of $G(n, m)$ and of $G_g(n, m)$.

Whereas the graph for $G_g(n, m)$ is a piecewise linear function, this is not the case for $G(n, m)$. For $m = \lfloor \mu \frac{n}{2} \rfloor$ with $\mu > 1$, the giant component of $G(n, m)$ whp has order $(1 + o(1))\beta n$, where β is the unique positive solution of the equation

$$1 - \beta = e^{-\mu\beta}.$$

This formula for $G(n, m)$ corresponds to the survival probability in a related Galton-Watson process (see e.g. [**64**]). In particular, as long as $\mu > 1$ is a constant, the largest component of $G(n, m)$ will leave a linear number of vertices uncovered, see Figure 3.1.

We prove Theorem 1.2.5 in Chapter 3.

1.3. Enumeration of cubic graphs

The second set of main results of this thesis concerns cubic graphs. We derive the number of cubic graphs embeddable on orientable surfaces of arbitrary genus. When the genus is constant, we prove asymptotic values for the number of cubic graphs and multigraphs. For the non-constant genus case, we prove coarser bounds and show at which growth rates of the genus major changes in the enumeration formula occur, in particular, for which growth rate of the genus the class of cubic graphs no longer has a growth constant.

More precisely, in the constant genus case (Section 1.3.1), we provide enumerative and structural results on cubic graphs and multigraphs embeddable on the surface $\mathbb{S}_g$ for general g. The number of all cubic graphs and multigraphs has been shown by Bender and Canfield [**6**]. Bodirsky, Kang, Löffler, and McDiarmid [**20**] showed the number of planar cubic graphs, and Kang and Łuczak [**66**] obtained the number of weighted cubic multigraphs. We extend these results to arbitrary genus. This result is a main ingredient in the proof of the phase transitions in Section 1.2.

When $g = g(n)$ is allowed to be a function in the number of vertices of the graph (Section 1.3.2), we provide upper and lower bounds for the number of cubic graphs embeddable on the corresponding surface. These bounds will show a transition for the number of cubic graphs on $\mathbb{S}_g$ from behaving like in the constant genus case from the previous main result (for $g = o((\log n)^{-2} n)$) to behaving like the general cubic graph case provided by Bender and Canfield [**6**] (for linear genus).

1.3.1. **Constant genus.** When comparing the number of planar cubic graphs (1.2) and general cubic graphs (1.1), we observe a wide discrepancy between these values. In the first main result of this section, we prove that the number of cubic graphs embeddable on a fixed orientable surface is much closer to the planar case than the general case.

THEOREM 1.3.1. *Let g be fixed. Then the number of cubic graphs embeddable on the orientable surface of genus g is given by*

$$|\mathcal{S}_g(2n)| = (c_g + o(1))n^{(5g-7)/2}\gamma_S^n(2n)!, \tag{1.3}$$

where c_g is a constant only depending on g and γ_S is an algebraic constant with first digits 3.133.

Interestingly, the constant γ_S does not depend on the genus and is the same constant as for the planar case. We derive the number of cubic multigraphs in a similar fashion.

THEOREM 1.3.2. *Let g be fixed. Then the number of cubic multigraphs embeddable on the orientable surface of genus g is given by*

$$|\mathcal{V}_g(2n)| = (d_g + o(1))n^{(5g-7)/2}\gamma_V^n(2n)!, \tag{1.4}$$

where d_g is a constant only depending on g and γ_V is an algebraic constant with first digits 3.986.

Again, the constant γ_V does not depend on the genus.

As a key intermediate step for the proofs of Theorems 1.2.1, 1.2.3 and 1.2.5, we need *weighted* cubic multigraphs. That is, we want to enumerate the class of cubic multigraphs, weighted by their *compensation factor*. The compensation factor of a graph was first introduced by Janson, Knuth, Łuczak and Pittel [**62**]. This factor is defined as the number of ways to orient and order all edges of the multigraph divided by $2^r r!$, which is equal to the number of such oriented orderings if all edges were distinguishable. For example, a double edge results in a factor $\frac{1}{2}$ and simple graphs are the only multigraphs with compensation factor one. For an exact definition, see Definition 2.1.1.

THEOREM 1.3.3. *Let g be fixed. Then the number of cubic multigraphs embeddable on the orientable surface of genus g weighted by their compensation factor is given by*

$$|\mathcal{W}_g(2n)| = (e_g + o(1))n^{(5g-7)/2}\gamma_W^n(2n)!, \tag{1.5}$$

where e_g is a constant only depending on g and $\gamma_W = \frac{79^{3/4}}{54^{1/2}} \approx 3.606$.

The constants c_g, d_g, and e_g from Theorems 1.3.1 to 1.3.3 can be obtained with arbitrary precision starting from the recursion (with respect to the genus) given by Bender, Gao, and Richmond [**11**] for the corresponding constant for simple maps.

Furthermore, the results on enumeration imply some interesting structural results as well.

THEOREM 1.3.4. *Let G be a graph chosen uniformly at random from all cubic (multi)graphs with $2n$ vertices embeddable on $\mathbb{S}_g$.*

(i) The probability that G is connected is bounded away from both zero and one.
(ii) The largest component H_1 of G has order $2n - O_p(1)$.
(iii) For $g \geq 1$, whp H_1 is not embeddable on $\mathbb{S}_{g-1}$, while $G \setminus H_1$ is planar.

This result is in contrast to the fact that a graph chosen uniformly at random from $\mathcal{S}(2n)$ is connected with high probability, as proved by Wormald [**107**]. Another difference between $\mathcal{S}(2n)$ and embeddable graphs $\mathcal{S}_g(2n)$ is that the latter has a growth constant whereas $\mathcal{S}(2n)$ does not. All theorems in this section are proved in Chapter 4.

1.3.2. **Growing genus.** Comparing the formula (1.1) for general cubic graphs with the formula (1.3) for cubic graphs embeddable on $\mathbb{S}_g$, we observe that the number of general cubic graphs is larger by a factor of order n^n and this does not change for any fixed genus g. We prove bounds on the number of cubic graphs on vertex set $[2n]$ embeddable on a surface of genus $g = g(n)$ for any growing function. These bounds give a characterisation of how fast the genus has to grow in order to deduce formulas similar either to the constant genus case or the unrestricted case.

THEOREM 1.3.5. *Let $g = g(n)$ and let $\mathcal{S}_g(2n)$ be the class of cubic graphs embeddable on the surface $\mathbb{S}_{g(n)}$. Then the following statements hold.*

(i) If $g = o((\log n)^{-2} n)$, $\mathcal{S}_g(2n)$ has the same growth constant γ_S as $\mathcal{S}_0(2n)$, i.e.

$$\lim_{n\to\infty} \left(\frac{|\mathcal{S}_{g(n)}(2n)|}{(2n)!} \right)^{\frac{1}{2n}} = \gamma_S$$

(ii) for $g \leq \frac{n-1}{2}$, there exist constants $0 < \alpha_1 \leq \alpha_2$ such that

$$\alpha_1^n n^{2g} (2n)! \leq |\mathcal{S}_g(2n)| \leq \alpha_2^n g^{-4g} n^{6g} (2n)!.$$

(iii) If $g > \frac{n-1}{2}$ all graphs are embeddable and $|\mathcal{S}_g(2n)| = |\mathcal{S}(2n)|$.

Moreover, $\mathcal{S}_g(2n)$ does not have a growth constant if $g = \omega\big((\log n)^{-1} n\big)$.

We see from these results that as long as $g = o((\log n)^{-2} n)$, the number of cubic graphs is closely related to the planar case. On the other hand, for g close to $\frac{n}{2}$, the results of Theorem 1.3.5(ii) coincide up to a factor α^n with the general case.

The proof of this theorem is provided in Chapter 5.

1.4. OVERVIEW

The main results of the thesis were stated in Sections 1.2 and 1.3. In Chapter 2 we provide some general definitions and introduce the methods utilised to prove the various results. Chapter 3 is based on [**67, 68**], proving the results stated in Section 1.2. Chapter 4 is based on [**44, 45**], proving the results on the constant genus case stated in Section 1.3. The proofs for non-constant genus are in Chapter 5, which will lead to results on component structures of random graphs on surfaces of non-constant genus [**34**]. The initial ideas for this proof stem from the author's research stay at the University of Oxford.

CHAPTER 2

Preliminaries and methods

2.1. Definitions and notations

A graph G is *simple* if it does not contain loops or multi-edges and is a *multigraph* otherwise. If in a multigraph there are more than two edges connecting the same pair of vertices, we call each pair of those edges a *double edge*. Therefore, every multi-edge consisting of r edges between the same two vertices contains $\binom{r}{2}$ double edges. If e is a loop incident to a vertex v, we say that v is the *base* of e. Similarly, we say that e is *based at* v. An edge that is neither a loop nor part of a double edge is a *single edge*. An edge e of a connected multigraph G is called a *bridge* if deleting e disconnects G.

A multigraph is called *cubic* if each vertex has degree three. By convention, we count a loop twice in the degree of its base. By Φ we denote the cubic multigraph with two vertices u, v and three edges between u and v (i.e. a triple edge). At various points, we will work with multigraphs weighted by the *compensation factor* introduced by Janson, Knuth, Łuczak, and Pittel [**62**], which is defined as follows.

Definition 2.1.1 (Compensation factor)**.** Given a multigraph M and an integer $i \geq 1$, denote by $e_i(M)$ the number of (unordered) pairs $\{u, v\}$ of vertices for which there are exactly i edges between u and v. Analogously, let $\ell_i(M)$ denote the number of vertices x for which there are precisely i loops based at x. Finally, let $\ell(M) = \sum_i i\ell_i(M)$ be the number of loops of M. The *compensation factor* of M is defined to be

$$w(M) := 2^{-\ell(M)} \prod_{i=1}^{\infty} (i!)^{-e_i(M)-\ell_i(M)}. \tag{2.1}$$

The compensation factor of a general graph is the product of the compensation factors of its connected components.

Given a connected *cubic* multigraph G, the compensation factor reduces to

$$w(G) = \begin{cases} \frac{1}{6} & \text{if } G = \Phi, \\ 2^{-(e_2(G)+\ell(G))} & \text{otherwise.} \end{cases}$$

This is used to avoid double counting in the construction in Section 2.2.2.

Definition 2.1.2 (Complex part, core, kernel, excess, deficiency)**.** Let H be a connected graph. We say that H is *unicyclic* if it contains precisely one cycle and we call H *complex* (also known as *multicyclic*) if it contains at least two cycles; the latter is the case if and only if H has more edges than vertices. If H is complex, we call

$$\mathrm{ex}(H) := |E(H)| - |V(H)|$$

the *excess* of H. For a non-connected graph G, we define $\mathrm{ex}(G)$ to be the sum of the excesses of its complex components (and set $\mathrm{ex}(G) = 0$ as a convention if G has no complex components). G is called *complex* if all its components are complex.

Let G be any graph. The union Q_G of all complex components of G is called the *complex part* of G. The *core* C_G of G is defined as the maximal subgraph of Q_G

of minimum degree at least two and the *kernel* K_G of G is a multigraph constructed from the core C_G by replacing all vertices of degree two in the following way. Every maximal path P in C_G consisting of vertices of degree two is replaced by an edge between the vertices of degree at least three that are adjacent to the end vertices of P. Given a graph G with n vertices, we denote the number of vertices of the complex part Q_G, the core C_G, and the kernel K_G by n_Q, n_C, and n_K, respectively.

The *deficiency* of G is defined as

$$\mathrm{d}(G) := 2|E(K_G)| - 3n_K = 2\,\mathrm{ex}(G) - n_K.$$

Definition 2.1.3 (Embeddings, genus of a graph, planarising and separating sets)**.** Let $\mathbb{N}$ be the set of non-negative integers. For $g \in \mathbb{N}$, we denote by $\mathbb{S}_g$ the orientable surface of genus g. An *embedding* of a multigraph G on $\mathbb{S}_g$ is a drawing of G on $\mathbb{S}_g$ without crossing edges. We consider G as a subset of $\mathbb{S}_g$, and therefore $\mathbb{S}_g \setminus G$ consists of connected components called *faces*. An embedding where additionally all faces are homeomorphic to open discs, or equivalently, where all faces are simply connected, is called a 2*-cell embedding*. Multigraphs that have an embedding are called *embeddable* on $\mathbb{S}_g$ and multigraphs that have a 2-cell embedding are called *strongly embeddable*.

By the *genus* of a given graph G we denote the smallest $g \in \mathbb{N}$ for which G is embeddable on $\mathbb{S}_g$. Graphs with genus zero are also called *planar*. For a graph G, we call a set $E' \subseteq E(G)$ such that $G' = (V(G), E(G) \setminus E')$ is planar a *planarising* edge set.

A 2-cell embedding of a strongly embeddable multigraph is also called a *map*. A *triangulation* is a map where each face is bounded by a triangle. These triangles might be degenerate, i.e., three loops with the same base, or a double edge and a loop based at one of the end vertices of the double edge, or a loop and an edge from the base of the loop to a vertex of degree one.

If S is the disjoint union of $\mathbb{S}_{g_1}, \dots, \mathbb{S}_{g_r}$ for non-negative integers $g_1, \dots, g_r$ and M_i is a 2-cell embedding of a graph G_i on $\mathbb{S}_{g_i}$ for each $i = 1, \dots, r$, then the induced function $N : (G_1 \cup \dots \cup G_n) \to S$ is called a *map* on S. Triangulations on S are defined analogously. We denote by $V(M)$, $E(M)$, and $F(M)$ the set of all vertices, edges, and faces of an embedding M, respectively.

We call a set $E' \subseteq E(M)$ *separating*, if the map $M' = (V(M), E')$ has at least two faces, i.e. if M' separates the surface.

Definition 2.1.4 (Dual maps, rooting a map, facewidth)**.** Let M be a map on a surface $\mathbb{S}$. We construct the *dual map* of M by first putting a vertex in each face of M, then for each edge e in M we draw an edge between the two (possibly coincident) vertices inside the faces on both sides of e while crossing e exactly once (and do not cross any other edges of M). The newly drawn edges should only intersect at their end points. Note that the dual map has multi-edges if two faces of the original (*primal*) map have more than one edge in common. It is well known that the dual of a map is also a map, see e.g. [**86**].

For each vertex $v \in V(M)$ of a map M, the edges and faces incident to v have a canonical cyclic order $e_0, f_0, e_1, f_1, \dots, e_{d-1}, f_{d-1}$ according to the way they are arranged around v (in counterclockwise direction). Note that faces can appear multiple times here and that a loop based at v will appear twice in this sequence. To avoid ambiguities, we distinguish the two ends of the loop in this sequence (e.g. by using half-edges or by orienting each loop). A triple $(v, e_i, e_{(i+1) \bmod d})$ of a vertex v and two consecutive edges $e_i, e_{i+1 \bmod d}$ in the cyclic sequence is called a *corner* (at v). We also say that $(v, e_i, e_{(i+1) \bmod d})$ is a corner *of the face* f_i. When we enumerate maps, we always work with maps with one distinguished corner, called the *root* of the map. If (v, e_i, e_{i+1}) is the root corner, we will call v the *root vertex*, e_i the *root edge*, and f_i the *root face*.

An *essential circle* on $\mathbb{S}_g$ is a circle that is not contractible to a point on $\mathbb{S}_g$. Let M be an embedding of a multigraph on $\mathbb{S}_g$. An *essential cycle* of M is a cycle of M which is an essential circle on the surface. The *facewidth* $\mathrm{fw}(M)$ of M is the minimal number of intersections of M with an essential circle on $\mathbb{S}_g$. The *edgewidth* $\mathrm{ew}(M)$ of M is defined as the minimal number of edges of an essential cycle of M. If $g = 0$, there are neither essential circles nor essential cycles and we use the convention $\mathrm{fw}(M) = \mathrm{ew}(M) = \infty$. The facewidth $\mathrm{fw}_g(G)$ of a multigraph G which is embeddable on $\mathbb{S}_g$ is defined as the *maximal* facewidth of all its embeddings on $\mathbb{S}_g$. If the genus is clear from the context we omit it and write $\mathrm{fw}(G)$. When we count multigraphs with restrictions on their facewidth we indicate the restriction by a superscript to the corresponding generating function, e.g. $G^{\mathrm{fw}\geq 2}(x)$ for the generating function of all multigraphs with facewidth at least two.

Definition 2.1.5 (Notations for special classes)**.** Throughout the thesis various classes of graphs and maps will appear at various points. These classes are defined here. For the proofs of Theorems 1.2.1, 1.2.3 and 1.2.5 in Chapter 3 we use:

- $\mathcal{G}_g$ the class of all graphs embeddable on $\mathbb{S}_g$;
- $\mathcal{Q}_g$ the class of all complex graphs in $\mathcal{G}_g$;
- $\mathcal{C}_g$ the class of all cores of graphs in $\mathcal{G}_g$;
- $\mathcal{K}_g$ the class of all kernels of graphs in $\mathcal{G}_g$;
- $\mathcal{U}$ the class of all graphs without complex components.

In other words, $\mathcal{Q}_g$ is the class of all complex graphs embeddable on $\mathbb{S}_g$; $\mathcal{C}_g$ consists of all complex graphs embeddable on $\mathbb{S}_g$ with minimum degree at least two; and $\mathcal{K}_g$ comprises all (weighted) multigraphs embeddable on $\mathbb{S}_g$ with minimum degree at least three. The empty graph lies in all the classes above by convention.

In the sections concerning cubic graphs and multigraphs, we use the following notations:

- $\mathcal{S}_g(n)$ the class of vertex-labelled cubic graphs on n vertices and m edges embeddable on $\mathbb{S}_g$;
- $\mathcal{V}_g(n)$ the class of vertex-labelled cubic multigraphs on n vertices and m edges embeddable on $\mathbb{S}_g$;
- $\mathcal{W}_g(n)$ the class of vertex-labelled weighted cubic multigraphs on n vertices and m edges embeddable on $\mathbb{S}_g$.

For a class $\mathcal{A}$ of (multi)graphs, we denote by $\overline{A}$ the subclass of $\mathcal{A}$ of connected (multi)graphs.

In order to express orders of components in a random graph when n tends to infinity, we use the following notation. Recall that an event holds *with high probability*, or whp for short, if it holds with probability tending to one as n tends to infinity.

Definition 2.1.6 (Landau notation)**.** Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and let $f\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function. For $c \in \mathbb{R}^+$ and $n \in \mathbb{N}$, consider the inequalities

$$|X_n| \leq cf(n), \tag{2.2}$$

$$|X_n| \geq cf(n). \tag{2.3}$$

We say that

(i) $X_n = O(f)$ whp, if there exists $c \in \mathbb{R}^+$ such that (2.2) holds whp;
(ii) $X_n = o(f)$ whp, if for every $c \in \mathbb{R}^+$, (2.2) holds whp;
(iii) $X_n = \Omega(f)$ whp, if there exists $c \in \mathbb{R}^+$ such that (2.3) holds whp;

(iv) $X_n = \omega(f)$ whp, if for every $c \in \mathbb{R}^+$, (2.3) holds whp;
(v) $X_n = \Theta(f)$ whp, if both $X_n = O(f)$ and $X_n = \Omega(f)$ whp;
(vi) $X_n = O_p(f)$, if for every $\delta > 0$, there exist $c_\delta \in \mathbb{R}^+$ and $N_\delta \in \mathbb{N}$ such that (2.2) holds for $c = c_\delta$ and $n \geq N_\delta$ with probability at least $1 - \delta$;
(vii) $X_n = \Theta_p(f)$, if for every $\delta > 0$, there exist $c_\delta^+, c_\delta^- \in \mathbb{R}^+$ and $N_\delta \in \mathbb{N}$ such that for $n \geq N_\delta$ with probability at least $1 - \delta$, both (2.2) holds for $c = c_\delta^+$ and (2.3) holds for $c = c_\delta^-$.

The special case of $X = O_p(1)$ is also known as X being *bounded in probability.*

Definition 2.1.7 (Generating functions, dominant singularities, Δ-analyticity)**.** If $\mathcal{A}$ is a class of maps, we write $\mathcal{A}(m)$ for the subclass of $\mathcal{A}$ containing all maps with exactly m edges. The generating function $\sum_m |\mathcal{A}(m)| y^m$ will be denoted by $A(y)$. If $\mathcal{B}$ is a class of (multi)graphs, we write $\mathcal{B}(n)$ for the subclass of $\mathcal{B}$ containing all (multi)graphs with exactly n vertices. The generating function $\sum_n \frac{|\mathcal{B}(n)|}{n!} x^n$ will be denoted by $B(x)$. For an ordinary generating function $F(z) = \sum_n f_n z^n$, we use the notation $[z^n]F(z) := f_n$. For an exponential generating function $H(z) = \sum \frac{h_n}{n!} z^n$, we write $[z^n]H(z) := \frac{h_n}{n!}$.

If two generating functions $F(z), H(z)$ satisfy $0 \leq [z^n]F(z) \leq [z^n]H(z)$ for all n, we say that F is *coefficient-wise smaller* than H, denoted by $F \preceq H$. The singularities of $F(z)$ with the smallest modulus are called *dominant singularities* of $F(z)$. As every generating function we consider in this paper always has non-negative coefficients $[z^n]F(z)$, there is a dominant singularity located on the positive real axis by Pringsheim's Theorem [**102**, pp. 214 ff.]. We denote this dominant singularity by ρ_F. If an arbitrary function $F : \mathbb{C} \to \mathbb{C}$ has a unique singularity with smallest modulus and this singularity lies on the positive real axis, then we also denote it by ρ_F. The function F converges on the open disc of radius ρ_F and thus corresponds to a holomorphic function on this disc. In many cases, this function can be holomorphically extended to a larger domain. Given $\rho, R \in \mathbb{R}$ with $0 < \rho < R$ and $\theta \in (0, \pi/2)$,

$$\Delta(\rho, R, \theta) := \{z \in \mathbb{C} \mid |z| < R \wedge |\arg(z - \rho)| > \theta\}$$

is called a Δ-*domain*. Here, $\arg(z)$ denotes the *argument* of a complex number, i.e. $\arg(0) := 0$ and $\arg(re^{it}) := t$ for $r > 0$ and $t \in (-\pi, \pi]$. We say that F is Δ-*analytic* if it is holomorphically extendable to some Δ-domain $\Delta(\rho_F, R, \theta)$.

Definition 2.1.8 (Dominant terms and functions)**.** A function F is *subdominant* to a function H if either $\rho_F > \rho_H$ or $\rho_F = \rho_H$ and $\lim_{z \to \rho_G} \frac{F(z)}{H(z)} = 0$. In the latter case, if both F and H are Δ-analytic, then in the above limit, z is taken from some fixed Δ-domain to which both F and H are holomorphically extendable. If F is subdominant to H, we also write $F(z) = o(H(z))$. Analogously we write $F(z) = O(H(z))$ if either $\rho_F > \rho_G$ or $\rho_F = \rho_G$ and $\limsup_{z \to \rho_G} \frac{|F(z)|}{|H(z)|} < \infty$.

Given a function $F(z)$ with a dominant singularity ρ_F, we say that a function $H(z) = c\big(1 - \rho_F^{-1} z\big)^{-\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}, c \in \mathbb{R} \setminus \{0\}$ or $H(z) = c \log\big(1 - \rho_F^{-1} z\big)$ is the *dominant term* of F if there is a decomposition

$$F(z) = P(z) + H(z) + o(H(z)),$$

where $P(z)$ is a polynomial. The dominant term, if it exists, is uniquely defined and Δ-analytic. If $H(z) = c(1 - \rho_F^{-1} z)^{-\alpha}$, the exponent $-\alpha$ is called the *dominant exponent* of F. If $H(z) = c \log\big(1 - \rho_F^{-1} z\big)$, then we say that F has the dominant exponent 0.

If we are counting rooted maps or multigraphs, the roots will be counted in the generating function unless stated otherwise. We will often mark vertices or edges of multigraphs or maps, which corresponds to applying the differential operator $z\frac{d}{dz}$ to the generating functions (with $z = x$ if vertices are marked and $z = y$ if edges are marked). To simplify notation we write δ_z for $z\frac{d}{dz}$ and δ_z^n for applying the operator $z\frac{d}{dz}$ repeatedly n times, which corresponds to marking n vertices or edges, while *allowing multiple marks*. We use the notation $F'(z) = \frac{dF}{dz}$ for the standard differential operator. Vice versa, we say that F is a *primitive* of F'.

2.2. Methods

To prove the main results stated in Sections 1.2 and 1.3 we utilise a variety of methods. In particular, we use combinatoric methods like double counting arguments and constructive decompositions, analytic methods (e.g. singularity analysis), maximising techniques, and probabilistic arguments (e.g. first moment method and Chernoff bounds).

More precisely, the main idea of the proofs of Theorems 1.2.1 and 1.2.3 is to enumerate all graphs embeddable on the surface $\mathbb{S}_g$ and determine the order of various parameters along the way. In order to enumerate $\mathcal{G}_g(n, m)$ we will give a constructive decomposition to cubic graphs. Starting from Theorem 1.3.1, we use maximising techniques (Section 2.2.4) and concentration results from probability theory (Section 2.2.5) to derive bounds for various parameters occurring in the decomposition. With the help of these parameters, the theorems are proved by double counting arguments (Section 2.2.1).

To show Theorem 1.3.1, we also use a constructive decomposition in order to relate cubic graphs to a special class of triangulations. These triangulations are enumerated via singularity analysis and the quadratic method. Singularity analysis is also used to prove parts of Theorem 1.3.1.

2.2.1. **Double counting.**

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes. Double counting is a method to prove bounds on $|\mathcal{A}|$ in terms of $|\mathcal{B}|$. In general, this is done in two steps. First, we provide a (family of) construction(s) from an element $A \in \mathcal{A}$ to at least a elements $B \in \mathcal{B}$. Second, we provide a construction in the reverse direction, showing that every element $B \in \mathcal{B}$ is obtained at most from b elements in $\mathcal{A}$. Then one can conclude that

$$|\mathcal{A}| \leq \frac{b}{a}|\mathcal{B}|.$$

By exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ in the constructions, we also derive lower bounds for $|\mathcal{A}|$.

This basic idea is a very powerful tool and can be applied in various different ways. Throughout the thesis, we use it mainly in two different ways. The first is to show that $|\mathcal{A}|$ is of smaller order than $|\mathcal{B}|$, by giving such constructions with $\frac{b}{a} = o(1)$. The other is to show upper and lower bounds for $|\mathcal{A}|$ which are close together. This yields good bounds on the exact value of $|\mathcal{A}|$.

One can also argue that the constructive decompositions in Section 2.2.2 are a special case of double counting. In that case we have exact values for a and b instead of upper and lower bounds resulting in

$$|\mathcal{A}| = \frac{b}{a}|\mathcal{B}|. \tag{2.4}$$

For examples of this, see Section 2.2.2.

Proving properties from enumeration results. When proving the phase transitions for embeddable graphs, we use double counting arguments to prove various structural results throughout the second phase transition. As input, we use the enumerative and concentration results for the number of vertices and edges in the complex part, the core, and the kernel deduced by the maximising techniques discussed in Section 2.2.4. With these values, we prove tight bounds on the number of *bad* graphs, which are graphs not having the property we want, showing that whp a random graph is not bad. The specifics differ slightly for each property (seen in more detail in the proofs in Chapter 3). In the following, let $\mathcal{B}$ be the class of bad graphs (the details differ from case to case). The main construction from a graph $B \in \mathcal{B}$ to $\mathcal{G}_g$ is as follows.

(i) Delete an edge in the non-complex part U of B;
(ii) add an edge between the largest component and one other component (chosen by some specific rules).

For the reverse direction we

(iii) delete a bridge;
(iv) add an edge between any two vertices outside the complex part.

The exact rules on the choice in step (ii) and thus also the estimates in step (iii) change for the different uses, but the main construction is used multiple times for various claims in the second phase transition. Thus, for the remainder of this argument let $m = (2 + n^{-2/5}\zeta)\frac{n}{2}$.

The first and most important application of this double counting scheme is to show the existence and size of the giant component. That is, we prove that the number of vertices outside the largest component is of the same order as the number n_U of vertices outside the complex part in the graph. In other words, the number of vertices that are in complex components but not in the giant component, is of at most the same order as the number of vertices outside the complex part. Together with the number of vertices outside the complex part, derived in Theorem 3.5.2, this will show the existence of the giant component.

We determine the order of vertices outside the largest component by the above double counting argument. Let $\alpha = \alpha(n)$ such that $\alpha = \omega(n_U)$, but $\alpha = o(n)$, and let $\mathcal{B}$ be the class of graphs where the largest component is too small, i.e. where $|H_1| \le n - \alpha$. We use the above scheme. In step (ii) we connect the largest component to any other component. There are $a \ge m_U \alpha(n-\alpha)$ possibilities for the construction from $\mathcal{B}$ to $\mathcal{G}_g$. For the reverse direction we delete any bridge from H_1. As all bridges are part of a spanning tree of H_1, there are at most $b = nn_U^2$ choices for the reverse direction. Therefore, with double counting we deduce that

$$|\mathcal{B}| \le \frac{n_U^2 n}{m_U \alpha(n-\alpha)} |\mathcal{G}_g(n,m)|.$$

Because $\alpha = \omega(n_U)$, we have

$$|\mathcal{B}| = o(|\mathcal{G}_g(n,m)|)$$

and thus the order of the giant component follows.

The second application of the scheme is to show that throughout the second phase transition in $G = G_g(n,m)$ all components except the giant component are planar whp.

To show this, let $\mathcal{B}$ be the class of graphs $G \in \mathcal{G}_g(n,m)$ where $G \setminus H_1(G)$ is not planar. When inserting the edge between the giant component and another component in step (ii) in the construction above, we choose the edge in such a way that it connects a vertex in the giant component with a vertex in a non-planar two-connected component. Because of this, there are only at most g possible bridges to choose to delete in step (iii) instead of n as before. With this improved factor we deduce strong enough bounds to show that

$$|\mathcal{B}| \le \frac{g n_U^2}{m_U n_U n} |\mathcal{G}_g(n,m)| = o(|\mathcal{G}_g(n,m)|),$$

proving that indeed all components except the giant component are planar.

With this knowledge we use the third application of the scheme to show that the giant component indeed has genus g. To achieve this, we choose the class $\mathcal{B}$ of bad graphs to be graphs where the giant component does not have genus g, i.e. where the giant component is embeddable on $\mathbb{S}_{g-1}$, and where the rest of the graph is planar. In this case, we insert the edge in step (ii) between any two vertices of the giant component. Since by adding one edge, the genus increases by at most one, this construction is feasible. In step (iii) we now delete any edge in the giant component that is *not* a bridge. There are still at most $m = (1+o(1))n$ such edges. We thus have

$$|\mathcal{B}| \leq \frac{mn_U^2}{m_U(n - O(n_U))^2} |\mathcal{G}_g(n,m)| = o(|\mathcal{G}_g(n,m)|),$$

proving that the giant component indeed has genus g.

Kernels and cubic kernels. Relating general kernels and cubic kernels is one of the central arguments of the proof. By two double counting arguments, we provide upper and lower bounds for the number of kernels (Lemma 3.4.5).

Let $l, d \in \mathbb{N}$ and let $\mathcal{K}_g(2l-d, 3l-d)$ be the class of all kernels with $2l-d$ vertices and $3l-d$ edges. Then

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \leq \frac{6^d}{d!}.$$

If in addition $d \leq \frac{2}{7}l$, then also

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \geq \frac{1}{216^d d!}.$$

For the exact details of the proof, see Section 3.7.1. Here is an overview of the construction.

For the upper bound we start with a cubic multigraph $K \in \mathcal{K}_g(2l, 3l)$. To construct a kernel in $\mathcal{K}_g(2l-d, 3l-d)$, we iteratively contract d edges of K. That is, we iteratively choose and delete an edge and identify its two end vertices. For the reverse construction, we iteratively take vertices of degree larger than three, split them into two vertices, add an edge between those vertices and distribute the edges to the new vertices in such a way that the graph is still embeddable on the same surface. The addition of the d new vertices is responsible for the factor $d!$. All other bounds can be shown to be of the form α^d, proving the upper bound.

The lower bound is a bit more intricate. For this construction we give constructions only between $\mathcal{K}_g(2l, 3l)$ and the subclass of $\mathcal{K}_g(2l-d, 3l-d)$ consisting of all graphs with maximal degree four. The construction of all cubic kernels from this subclass is achieved by splitting all vertices of degree four into two vertices of degree three, as for the upper bound. The upper bound on the degree is necessary in order to prove a bound on the change in the compensation factor in this construction. For the reverse direction we still contract edges. In order for the resulting graph to have maximal degree at most four, the chosen edges have to be a matching. Thus we need a bound on the number of such matchings, but again, we deduce such bounds showing the claimed result.

Cubic graphs with non-constant genus. All of the bounds in Chapter 5 are proved by double counting between various classes of graphs and maps. The main idea is to start with a class with a known number of elements and compare that class to the goal. In our case, this goal is the class of cubic graphs (or multigraphs) with $2n$ vertices embeddable on a surface of genus $g = g(n)$. We use two such starting points. On the one hand, we compare $S_{g(n)}(2n, 3n)$ with planar cubic graphs. This works while the genus is not growing too fast and results in tight bounds. For a genus growing faster than this, we compare to the class of unicellular maps with given degree sequence enumerated by Walsh and Lehman [**105**]. The advantage of this class is that it is even enumerated for non-constant genus. Comparing cubic graphs to this class requires various intermediate steps which in turn results in weaker bounds than the first construction, but for a larger range of the genus.

With these initial classes and constructions between these classes and cubic graphs, we derive the claimed bounds for cubic graphs with the help of (2.4) by plugging in the corresponding values.

2.2.2. **Constructive decomposition.** Let $\mathcal{A}$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ be combinatorial classes. A constructive decomposition between $\mathcal{A}$ and the classes $\mathcal{B}_1, \ldots, \mathcal{B}_k$ is a reversible construction from elements $A \in \mathcal{A}$ to elements $B \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_k$. In general, with such constructions, either one of two things might be derived – either we derive an equation relating the generating functions $A(z)$ and $B_i(z)$ of $\mathcal{A}$ and $\mathcal{B}_i$, respectively, or an equation directly relating $|\mathcal{A}(n)|$ and $|\mathcal{B}_i(n)|$. We use both possibilities, the first to enumerate cubic multigraphs, and the second to derive the number of all graphs from cubic multigraphs.

One can argue that a constructive decomposition is a special case of double counting. By providing constructions in a double counting argument such that equality holds in (2.4), the double counting argument becomes a constructive decomposition. Nonetheless, we treat it as two separate methods, as they are in general used for different things and the arguments change slightly depending on whether the goal is (asymptotic) equality or not.

Cubic graphs and triangulations. The constructive decomposition used to derive equations for the generating function of cubic graphs has already been used multiple times. The idea is to use Whitney's Theorem (or a variation of it) to relate 3-connected graphs to corresponding maps. Whitney's Theorem states that a 3-connected planar graph has a unique embedding on the sphere up to orientation (see [**106**]). That is, there is a 1-to-2 relation between 3-connected planar graphs and 3-connected maps. As additionally the *dual* maps of 3-connected maps on the sphere are exactly the simple triangulations of the sphere, this step is very easy in the planar case. In the non-planar case, Whitney's Theorem does not hold and thus additional constraints are required. The construction therefore consists of the following steps:

(i) Construct all graphs as a set of connected graphs;
(ii) Construct all connected graphs from 2-connected *blocks*;
(iii) Construct 2-connected graphs from 3-connected graphs;
(iv) Relate 3-connected graphs and 3-connected maps.

In [**20**], this construction was used in order to enumerate *planar* cubic graphs. In order to adapt this scheme for arbitrary (constant) genus, every step of the construction has to be altered. In step (i), we have to consider that the resulting graph has as genus the sum of the genera of all its connected components. Thus we have to sum over all possible genus partitions on the connected components, which in turn relate to integer partitions of all integers $g' \leq g$. Doing the calculations shows also that the dominant term in this sum is from the graphs where one component has genus g and all other components are planar (see Section 4.4.4).

For the other steps in the construction we use the *facewidth* of a graph as an additional parameter. There are various helpful results regarding the structure of embeddable graphs with respect to their facewidth. In step (ii), we use the fact that if the facewidth of a connected (non-planar) graph is not too small, it has a unique non-planar 2-connected component (see Lemma 4.2.8, [**99**]). Together with the fact that almost all graphs have large facewidth (see Lemma 4.4.7), we infer a valid construction, namely taking this one 2-connected component of genus g and replacing its edges by a sequence of pendant *planar* connected components (see Figure 4.3). These planar components are already known and equations for their generating functions are given in Proposition 4.4.9.

Similarly, for step (iii), we have a theorem stating that if the facewidth of a 2-connected graph is too large, it has a unique non-planar 3-connected component. Again almost all cubic graphs have large enough facewidth (Lemma 4.4.7) and we construct 2-connected graphs by replacing the edges in a 3-connected graph of genus g by 2-connected planar networks (see Lemma 4.4.4).

Step (iv) is a bit more intricate. We use an extension of Whitney's Theorem by Robertson and Vitray stating that there exists a unique embedding (up to orientation) of a 3-connected graph on $\mathbb{S}_g$ if its facewidth is at least $2g+3$ (Lemma 4.2.8). Again we show that almost all graphs satisfy this condition (see Section 4.4.1). In order to enumerate the resulting cubic maps, we use their dual map, which is a triangulation. In contrast to the planar case, these triangulations are not (just) all simple triangulations. Nonetheless, a simple description of these triangulations is possible, see Proposition 4.3.1. Enumerating these triangulations is done by a mixture of decompositions, singularity analysis and the quadratic method (for details, see Section 4.5), following the approach of Bender and Canfield for enumerating simple triangulations on surfaces of arbitrary genus [**7**].

Reducing graphs to cubic multigraphs. The first part of the construction reduces a graph $G \in \mathcal{G}_g(n,m)$ to its kernel. This is done via the following intermediate steps:

(B1) Divide G into its complex components Q_G and its tree and unicyclic components U_G;
(B2) recursively delete vertices of degree one from Q_G to obtain the 2-core C_G of G;
(B3) remove vertices of degree two in C_G, i.e. replace paths where all internal vertices have degree two by a single edge to obtain the kernel K_G of G.

Comparing general kernels and cubic kernels is done by a double counting argument (see Section 2.2.1).

Vice versa, we can construct a graph on $\mathbb{S}_g$ by performing the reverse constructions.

(C1) Pick a kernel, i.e. a multigraph with minimum degree at least three that is embeddable on $\mathbb{S}_g$, and subdivide its edges to obtain a core;
(C2) to every vertex v of the core, attach a rooted tree T_v (possibly only consisting of one vertex) by identifying v with the root of T_v, so as to obtain a complex graph;
(C3) add trees and unicyclic components to obtain a general graph embeddable on $\mathbb{S}_g$.

The steps (C2) and (C3) are exactly reversing the steps (B2) and (B1), respectively, and we thus derive exact equations. In the third step we have to account for overcounting. This happens when the construction yields loops or multiple edges, as there are multiple ways in the reverse direction to end up at the same result. To avoid this overcounting, we use multigraphs weighted by the compensation factor given in Definition 2.1.1. Recall that for a multigraph M the compensation factor of M is defined to be

$$w(M) := 2^{-\ell(M)} \prod_{i=1}^{\infty} (i!)^{-e_i(M)-\ell_i(M)},$$

where $e_i(M)$ denotes the number of pairs $\{u,v\}$ of vertices for which there are exactly i edges between u and v, $\ell_i(M)$ denotes the number of vertices x for which there are precisely i loops at x, and $\ell(M) = \sum_i i\ell_i(M)$ denotes the number of loops of M. In (B3), the compensation factor enables us to distinguish multiple edges and loops at the same vertex (because of the factors $1/i!$) as well as the different orientations of loops (because of the factor $2^{-\ell(M)}$). This fact ensures that there is no overcounting in (B3). Indeed, if a core C has kernel K, then C can be constructed from K by subdividing edges in precisely $\frac{1}{w(K)}$ different ways; thus, assigning weight $w(K)$ to K prevents overcounting.

The construction of graphs in $\mathcal{G}_g$ from their kernel via the core and complex part as described in (C1)–(C3) can be translated to relations between the numbers of graphs in the previously defined classes. Starting from $\mathcal{G}_g(n,m)$, (C3) immediately gives rise to the identity

$$|\mathcal{G}_g(n,m)| = \sum_{n_Q,l} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, n_Q+l)| \cdot |\mathcal{U}(n_U, m_U)|, \tag{2.5}$$

where $n_U = n - n_Q$ and $m_U = m - n_Q - l$. Indeed, for each fixed number n_Q of vertices in the complex part and each fixed excess l,

- the binomial coefficient counts the possibilities for which vertices lie in the complex part,
- $|\mathcal{Q}_g(n_Q, n_Q+l)|$ counts the complex part s with n_Q vertices and n_Q+l edges, and
- $|\mathcal{U}(n_U, m_U)|$ counts all possible arrangements of non-complex components.

If $|\mathcal{Q}_g(n_Q, n_Q+l)|$ and $|\mathcal{U}(n_U, m_U)|$ are known, then we can use (2.5) to determine $|\mathcal{G}_g(n,m)|$. Determining $|\mathcal{Q}_g(n_Q, n_Q+l)|$ turns out to be quite a challenging task, to which we devote a substantial part of this thesis. The number $|\mathcal{U}(n_U, m_U)|$, on the other hand, can be determined using known results (see Lemma 3.3.1).

2.2.3. **Generating functions.** Let $\mathcal{A}$ be a class of combinatorial objects, $\mathcal{A}(n)$ its subclass of all elements of size n, and $a_n = |\mathcal{A}(n)|$. The generating function $A(z) = \sum a_i \frac{z^i}{i!}$ of $\mathcal{A}$ is a useful tool for finding asymptotic values of a_n. Methods to derive these asymptotics include e.g. singularity analysis, the quadratic method, both used in this thesis, the saddle-point method (as an alternative to singularity analysis; see e.g. [**47**] for a general overview or [**3**] for an application in map enumeration), or local and global limit theorems (used to enumerate $G_g(n,m)$ in the case $m = (1+\mu)n$; see e.g. [**59**]).

Triangulations and the quadratic method. From the constructive decomposition in Section 2.2.2 we see that in order to enumerate cubic multigraphs, we first want to derive the number of triangulations in the class $\mathcal{M}_g(n)$ of triangulations on $\mathbb{S}_g$ with no separating double edge and no separating pair of loops. Indeed, we show in Proposition 4.3.4 that

$$M_g(z) \cong c_g\left(1-\rho_M^{-1}z\right)^{-5g/2+3/2} + O\Big(\left(1-\rho_M^{-1}z\right)^{-5g/2+7/4}\Big).$$

We prove this in Section 4.5 for simple triangulations and triangulations without separating loops and double edges. The statement for $M_g(z)$ then follows, as it is sandwiched between the other two classes. In order to prove these results, we follow an idea of Bender and Canfield [**7**], who showed an asymptotic enumeration result for simple triangulations, albeit not in the form stated here.

The main idea is to add additional classes of maps and connect them using equations deduced by decomposing maps into one another. For these additional classes, we allow markings in some faces and we allow one face (the root face) to not necessarily be a triangle (see Section 4.5.2 for the exact definitions). In order to keep track of this, we also use additional parameters and thus additional variables in the generating function. Then we use induction on the genus g to prove asymptotic results for the generating functions. In order to do this, we inductively plug in asymptotic properties we already know, and determine which terms in the equations contribute significantly and which do not. That is, we determine the terms that are dominating and derive the desired asymptotic properties for these. For the base case of $g = 0$ we use what is known as the quadratic method (see e.g. [**60**]).

Suppose one has two generating functions $A(z)$, $B(z,u)$ with

$$(f_1(z,u,A(z))B(z,u) + f_2(z,u,A(z)))^2 = f_3(z,u,A(z)),$$

where f_1, f_2, f_3 are given functions. Then the quadratic method is a way to prove asymptotic estimates for the coefficients of $A(z)$ and $B(z,u)$. The idea is to choose $u = u(z)$ in such a way that $f_3(z,u(z),A(z)) = 0$. As the left hand side is a square, that means that also $\frac{\partial}{\partial u} f_3(z,u(z),A(z)) = 0$. From these two equations, one then determines the functions $u(z)$ and $A(z)$ (with methods depending on the function f_3) and from there also $B(z,u)$. We use this in Lemma 4.5.2 on the generating functions of simple planar triangulations and planar quasi-triangulations, where the second variable is the number of vertices on the root face.

Cubic graphs and singularity analysis. Singularity analysis is a method to derive asymptotic estimates on a_n from $A(z)$, even if $A(z)$ is given implicitly (e.g. in Flajolet and Sedgewick [**47**]) or as one function in a system of equations for various generating functions simultaneously (e.g. shown by Drmota [**35**]). The central theorem of the method is the so-called transfer theorem, which allows the evaluation of coefficients of generating functions.

THEOREM 2.2.1 ([**46, 47**]). *Let $A(z)$ be a Δ-analytic generating function and let $\alpha, \beta \in \mathbb{R}$ such that*

$$A(z) = O\left(\left(\log \frac{1}{1-z}\right)^{\beta} (1-z)^{-\alpha}\right).$$

Then

$$[z^n]A(z) = O\left((\log n)^{\beta} n^{\alpha-1}\right).$$

The same statement is also true when replacing O by o throughout the theorem.

This statement holds, when the dominant singularity of $A(z)$ is one. Otherwise this theorem can be used after rescaling the function ($B(z) = A(\rho z)$ and $[z^n]A(z) = \rho^{-n}[z^n]B(z)$, where ρ is the dominant singularity of $A(z)$).

We use a refined version of this method given in Theorem 4.2.3 in order to deal with the generating functions of various graph classes used in the construction from triangulations to cubic graphs. Additionally, we also use methods of dealing with additionally differentiated or integrated functions (see Lemma 4.2.4). For more details on these methods, see e.g. [**36, 47**].

The constructive decomposition described in Section 2.2.2 results in equations for the generating function of cubic multigraphs embeddable on $\mathbb{S}_g$ in terms of the corresponding generating functions for 2- and 3-connected multigraphs, which in turn are given by equations containing the generating functions for triangulations. From the quadratic method and the results in Section 4.3, we deduce important properties of the generating functions of triangulations and thus of 3-connected maps on $\mathbb{S}_g$. We then work backwards step-by-step through the construction described in Section 2.2.2 to transfer the properties from maps via 3- and 2-connected graphs to connected and general cubic graphs. We describe all these construction steps in terms of generating functions (for the exact details, see Section 4.4.2). From those equations, we use singularity analysis to derive the results of Theorem 1.3.1. Analogous arguments also work for Theorems 1.3.2 and 1.3.3.

2.2.4. **Maximising techniques.** Suppose we have a sum $A(n) = \sum_{i \in I} a_i(n)$ of non-negative values $a_i(n)$ and we want to prove good upper and lower bounds on this sum. We use the following method. Let $i_0 = i_0(n) \in I$ be the index at which $a_i(n)$ is maximised. This value is calculated (approximately) by deriving a root of the derivative of $a_i(n)$ with respect to i. Thus we provide a first trivial bound:

$$a_{i_0}(n) \leq A(n) \leq |I| a_{i_0}(n).$$

In some cases, these bounds are already very good, in particular when $|I|$ is small. Otherwise, we reparametrise $i = i_0 + r$ and write

$$A(n) = a_{i_0}(n) \sum_{r \in I'} \frac{a_{r+i_0}(n)}{a_{i_0}(n)}, \tag{2.6}$$

where $I' = \{i : i + i_0 \in I\}$, and analyse this sum. This analysis is different from case to case, but in general it is possible to derive bounds by splitting the sum into two parts. One part $I_1 \subseteq I'$ consists of summands far away from the optimum, such that

$$|I_1| \sum_{r \in I_1} \frac{a_{r+i_0}(n)}{a_{i_0}(n)} = o(1).$$

Then this part only contributes a factor $(1 + o(1))$ to the total result.

For the second part $I_2 = I' \setminus I_1$, $a_{r+i_0}(n)$ and $a_{i_0}(n)$ are in general close to each other. Thus we write

$$\frac{a_{r+i_0}(n)}{a_{i_0}(n)} = 1 - b_r(n) = \exp(\log(1 - b_r(n)))$$

and then use a Taylor expansion on $\log(1 - b_r(n))$. By the choice of i_0 as the maximal value, the linear term in this expansion will be negligible. Additionally, as the $b_r(n)$ are close to 0, the error bounds are good and the sum only depends on the leading exponent α in the expansion (in our case two or three). Therefore, it remains to bound a sum of the form

$$\sum_{r \in I_2} \exp(-r^\alpha),$$

which is very well known. This method is in a way a discrete version of the saddle-point method for integrals.

In this thesis, the constructive decomposition of graphs in $\mathbb{S}_g(n, m)$ to cubic kernels (see Section 2.2.2) results in a quadruple sum, the parameters being the number of vertices in the complex part, the number of vertices in the core, the excess and the deficiency (n_Q, n_C, l, and d, respectively). For these sums, the above method is used for n_C (see Lemma 3.4.8) and l (see Lemma 3.4.15) directly with $\alpha = 2$. n_Q uses the same general idea, but as we calculate that $\alpha = 3$, the details are a bit more intricate (see Lemmas 3.4.13 and 3.4.14). Finally, for the sum over d this method is not necessary, as we bound the sum over d directly by binomial sums for which explicit formulas are known (see Lemma 3.4.9).

2.2.5. **Probabilistic bounds.** In this thesis, arguments from probability theory are used in two different ways. First, we use first moment methods in order to extend properties known for the kernel to general graphs. Secondly, we use concentration results on various probability distributions in order to derive good bounds for the sums during the maximisation process described in Section 2.2.4.

First moment method. The first moment method is a simple but powerful method using probability theory to show the existence or non-existence of combinatorial objects. The main idea is to define a probability distribution on the set of all possible objects (in our case the uniform model on all embeddable graphs), calculating the expectation of the property we want and then using Markov's inequality to show the (non)-existence of that property. This is one of the initial methods used in what is nowadays known as the *probabilistic method* first used by Erdős [**39**].

Markov's inequality states that for a random variable X only taking non-negative values, the probability that X is large is bounded from above

$$\mathbb{P}\left[X \geq a\right] \leq \frac{\mathbb{E}\left[X\right]}{a}. \tag{2.7}$$

Suppose now we have a sequence X_n of such random variables with expectations $\mathbb{E}\left[X_n\right] = f(n)$. Then (2.7) shows immediately that $X_n = O_p(f(n))$. Indeed, for $a = c_\delta f(n)$, (2.7) shows that $\mathbb{P}\left[X \geq c_\delta f(n)\right] \leq \frac{1}{c_\delta}$ and thus the definition of O_p is satisfied with $\delta = \frac{1}{c_\delta}$. We will use this fact throughout the thesis.

The main use of this method in our case is in the first supercritical regime and in the regime between the two phase transitions. There we use it to derive estimates for the order of the giant component of the core and the complex part of the graph. From the results shown in Chapter 4, we already know that cubic kernels have one giant component of genus g and all other components are planar. Throughout the first phase transition and the intermediate regime we know that the kernel of an embeddable graph is cubic with positive probability (with high probability in the first supercritical regime), see Theorem 3.5.1. Thus we know that with high probability the kernel of $G_g(n,m)$ has one component of order $n_K - O(1)$ and genus g, where n_K is the number of vertices in the kernel. In order to derive bounds for the order of the largest component of the core and the complex part, we use a random construction and then the bounds from Markov's inequality above. We construct the core from the kernel by adding vertices at random to the edges in the kernel. We show in Theorem 3.5.4 that this does indeed yield a valid core. Then all of the $n_C - n_K$ new vertices have a probability of $O\left(\frac{1}{n_K}\right)$ of *not* lying in the largest component. Therefore, the expected combined order of all components except the largest component is $\frac{n_C - n_K}{n_K}$. By Markov's inequality and the above argument, this leads to the claimed order of the core. Similarly, by constructing the complex part from the core by attaching a random forest to the vertices in the core, we deduce the claimed order of the complex part in the same way. For more details, see Theorem 3.5.4.

Concentration. Returning to the four sums discussed in Section 2.2.4, we not only want to calculate the value of the sum, but also the indices giving the main contribution to the corresponding sums. That is, for a sum $\sum_{i \in I} a_i(n)$ we want to find an index set $I' \subseteq I$ (as small as possible) such that

$$\sum_{i \in I'} a_i(n) = (1 - o(1)) \sum_{i \in I} a_i(n).$$

That is, we want to find the part of the sum where the sum is concentrated. We do this by using known concentration results from various probability distributions, in particular the normal distribution and the binomial distribution. The corresponding inequalities are known as Chernoff bounds (e.g. [**2**]).

Lemma 2.2.2. *Let X be a Gaussian random variable with expectation $\mathbb{E}[X]$ and variance σ^2, then*

$$\mathbb{P}\Big[\big|X - \mathbb{E}[X]\big| \geq t\Big] \leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right).$$

If X is a Binomial random variable, then

$$\mathbb{P}\Big[\big|X - \mathbb{E}[X]\big| \geq t\Big] \leq 2\exp\left(-\frac{t^2}{2\big(\mathbb{E}[X] + \frac{t}{3}\big)}\right).$$

The first of these bounds is used to derive concentration results for the sum over n_C and the second bound for the sum over d.

As discussed already in Section 2.2.4, we reduce the sum over n_C to a sum of the form

$$\sum_{n_C \in I} a(n; n_C) = a(n; \overline{n_C}) \sum_{n_C \in I} \exp\big(-\alpha(n_C - \overline{n_C})^2\big).$$

Normalising this sum results in an integral over a density function of a Gaussian random variable. Thus the corresponding Chernoff bound shows that the contribution of summands far from the mean is negligible (see Lemma 3.4.8(iii) for more details).

Similarly, the sum over all possible deficiencies is bounded from above and below by sums of the form

$$\sum_d \binom{2l}{d} \alpha_1(l,n)^d \leq \sum_d s_d(l,n) \leq \sum_d \binom{2l}{d} \alpha_2(l,n)^d.$$

Normalising results in this sum over the density function of a binomial random variable and again the stated Chernoff bound proves the range of the main contribution (see Lemma 3.4.9(iii)).

The main contributions of the other sums are derived by directly comparing upper and lower bounds without the help of probability density functions.

CHAPTER 3

Phase transitions

3.1. Introduction and results

3.1.1. **Background and motivation.** In their seminal work [**40, 41**], Erdős and Rényi introduced the uniform random graph model $G(n, m)$ (often called *Erdős–Rényi random graph*), thus laying the foundations for the classical theory of random graphs. Amongst other results, Erdős and Rényi described the *emergence of the giant component* in $G(n, m)$, a phenomenon that subsequently became one of the most extensively studied properties of random graphs. Erdős and Rényi [**41**] proved that for constant average degree μ, the largest component of $G(n, \mu \frac{n}{2})$ has logarithmically many vertices for $\mu < 1$, order $n^{2/3}$ vertices when $\mu = 1$, and linearly many vertices when $\mu > 1$. Erdős and Rényi interpreted this behaviour as a 'double jump' of the *order* (that is, the number of vertices in) the largest component of $G(n, \mu \frac{n}{2})$ as the average degree increases from $\mu < 1$ to $\mu = 1$ and further to $\mu > 1$. This phenomenon was considered by Erdős and Rényi to be 'one of the most striking facts concerning random graphs'.

Bollobás [**22**] refined the result of Erdős and Rényi by considering the range 'close to' the point of the phase transition, that is, when $s := m - \frac{n}{2}$ is sublinear. Bollobás' result, which was later improved by Łuczak [**75**], shows that the order of the largest component changes gradually, depending on whether s has order at most $n^{2/3}$ (known as the *critical regime*) or if s has larger order and $s > 0$ (the *supercritical regime*) or $s < 0$ (the *subcritical regime*). In the subcritical regime, whp (that is, with probability tending to 1 as $n \to \infty$) all components of $G(n, m)$ have order $o(n^{2/3})$. In the critical regime, several components of order $\Theta_p(n^{2/3})$ appear simultaneously (see Definition 2.1.6 for a definition of Θ_p). Finally, in the supercritical regime, a giant component of order $\omega(n^{2/3})$ appears, and every other component has order $o(n^{2/3})$.

In addition to the order of connected components, the results of Bollobás [**22**] and Łuczak [**75**] also state whether these components are *trees*, *unicyclic*, or *complex*. Here, a connected graph is called *complex* (or *multicyclic*) if it contains at least two cycles.

Theorem 3.1.1 ([**22, 75**]). *Let $m = \left(1 + \lambda n^{-1/3}\right) \frac{n}{2}$, where $\lambda = o(n^{1/3})$, and let $H_i = H_i(G)$, $i = 1, 2, \ldots$, be the i-th largest component of $G = G(n, m)$.*

(i) If $\lambda \to -\infty$, then for every $i \geq 1$ whp H_i is a tree and has order

$$(2 + o(1)) \frac{n^{2/3}}{\lambda^2} \log\left(-\lambda^3\right).$$

(ii) If $\lambda \to c$ for a constant $c \in \mathbb{R}$, then the probability that G has complex components is bounded away both from 0 and 1. For every $i \geq 1$ the order of H_i is

$$\Theta_p\left(n^{2/3}\right).$$

Furthermore, the probability that H_i is complex is bounded away both from 0 and 1.

(iii) If $\lambda \to \infty$, then whp the largest component H_1 of G is complex and has order

$$(2 + o(1))\, \lambda\, n^{2/3}.$$

For $i \geq 2$, whp H_i is a tree of order $o(n^{2/3})$.

Subsequently, Aldous [**1**] further improved the result for the critical regime using multiplicative coalescent processes and inhomogeneous Brownian motion. In the supercritical regime and in the regime $\mu > 1$, *central limit theorems* and *local limit theorems* provide stronger concentration results for the order and the *size* (that is, the number of edges) of the largest component. The methods used for these results range from counting techniques [**94, 101**] over Fourier analysis [**5**] to probabilistic methods such as Galton-Watson branching processes [**25**], two-round exposure [**4**], or random walks and martingales [**24**].

Since the pioneering work of Erdős and Rényi, various random graph models have been introduced and studied. A particularly interesting model are random *planar* graphs or, more generally, random graphs that are embeddable on a fixed two-dimensional surface. Here, a graph G is called *embeddable* on a surface $\mathbb{S}$ if G can be drawn on $\mathbb{S}$ without crossing edges.

Graphs embeddable on a surface and graphs *embedded* on a surface—also known as *maps*—have been studied extensively since the pioneering work of Tutte (see e.g. [**104**]) in view of asymptotic properties [**15, 19, 20, 30, 37, 38, 48, 55, 56, 57, 59, 66, 80, 81, 82, 90, 91**], random sampling [**16, 17, 18, 19, 49, 100**], and enumeration [**30, 59, 82**].

We call a graph *planar* if it is embeddable on the sphere and denote by $P(n,m)$ the graph chosen uniformly at random from the class $\mathcal{P}(n,m)$ of all planar graphs with vertex set $[n] = \{1, \ldots, n\}$ and m edges. Kang and Łuczak [**66**] proved that $P(n,m)$ features a similar phase transition as $G(n,m)$, that is, the giant component emerges at $m = \frac{n}{2} + O(n^{2/3})$.

Theorem 3.1.2 ([**66**]). *Let $m = \left(1 + \lambda\, n^{-1/3}\right)\frac{n}{2}$, where $\lambda = \lambda(n) = o(n^{1/3})$, and let $H_i = H_i(G)$, $i = 1, 2, \ldots,$ be the i-th largest component of $G = P(n,m)$. For every $i = 1, 2, \ldots$ whp*

$$|H_i| = \begin{cases} (2 + o(1))\frac{n^{2/3}}{\lambda^2}\log\left(-\lambda^3\right) & \text{if } \lambda \to -\infty, \\ \Theta\left(n^{2/3}\right) & \text{if } \lambda \to c \in \mathbb{R}, \\ (1 + o(1))\, \lambda\, n^{2/3} & \text{if } \lambda \to \infty \text{ and } i = 1, \\ \Theta(n^{2/3}) & \text{if } \lambda \to \infty \text{ and } i \geq 2. \end{cases}$$

The main difference to the Erdős–Rényi random graph lies in the case $\lambda \to \infty$. In this regime, the largest component of $P(n,m)$ is roughly half as large as the largest component of $G(n,m)$. On the other hand, the order of the second largest component (or more generally, of the i-th largest component for every fixed $i \geq 2$) is much larger in $P(n,m)$ than in $G(n,m)$.

This behaviour, however, is not the most surprising feature of random planar graphs. In fact, Kang and Łuczak [**66**] discovered that there is a second phase transition at $m = n + O(n^{3/5})$, which is when the giant component covers almost all vertices. Such a behaviour is not observed for Erdős–Rényi random graphs, where the number of vertices outside the giant component is linear in n as long as m is linear.

Theorem 3.1.3 ([**66**]). *Let $m = \left(2 + \zeta\, n^{-2/5}\right)\frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$. Then whp the largest component H_1 of $P(n,m)$ is complex and*

$$n - |H_1| = \begin{cases} (1+o(1))|\zeta| n^{3/5} & \text{if } \zeta \to -\infty, \\ \Theta\left(n^{3/5}\right) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta\left(\zeta^{-3/2}\, n^{3/5}\right) & \text{if } \zeta \to \infty \text{ and } \zeta = o(n^{1/15}). \end{cases}$$

Given that this second phase transition has only been observed for random planar graphs, the fundamental question that is raised by Theorem 3.1.3 is whether this is an intrinsic phenomenon of planar graphs.

Question 3.1.4. *Which other classes of graphs feature a phase transition analogous to Theorem 3.1.3?*

Canonical candidates for classes that lie 'between' $\mathcal{P}(n,m)$ and $\mathcal{G}(n,m)$ are graphs that are embeddable on a surface of fixed positive genus. In this chapter, we consider graphs embeddable on the *orientable* surface $\mathbb{S}_g$ with genus $g \geq 0$. Let $\mathcal{G}_g(n,m)$ be the class of graphs with vertex set $[n]$ and m edges that are embeddable on $\mathbb{S}_g$. (Of course, $\mathcal{G}_0(n,m) = \mathcal{P}(n,m)$.) One of the main results of this chapter is that for every fixed g, the answer to Question 3.1.4 is positive for the class $\mathcal{G}_g(n,m)$.

For $m = \left\lfloor \mu \frac{n}{2} \right\rfloor$ with $\mu \in (2,6)$, that is, when the average degree is bounded away both from 2 and 6, Giménez and Noy [**59**] showed, among several other results, that whp $P(n,m)$ has a component that covers all but finitely many vertices. Observe that Theorem 3.1.3 leaves a gap of order $\Theta(n^{1/3})$ to the 'dense' regime considered by Giménez and Noy. Subsequently, Chapuy, Fusy, Giménez, Mohar, and Noy [**30**] proved analogous results in the dense regime for $G_g(n,m)$.

3.1.2. **Main results.** In this chapter we determine the component structure of $G_g(n,m)$ for arbitrary $g \geq 0$ in the 'sparse' regime $m \leq (1+o(1))n$. In terms of phase transitions, the component structure of $G_g(n,m)$ features particularly interesting phenomena in this regime, similar to $P(n,m)$. To derive these phenomena, we use a wide range of complementary methods from various fields (see Section 3.1.4 for more details).

The main results of this chapter are fourfold. We determine the order and structure of the largest components of a graph $G_g(n,m)$ chosen uniformly at random from $\mathcal{G}_g(n,m)$, where the number m of edges is a) around $\frac{n}{2}$, b) around n, or c) in between the previous two regimes. Lastly, similar to the 'symmetry rule' for $G(n,m)$, we derive d) the relation between the numbers of edges and vertices outside the giant component.

Our first main result describes the appearance of the unique giant component in $G_g(n,m)$ when the average degree is around one. Similar to various random graph models including Erdős–Rényi random graphs and random planar graphs (see Theorems 3.1.1 and 3.1.2), the critical range for the number of edges for the appearance of the giant component is $m = \frac{n}{2} + O(n^{2/3})$. Below this range, the i-th largest component (for each $i \geq 1$) of $G_g(n,m)$ whp is a tree of order $o(n^{2/3})$. In the critical range, several components of order $\Theta_p(n^{2/3})$ appear simultaneously. After the critical range, $G_g(n,m)$ whp has a unique component of order $\omega(n^{2/3})$ which in addition is complex and has genus g, that is, it is embeddable on $\mathbb{S}_g$, but not on $\mathbb{S}_{g-1}$.

Theorem 3.1.5. *Let $m = \left(1 + \lambda\, n^{-1/3}\right)\frac{n}{2}$, where $\lambda = \lambda(n) = o(n^{1/3})$, and denote by $H_i = H_i(G)$, $i = 1, 2, \dots$, the i-th largest component of $G = G_g(n,m)$.*

(i) If $\lambda \to -\infty$, then for every $i \geq 1$ whp H_i is a tree of order

$$(2+o(1))\frac{n^{2/3}}{\lambda^2}\log\bigl(-\lambda^3\bigr).$$

(ii) If $\lambda \to c$ for a constant $c \in \mathbb{R}$, then the probability that G has complex components is bounded away both from 0 and 1. The i-th largest component has order

$$\Theta_p\Big(n^{2/3}\Big).$$

(iii) If $\lambda \to \infty$, then whp H_1 has genus g, is complex, and has order

$$\lambda\, n^{2/3} + O_p(n^{2/3}).$$

The rest $G \setminus H_1$ of the graph is planar whp and has $O_p(1)$ complex components, each of which has order $O_p(n^{2/3})$.

For $i \geq 2$, we have $|H_i| = \Theta_p(n^{2/3})$. The probability that G has at least i complex components is bounded away both from 0 and 1.

Comparing the special case of $g = 0$ in Theorem 3.1.5 with Theorem 3.1.2, the following discrepancies are apparent. Firstly, in the critical regime $\lambda \to c \in \mathbb{R}$, Theorem 3.1.5(ii) yields components of order $\Theta_p(n^{2/3})$ compared to $\Theta(n^{2/3})$ in Theorem 3.1.2. The same holds for the orders of H_i for $i \geq 2$ in the supercritical regime $\lambda \to \infty$. Both points are due to minor mistakes in [**66**]; the proofs given there in fact give order $\Theta_p(n^{2/3})$ instead of the claimed $\Theta(n^{2/3})$. Secondly, the error term in the order of the giant component given in Theorem 3.1.5(iii) is stronger than the one from Theorem 3.1.2. Finally, Theorem 3.1.5(iii) tells us that for positive genus, the giant component is not only the unique largest component but also the unique *non-planar* one.

Our second main result describes the time when the giant component covers almost all vertices. This happens when the average degree is around two; or more precisely, when the number of edges is $m = n + O(n^{3/5})$. Here, the number of vertices *outside* the giant component changes from $\omega(n^{3/5})$ for m below the critical range to $\Theta(n^{3/5})$ within the critical range to $o(n^{3/5})$ beyond the critical range.

THEOREM 3.1.6. *Let $m = \bigl(2+\zeta\, n^{-2/5}\bigr)\frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$. Then whp the largest component $H_1 = H_1(G)$ of $G = G_g(n,m)$ has genus g, is complex, and*

$$n - |H_1| = \begin{cases} (1+o(1))|\zeta| n^{3/5} & \text{if } \zeta \to -\infty, \\ \Theta\bigl(n^{3/5}\bigr) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta\Bigl(\zeta^{-3/2}\, n^{3/5}\Bigr) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

Whp all other components of G are planar. For $i \geq 2$, we have

$$|H_i| = \begin{cases} \Theta_p\bigl(|\zeta|^{2/3} n^{2/5}\bigr) & \text{if } \zeta \to -\infty, \\ \Theta_p\bigl(n^{2/5}\bigr) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta_p\bigl(\zeta^{-1}\, n^{2/5}\bigr) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

The main improvement of Theorem 3.1.6 in comparison to Theorem 3.1.3 (the corresponding result for $g = 0$) is that Theorem 3.1.3 only deals with the case $\zeta = o(n^{1/15})$ and therefore leaves a gap to the dense regime $m = \lfloor \mu n \rfloor$ with $\mu \in (1,3)$ that has been covered in [**30, 59**]. Theorem 3.1.6 closes this gap up to a factor $(\log n)^{2/3}$. Moreover, we show that the giant component is the unique non-planar component (in fact, it even has the maximal possible genus) and derive the order of the i-th largest component.

Additionally, Theorem 3.1.6 provides a correction of the proof given in [**66**] for the number of vertices outside the giant component. In [**66**], the upper bound on this number was obtained with the help of an intermediate result (Theorem 2(iv) in [**66**]) about the structure of the complex part (see Definition 2.1.2 for a definition). However, this intermediate result does *not* apply in the regime $m \sim n$.

In the dense regime, that is, when the average degree is strictly larger than two, the giant component covers all but finitely many vertices [**30, 59**]. This coincides with Theorem 3.1.6 when $\zeta \to n^{2/5}$.

Our third main result covers the case when the number of edges is between the regimes of the two phase transitions, that is, the average degree of the graph is between one and two. In this 'intermediate' regime, the largest component is complex, has genus g, and its order is linear both in n and in the average degree of the graph.

THEOREM 3.1.7. *Let $m = \mu \frac{n}{2}$, where $\mu = \mu(n)$ converges to a constant in $(1, 2)$, and let $H_i = H_i(G)$, $i = 1, 2, \dots$, be the i-th largest component of $G = G_g(n, m)$. Then whp H_1 has genus g, is complex, and has order*

$$|H_1| = (\mu - 1)n + O_p\left(n^{2/3}\right).$$

Whp all other components of G are planar. For $i \geq 2$, we have $|H_i| = \Theta_p(n^{2/3})$.

Theorem 3.1.7 covers the ground between the supercritical regime of Theorem 3.1.5 and the subcritical regime of Theorem 3.1.6 in a 'smooth' manner in the sense that (a) if we set $\mu = 1 + \lambda n^{-1/3}$ in Theorem 3.1.7, the order of H_1 coincides with the supercritical case of Theorem 3.1.5 and (b) for $\mu = 2 + \zeta n^{-2/5}$ in Theorem 3.1.7, we get the same value for $n - |H_1|$ as in the subcritical case of Theorem 3.1.6.

In the intermediate regime, or more generally, for $m = \mu \frac{n}{2}$ with $\mu > 1$, the classical Erdős–Rényi random graph $G(n, m)$ whp has a largest component of order $(1 + o(1))\beta n$, where β is the unique positive solution of the equation

$$1 - \beta = e^{-\alpha\beta}.$$

In particular, as long as $\mu > 1$ is a constant, the largest component of $G(n, m)$ will leave a linear number of vertices uncovered, see Figure 3.1. Indeed, Karp [**71**] proved that the components of $G(n, m)$ can be explored via a Galton-Watson branching process with offspring distribution $\mathrm{Po}(\mu)$; the survival property of such a process is given by β above, yielding order $(1 + o(1))\beta n$ of the largest component. For graphs on surfaces, however, there is no such simple approach to explore components.

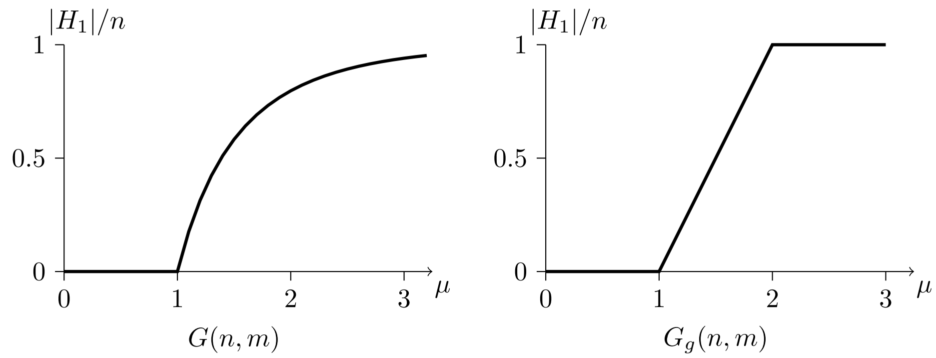

FIGURE 3.1. Rescaled order of the largest component of $G(n, m)$ and of $G_g(n, m)$.

Another well-known phenomenon in terms of the emergence of the giant component is the so-called *symmetry rule.* This rule states that for $G = G(n,m)$ in the supercritical regime $m = \left(1+\lambda\, n^{-1/3}\right)\frac{n}{2}$ with $\lambda \to \infty$ (but $\lambda = o\left(n^{1/3}\right)$), the graph $R(G) = G \setminus H_1(G)$ obtained by deleting the giant component behaves like $G(n_R, m_R)$ with $m_R = \left(1+\lambda_R\, n_R^{-1/3}\right)\frac{n_R}{2}$, where $\lambda_R \to -\infty$, that is, like in the subcritical regime. For even larger m, the fraction m_R/n_R is bounded by a constant smaller than $\frac{1}{2}$.

For $G = G_g(n,m)$, such a result does *not* hold. In fact, throughout the ranges covered by Theorems 3.1.5 to 3.1.7, the graph $R(G)$ behaves like in the critical regime of Theorem 3.1.5. Roughly speaking, the fraction of edges that could not be 'inserted' into the giant component due to the embeddability causes yet another interesting phenomenon outside the giant component, resulting in making $R(G)$ 'critical'.

Theorem 3.1.8. *Suppose that*

$$\frac{n}{2} + O\left(n^{2/3}\right) \le m \le n + o\left((\log n)^{-2/3} n\right)$$

(i.e. m is at least as big as in the critical regime of Theorem 3.1.5 and at most as big as in Theorem 3.1.6). For $G = G_g(n,m)$, we denote by n_R and m_R the number of vertices and edges in $R(G) = G \setminus H_1(G)$, respectively. Then

$$m_R = \frac{n_R}{2} + O_p\left(n_R^{2/3}\right).$$

A central ingredient in the proof of Theorem 3.1.8 will be the size of the giant component (that is, the number of its edges), which we determine in Corollary 3.5.8. At the same time, we shall derive in Section 3.5 various other parameters of $G_g(n,m)$, such as the order of its complex part, core, and kernel, as well as its excess and deficiency. See Section 3.2.2 for definitions of these concepts.

3.1.3. **Related work.** The order of the largest component of the Erdős–Rényi random graph $G(n,m)$ at the time of the phase transition has been extensively studied [**22, 24, 75, 79, 94**]. Most of the results have been proved using purely probabilistic arguments (e.g. random walks, martingales, Galton-Watson branching processes), leading to even stronger results than the ones stated in Theorem 3.1.1, e.g. about the limiting distribution of the order and size of the largest component [**4, 5, 24, 26**]. In the case of $G_g(n,m)$, the additional condition of the graph being embeddable on $\mathbb{S}_g$ makes it virtually impossible to use the same techniques in order to derive such strong results.

In the case of $G(n,m)$, several results have been proved via the *random graph process* that adds one new edge at a time. In the case of $G_g(n,m)$, a similar process can be defined that 'rejects' prospective new edges if adding them violates embeddability (see [**58**] for the planar case). However, the probability distribution of the graphs appearing in this process differs from that of $G_g(n,m)$; for instance, the planar graph process is connected whp for $m = (1+\varepsilon)n$, while $G_0(n,m)$ has a positive probability to be disconnected.

Comparing Theorems 3.1.1 and 3.1.5, the main differences appear when the giant component arises in the *supercritical regime*, that is, when $\lambda \to \infty$. Firstly, the order of the giant component is only about half as large in $G_g(n,m)$ as it is in $G(n,m)$. Secondly, the i-th largest component H_i for fixed $i \ge 2$ is much larger in $G_g(n,m)$ than in $G(n,m)$. These two differences are closely related for the following reason. In $G(n,m)$, the number n_R of vertices and m_R of edges *outside* the giant component lie in the subcritical regime due to the symmetry rule; thus $G(n_R, m_R)$ only has small components. In $G_g(n,m)$ on the other hand, the smaller order (and size) of the giant component enforces m_R to be in the *critical* regime by Theorem 3.1.8, thus resulting in larger orders for H_i with $i \geq 2$. Lastly, while each such H_i is a tree whp for the Erdős–Rényi random graph, this is not necessarily the case for $G_g(n, m)$.

Planar graphs and graphs embeddable on $\mathbb{S}_g$ have been investigated separately for the 'sparse' regime $m \leq n + o(n)$ [**66**] and for the 'dense' regime $m = \lfloor \mu n \rfloor$ with $\mu \in (1, 3)$ [**30, 59**]. From a random graph point of view, in particular when the giant component is considered, the sparse regime is more interesting. In this regime, Kang and Łuczak [**66**] supplied new resourceful proof methods—some of which we apply in a somewhat similar fashion in this chapter—combining probabilistic and graph theoretic methods with techniques from enumerative and analytic combinatorics. On the other hand, minor mistakes in [**66**] led to results that featured order terms that claimed to be stronger than what has actually been proved. One contribution of this chapter is to correct and strengthen these results from [**66**].

In the dense regime, Giménez and Noy [**59**] and Chapuy, Fusy, Giménez, Mohar, and Noy [**30**] use techniques from analytic combinatorics to prove various limit laws for graphs embeddable on $\mathbb{S}_g$, e.g. regarding the number of components, the order of the largest component, and the chromatic and list-chromatic number. Their main method is to define a generating function that describes the graph parameter in question and to apply so-called Quasi-powers theorems (see [**47**, Chapter IX] for an overview) to these functions in order to prove that the random variable corresponding to the graph parameter converges in distribution to a Poisson or to a Gaussian random variable.

The advantage of this technique is that it can be applied to derive limit laws for various graph parameters. However, this particular technique is limited to a) the class $\mathcal{G}_g(n)$ of n-vertex graphs embeddable on $\mathbb{S}_g$, in other words, graphs with n vertices and an *arbitrary* number of edges, and b) the class $\mathcal{G}_g(n, \lfloor \mu n \rfloor)$, where μ is a constant (and $\mu > 1$ in [**30, 59**]). A random graph chosen from the class $\mathcal{G}_g(n)$ is averaged over all graphs with an arbitrary number of edges and thus not appropriate when we look at a specific range of m.[1] Furthermore, the class $\mathcal{G}_g(n, \lfloor \mu n \rfloor)$ scales the number $\lfloor \mu n \rfloor$ of edges as a linear function in n—this is not fine enough in order to capture the changes that take place within the critical windows, which have length $\Theta(n^{2/3})$ for Theorem 3.1.5 and $\Theta(n^{3/5})$ for Theorem 3.1.6. In terms of critical behaviour these techniques are therefore not applicable.

3.1.4. **Proof techniques and outline.** The techniques used in this chapter are novel in comparison to the vast majority of papers on random graphs. Classical random graph results are usually proved with the help of probabilistic arguments such as first and second moment methods, independence of random variables, or martingales. On the other hand, papers about random graphs on surfaces, e.g. [**30, 59**], use singularity analysis of generating functions. In contrast, we combine various complementary methods to prove our results.

The starting point of our proofs are *constructive decompositions* of graphs, a method mostly used in enumerative combinatorics. Every graph in $\mathcal{G}_g(n, m)$ can be decomposed into its complex components and non-complex components, which then can further be decomposed into smaller parts. The most important structures occurring in this decomposition are the so-called *core* and *kernel* of the graph. The decomposition is *constructive* in the sense that every graph can be constructed in a unique way starting from its kernel via its core and complex components (see Section 3.3.1).

[1]In fact, the properties of a random graph chosen from $\mathcal{G}_g(n)$ are dominated by the graphs whose edge density is quite large, more precisely, when $\mu \approx 2.21$ [**30, 59**].

We interpret the aforementioned constructive decomposition in terms of *combinatorial counting*, in other words, we represent the number of graphs in the class $\mathcal{G}_g(n,m)$ as a sum of subclasses that are involved in the decomposition. We proceed by determining the main contributions to the sum using a combinatorial variant of *Laplace's method* from complex analysis, a technique to derive asymptotic estimates of integrals that depend on a parameter n tending to infinity. To illustrate how we apply this approach, assume that we want to analyse a sum of the form

$$A(n) = \sum_{i\in I} B(i)C(n-i),$$

where i is a parameter related to one of the substructures occurring in the constructive decomposition, e.g. the order of the core, say. We rewrite $A(n)$ as

$$A(n) = \sum_{i\in I} \exp(f(i))$$

with $f(i) = \log(B(i)C(n-i))$ and then estimate the exponent $f(i)$ in order to determine the *main contribution* to $A(n)$ in the following sense. We determine a set $J \subset I$ so that the partial sum over all $i \in I \setminus J$ (the *tail* of the sum) is of smaller order than the total sum (see Definition 3.3.3 for a formal definition). The probabilistic interpretation of this main contribution is that $G_g(n,m)$ whp has its corresponding parameter i in the set J. In our example, this will tell us the 'typical' order of the core of $G_g(n,m)$.

The exact method how we estimate the value of the tail and compare it to the total value of the sum will differ from case to case. In some cases, rough bounds provided by maximising techniques will suffice; in other cases, we need better bounds, which we derive by using Chernoff bounds or by bounding the sums via integrals. Systematic applications of these techniques enable us to derive the exact ranges of the main contributions. From the main contributions, we deduce the orders of components, component structure, and other structural properties of $G_g(n,m)$ by applying both combinatorial methods (e.g. double counting) and basic probabilistic techniques (e.g. Markov's inequality).

This chapter is organised as follows. After presenting the necessary notation and definitions in Section 3.2, we give an overview of the proof strategy in Section 3.3; in particular, we derive the aforementioned representation of $|\mathcal{G}_g(n,m)|$ as a sum. In Section 3.4, we determine the main contributions to this sum using the techniques mentioned above. From these results, we derive structural properties of $G_g(n,m)$ in Section 3.5. Sections 3.6 and 3.7 are devoted to the proofs of the main results and the auxiliary results, respectively. Finally, we discuss various open questions in Section 3.8.

3.2. Preliminaries

3.2.1. **Graphs on surfaces.** Given a graph G, we denote its vertex set and its edge set by $V(G)$ and $E(G)$, respectively, and call $|V(G)|$ its *order* and $|E(G)|$ its *size*. All graphs in this chapter are vertex-labelled, that is, $V(G) = [n]$ for some $n \in \mathbb{N}$. Let $g \in \mathbb{N}$ be fixed. An *embedding* of a graph G on $\mathbb{S}_g$, the orientable surface of genus g, is a drawing of G on $\mathbb{S}_g$ without crossing edges. If G has an embedding on $\mathbb{S}_g$, we call G *embeddable* on $\mathbb{S}_g$. Clearly, embeddability is monotone in g, i.e. every graph that is embeddable on $\mathbb{S}_g$ is also embeddable on $\mathbb{S}_{g+1}$. By the *genus* of a given graph G we denote the smallest $g \in \mathbb{N}$ for which G is embeddable on $\mathbb{S}_g$. Graphs with genus zero are also called *planar*.

Let H be a connected graph embeddable on $\mathbb{S}_g$. We say that H is *unicyclic* if it contains precisely one cycle and we call H *complex* (also known as *multicyclic*) if it contains at least two cycles; the latter is the case if and only if H has more edges than vertices. If H is complex, we call

$$\mathrm{ex}(H) := |E(H)| - |V(H)|$$

the *excess* of H. For a non-connected graph G, we define $\mathrm{ex}(G)$ to be the sum of the excesses of its complex components (and set $\mathrm{ex}(G) = 0$ as a convention if G has no complex components). G is called *complex* if all its components are complex.

3.2.2. **Complex part, core, and kernel.** Let G be any graph. The union Q_G of all complex components of G is called the *complex part* of G. The *core* C_G of G is defined as the maximal subgraph of minimum degree at least two of Q_G. The core can also be obtained from the complex part by recursively deleting vertices of degree one (in an arbitrary order). Vice versa, the complex part can be constructed from the core by attaching trees to the vertices of the core. Finally, the *kernel* K_G of G is constructed from the core C_G by replacing all vertices of degree two in the following way. Every maximal path P in C_G consisting of vertices of degree two is replaced by an edge between the vertices of degree at least three that are adjacent to the end vertices of P. By this construction, loops and multiple edges can occur. Reversing the construction, the core arises from the kernel by subdividing edges.

It is important to note that K_G is non-empty as soon as Q_G is, because each component of the complex graph Q_G contains a non-empty core with at least one vertex of degree at least three. Furthermore, K_G has minimum degree at least three and might contain loops and multiple edges. Observe that G is embeddable on $\mathbb{S}_g$ if and only if K_G is. In particular, G and K_G have the same genus. Also observe that $\mathrm{ex}(G) = \mathrm{ex}(Q_G)$ by definition and $\mathrm{ex}(K_G) = \mathrm{ex}(C_G) = \mathrm{ex}(Q_G)$, because subdividing edges and attaching trees changes the number of vertices and edges by the same amount.

Given a graph G with n vertices, we denote the number of vertices of the complex part Q_G, the core C_G, and the kernel K_G by n_Q, n_C, and n_K, respectively. The number of edges of Q_G, C_G, and K_G satisfy

$$|E(Q_G)| = n_Q + \mathrm{ex}(G), \qquad |E(C_G)| = n_C + \mathrm{ex}(G), \qquad |E(K_G)| = n_K + \mathrm{ex}(G).$$

The kernel has minimum degree at least three by definition and thus has at least $\frac{3}{2}n_K$ edges. A kernel is called *cubic* if all its vertices have degree three; in that case, it has precisely $\frac{3}{2}n_K$ edges. The *deficiency* of G is defined as

$$\mathrm{d}(G) := 2|E(K_G)| - 3n_K = 2\,\mathrm{ex}(G) - n_K.$$

Clearly, the deficiency is always non-negative and $\mathrm{d}(G) = 0$ if and only if the kernel K_G is either empty or cubic. The definition of the excess and deficiency of a graph immediately implies the following relation between the deficiency, the excess, and the number of vertices and edges of the kernel.

Lemma 3.2.1. *Given a graph G, the numbers n_K of vertices and m_K of edges in the kernel K_G of G are*

$$n_K = 2\,\mathrm{ex}(G) - \mathrm{d}(G) \qquad \textit{and} \qquad m_K = 3\,\mathrm{ex}(G) - \mathrm{d}(G).$$

□

3.2.3. **Useful bounds.** We will frequently use the following widely known formulas.

$$1+x = \exp\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right) \qquad \text{if } x = o(1), \tag{3.1}$$

$$1+x \le \exp(x), \tag{3.2}$$

$$1+x \ge \exp\left(x - \frac{x^2}{2}\right) \qquad \text{if } x \ge 0, \tag{3.3}$$

To derive bounds for the factorial $n!$ and the falling factorial $(k)_i := k!/(k-i)!$ we shall use the inequalities

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le e\sqrt{n}\left(\frac{n}{e}\right)^n, \tag{3.4}$$

$$k^i \exp\left(-\frac{i^2}{2(k-i)}\right) \le (k)_i \le k^i \exp\left(-\frac{i(i-1)}{2k}\right). \tag{3.5}$$

For $1 \le k \le n-1$ we will also use refined bounds for the binomial coefficient obtained by applying (3.4) thrice.

$$\frac{\sqrt{2\pi}n^{n+1/2}}{e^2 k^{k+1/2}(n-k)^{n-k+1/2}} \le \binom{n}{k} \le \frac{e n^{n+1/2}}{2\pi k^{k+1/2}(n-k)^{n-k+1/2}}. \tag{3.6}$$

We shall also use the inequality

$$\frac{1}{a+b} \ge \frac{1}{a} - \frac{b}{a^2} \quad \text{if } a \ne 0, a+b > 0. \tag{3.7}$$

Finally, we need some well known inequalities from probability theory. Given a random variable X, we denote by $\mathbb{E}[X]$ its expectation and by σ^2 its variance. For a non-negative random variable X and any $t > 0$, Markov's inequality states that

$$\mathbb{P}[X \ge t] \le \frac{\mathbb{E}[X]}{t}. \tag{3.8}$$

In terms of Chernoff bounds, we shall need the two special cases of normal distributions and binomial distributions. For a Gaussian random variable X, we have, for any given $t > 0$,

$$\mathbb{P}\Big[|X - \mathbb{E}[X]| \ge t\Big] \le 2\exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{3.9}$$

If X is a binomial random variable, then

$$\mathbb{P}\Big[|X - \mathbb{E}[X]| \ge t\Big] \le 2\exp\left(-\frac{t^2}{2\big(\mathbb{E}[X] + \frac{t}{3}\big)}\right). \tag{3.10}$$

3.3. Proof strategy

3.3.1. **Decomposition and construction.** Throughout the chapter, let $g \in \mathbb{N}$ be fixed. We have seen in Section 3.2.2 that any graph that is embeddable on $\mathbb{S}_g$ can be decomposed into a) its complex part and b) trees and unicyclic components. The complex part can then further be decomposed so as to obtain the core and the kernel. Vice versa, we can construct a graph on $\mathbb{S}_g$ by performing the reverse constructions.

Construction. The following steps construct every graph embeddable on $\mathbb{S}_g$.

(C1) Pick a kernel, i.e. a multigraph with minimum degree at least three that is embeddable on $\mathbb{S}_g$ and subdivide its edges to obtain a core;

(C2) to every vertex v of the core, attach a rooted tree T_v (possibly only consisting of one vertex) by identifying v with the root of T_v, so as to obtain a complex graph;

(C3) add trees and unicyclic components to obtain a general graph embeddable on $\mathbb{S}_g$.

To avoid overcounting in (C1) if the kernel has loops or multiple edges, multigraphs will always be weighted by the *compensation factor* introduced by Janson, Knuth, Łuczak, and Pittel [**62**], which is defined as follows. Given a multigraph M and an integer $i \geq 1$, denote by $e_i(M)$ the number of (unordered) pairs $\{u, v\}$ of vertices for which there are exactly i edges between u and v. Analogously, let $\ell_i(M)$ denote the number of vertices x for which there are precisely i loops at x. Finally, let $\ell(M) = \sum_i i\ell_i(M)$ be the number of loops of M. The compensation factor of M is defined to be

$$w(M) := 2^{-\ell(M)} \prod_{i=1}^{\infty} (i!)^{-e_i(M)-\ell_i(M)}. \tag{3.11}$$

In (C1), the compensation factor enables us to distinguish multiple edges and loops at the same vertex (because of the factors $1/i!$) as well as the different orientations of loops (because of the factor $2^{-\ell(M)}$). This fact ensures that there is no overcounting in (C1). Indeed, if a core C has kernel K, then C can be constructed from K by subdividing edges in precisely $\frac{1}{w(K)}$ different ways; thus, assigning weight $w(K)$ to K prevents overcounting.

We denote by

- $\mathcal{G}_g$ the class of all graphs embeddable on $\mathbb{S}_g$;
- $\mathcal{Q}_g$ the class of all complex parts of graphs in $\mathcal{G}_g$;
- $\mathcal{C}_g$ the class of all cores of graphs in $\mathcal{G}_g$;
- $\mathcal{K}_g$ the class of all kernels of graphs in $\mathcal{G}_g$;
- $\mathcal{U}$ the class of all graphs without complex components.

In other words, $\mathcal{Q}_g$ is the class of all complex graphs embeddable on $\mathbb{S}_g$; $\mathcal{C}_g$ consists of all complex graphs embeddable on $\mathbb{S}_g$ with minimum degree at least two; and $\mathcal{K}_g$ comprises all (weighted) multigraphs embeddable on $\mathbb{S}_g$ with minimum degree at least three. The empty graph lies in all the classes above by convention.

If $n, m \in \mathbb{N}$ are fixed, we write $\mathcal{G}_g(n, m)$ for the subclass of $\mathcal{G}_g$ containing all graphs with exactly n vertices and m edges. By $G_g(n, m)$ we denote a graph chosen uniformly at random from all graphs in $\mathcal{G}_g(n, m)$. We use the corresponding notation also for the other classes defined above.

The construction of graphs in $\mathcal{G}_g$ from their kernel via the core and complex part as described in (C1)–(C3) can be translated to relations between the numbers of graphs in the previously defined classes. Starting from $\mathcal{G}_g(n, m)$, (C3) immediately gives rise to the identity

$$|\mathcal{G}_g(n, m)| = \sum_{n_Q, l} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, n_Q + l)| \cdot |\mathcal{U}(n_U, m_U)|, \tag{3.12}$$

where $n_U = n - n_Q$ and $m_U = m - n_Q - l$. Indeed, for each fixed number n_Q of vertices in the complex part and each fixed excess l

- the binomial coefficient counts the possibilities which vertices lie in the complex part,
- $|\mathcal{Q}_g(n_Q, n_Q + l)|$ counts the complex parts with n_Q vertices and $n_Q + l$ edges, and
- $|\mathcal{U}(n_U, m_U)|$ counts all possible arrangements of non-complex components.

If $|\mathcal{Q}_g(n_Q, n_Q + l)|$ and $|\mathcal{U}(n_U, m_U)|$ are known, then we can use (3.12) to determine $|\mathcal{G}_g(n, m)|$. Determining $|\mathcal{Q}_g(n_Q, n_Q + l)|$ turns out to be quite a challenging task, to which we devote a substantial part of this chapter. The number $|\mathcal{U}(n_U, m_U)|$, on the other hand, can be determined using known results.

3.3.2. **Graphs without complex components.** The class $\mathcal{U}$ of graphs without complex components (i.e. each component is either a tree or unicyclic) has been studied by Britikov [**28**] and by Janson, Knuth, Łuczak, and Pittel [**62**], who determined the number of graphs in $\mathcal{U}(n, m)$ for different regimes of m.

Lemma 3.3.1. *Let $m = (1 + \lambda n^{-1/3})\frac{n}{2}$ with $\lambda = \lambda(n) < n^{1/3}$ and let $\rho(n, m)$ be such that*

$$|\mathcal{U}(n, m)| = \binom{\binom{n}{2}}{m}\rho(n, m).$$

There exists a constant $c > 0$ such that for

$$f(n, m) = c\left(\frac{2}{e}\right)^{2m-n}\frac{m^{m+1/2}n^{n-2m+1/2}}{(n-m)^{n-m+1/2}},$$

we have

- *(i)* $\rho(n, m) = 1 + o(1)$, *if* $\lambda \to -\infty$;
- *(ii) for each $a \in \mathbb{R}$, there exists a constant $b = b(a) > 0$ such that $\rho(n, m) \geq b$ whenever $\lambda \leq a$;*
- *(iii)* $\rho(n, m) \leq n^{-1/2} f(n, m)$ *if* $\lambda > 0$ *and* $\lambda = o(n^{1/12})$;
- *(iv)* $\rho(n, m) \leq f(n, m)$ *if* $\lambda > 0$.

Lemma 3.3.1(i), (ii), and (iii) are proven in [**28**] and [**62**], but (iv) is a slight extension of the results in [**62**] which we prove in Section 3.7 along the following lines. Inspired by the proof of (iii) in [**62**], we bound $\rho(n, m)$ by a contour integral and prove that this integral has value at most $f(n, m)$ for all $\lambda > 0$.

Clearly, every graph in $\mathcal{U}$ is planar and thus also embeddable on $\mathbb{S}_g$. This fact, together with Lemma 3.3.1 and Theorem 3.1.1(i) and (ii) will be enough to prove Theorem 3.1.5(i) and (ii). For all other regimes, Lemma 3.3.1 will provide a very useful way to bound the number $|\mathcal{U}(n, m)|$ in (3.12).

3.3.3. **Complex parts.** For the number $|\mathcal{Q}_g(n_Q, n_Q + l)|$, we analyse (C1)–(C2) in order to derive an identity similar to (3.12). Firstly, we need to sum over all possible numbers n_C of vertices in the core; the number of edges in the core is then given by $n_C + l$. For fixed n_Q, n_C, and l, we have

- $\binom{n_Q}{n_C}$ choices for which vertices of the complex part lie in the core,
- $|\mathcal{C}_g(n_C, n_C + l)|$ ways to choose a core, and
- $n_C n_Q^{n_Q - n_C - 1}$ possibilities to attach n_C rooted trees with n_Q vertices in total to the vertices of the core.

By (C2), we thus deduce that

$$|\mathcal{Q}_g(n_Q, n_Q + l)| = \sum_{n_C}\binom{n_Q}{n_C}|\mathcal{C}_g(n_C, n_C + l)|n_C n_Q^{n_Q - n_C - 1}. \tag{3.13}$$

In order to determine $|\mathcal{C}_g(n_C, n_C + l)|$, recall that by Lemma 3.2.1, the number of vertices and edges in the kernel depend only on the excess and deficiency of the graph. Thus, we choose the deficiency d as the summation index. The number of ways to construct a core from a kernel according to (C1) cannot be described in an easy fashion like the constructions in (C2) and (C3). We will investigate this construction step in more detail in Lemma 3.4.9. For a kernel $K \in \mathcal{K}_g(2l - d, 3l - d)$, consider the number of different ways to subdivide its edges that result in a core with n_C vertices and $n_C + l$ edges. Denote by $\varphi_{n_C,l,d}$ the average of this number, taken over all kernels in $\mathcal{K}_g(2l-d, 3l-d)$. With this notation, we deduce from (C1) that

$$|\mathcal{C}_g(n_C, n_C+l)| = \sum_d \binom{n_C}{2l-d} |\mathcal{K}_g(2l-d, 3l-d)| \varphi_{n_C,l,d}. \tag{3.14}$$

Recall that the multigraphs in $\mathcal{K}_g$ are weighted. Accordingly, $|\mathcal{K}_g(2l-d, 3l-d)|$ does not denote the *number* of these multigraphs, but the *sum of their weights.*

3.3.4. **Analysing the sums.** In each of (3.12), (3.13), and (3.14), we may assume that the parameters n_Q, n_C, l, d of the sums only take those values for which the summands are non-zero.

Definition 3.3.2. We call values for a parameter (or a set of parameters) *admissible*, if there exists at least one graph satisfying these values for the corresponding parameters.

The definition of the parameters, together with Lemma 3.2.1, directly yield the following necessary conditions for admissibility.

(A1) $0 \leq n_Q \leq n$;
(A2) $0 \leq n_C \leq n_Q$;
(A3) $0 \leq l \leq m - n_Q$;
(A4) $l = 0$ if and only if $n_Q = 0$;
(A5) $l \leq 2n_C + 6(g-1)$;
(A6) $0 \leq d \leq 2l$.

Inequality (A5) is due to Euler's formula applied to the core. These bounds will frequently be used; if we use other bounds, we will state them explicitly.

On the first glance, the sole application of (3.12), (3.13), and (3.14) seems to be to determine the number of graphs with given numbers of vertices and edges in the classes $\mathcal{G}_g$, $\mathcal{Q}_g$, and $\mathcal{C}_g$. However, we shall use these sums to derive *typical structural properties* of graphs chosen uniformly at random from one of these classes.

Our plan to derive such properties from the sums (3.12), (3.13), and (3.14) is as follows. Once we have determined the values $|\mathcal{K}_g(2l-d, 3l-d)|$ and $\varphi_{n_C,l,d}$, we consider the parameters n_Q, n_C, l, d of the sums one after another. For each parameter i, we seek to determine which range for i provides the 'most important' summands. To make this more precise, let us introduce the following notation.

Definition 3.3.3. For every $n \in \mathbb{N}$, let $I(n), I_0(n) \subset \mathbb{N}$ be finite index sets with $I_0(n) \subseteq I(n)$. For each $i \in I(n)$, let $A_i(n) \geq 0$. We say that $I_0(n)$ *provides the main contribution to* the sum

$$\sum_{i \in I(n)} A_i(n)$$

if

$$\sum_{i \in I(n) \setminus I_0(n)} A_i(n) = o\left(\sum_{i \in I(n)} A_i(n) \right),$$

where $n \to \infty$. The sum over $i \in I(n) \setminus I_0(n)$ is then called the *tail* of $\sum A_i(n)$.

We shall determine index sets $I_Q(n)$, $I_C(n)$, $I_l(n)$, $I_d(n)$ that provide the main contributions to the sums in (3.12)–(3.14) over n_Q, n_C, l, and d, respectively. This will yield statements about the size of these values in the following way. For fixed $m = m(n)$, the index set $I_C(n)$, for example, will be of the type $[c_1 f(n), c_2 f(n)]$ for certain constants $0 < c_1 < c_2$ and a certain function $f = f(n)$. This implies that if $G = G_g(n, m)$, then whp $n_C \in I_C(n)$ and thus $n_C = \Theta(f)$.

The main challenge is to find the 'optimal' intervals $I_Q(n)$, $I_C(n)$, $I_l(n)$, $I_d(n)$ in view of Definition 3.3.3 in the sense that they should be a) large enough so as to provide the main contribution and b) as small as possible so as to yield stronger concentration results. To achieve these two antipodal goals is a difficult task whose solution will differ from case to case. In order to prove that a given interval indeed provides the main contribution to a sum, we bound the tail of the sum using various complementary methods including maximising techniques (e.g. Lemmas 3.4.8, 3.4.13, 3.4.16 and 3.4.17), Chernoff bounds (Lemmas 3.4.8 and 3.4.9), and approximations by integrals (Lemmas 3.4.14 and 3.4.19).

Determining the main contributions to (3.12), (3.13), and (3.14) will yield structural statements like the typical order of the complex part, the core, and the kernel of $G = G_g(n,m)$. In order to derive the component structure of G, we further apply combinatorial techniques like double counting (e.g. Theorem 3.1.6 and Lemma 3.4.5) and probabilistic methods such as Markov's inequality (Theorem 3.5.4).

3.4. Kernels, cores, and complex parts

For the remainder of the chapter, let $n, m, n_Q, n_C, l, d \in \mathbb{N}$ be such that $m = m(n) \leq (1+o(1))n$ and such that n_Q, n_C, l, and d are admissible (in terms of Definition 3.3.2). Furthermore, set $n_U = n - n_Q$ and $m_U = m - n_Q - l$.

The aim of this section is to determine the main contributions (in the sense of Definition 3.3.3) to the sums in (3.12), (3.13), and (3.14). In other words, we derive the typical orders of the complex part and the core of $G = G_g(n,m)$, as well as the excess and the deficiency of G. These orders will be the main ingredients for the proofs of Theorems 3.1.5–3.1.7. For all results in this section, we defer the proofs to Section 3.7.

3.4.1. **Kernels.** Throughout this section, we assume $l \geq 1$. As a basis of our analysis of (3.12), (3.13), and (3.14), we first have to determine the sum $|\mathcal{K}_g(2l-d, 3l-d)|$ of weights of the multigraphs in $\mathcal{K}_g(2l-d, 3l-d)$. We start with the case when the kernel is cubic (or equivalently, $d = 0$). The number of cubic kernels was determined in [**45**] by Fang, Kang, Sprüssel, and the author of this thesis.

Theorem 3.4.1 ([**45**]). *The number of cubic multigraphs with $2l$ vertices and $3l$ edges embeddable on $\mathbb{S}_g$, weighted by their compensation factor, is given by*

$$|\mathcal{K}_g(2l, 3l)| = \left(1 + O\left(l^{-1/4}\right)\right) e_g l^{5g/2-7/2} \gamma_K^{2l} (2l)!\,,$$

where $\gamma_K = \frac{79^{3/4}}{54^{1/2}} \approx 3.606$ and $e_g > 0$ is a constant depending only on g.

The number of *connected* cubic kernels will be of interest as well.

Theorem 3.4.2 ([**45**]). *The number of* connected *multigraphs in $\mathcal{K}_g(2l, 3l)$, weighted by their compensation factor, is*

$$\left(1 + O\left(l^{-1/4}\right)\right) c_g l^{5g/2-7/2} \gamma_K^{2l} (2l)!\,,$$

where γ_K is as in Theorem 3.4.1 and $c_g > 0$ is a constant depending only on g.

In particular, Theorems 3.4.1 and 3.4.2 imply that $K_g(2l, 3l)$ is connected with probability tending to $\frac{c_g}{e_g} > 0$; in other words, the probability that a random cubic (planar) kernel is connected is bounded away from zero, in contrast to the well-known fact that a random cubic general (not necessarily planar) kernel is connected whp (see e.g. [**77**, Lemma 1(i)]).

Before we consider kernels with non-zero deficiency, we shall look at the structure of cubic kernels. We aim to find the giant component of $G_g(n,m)$ and prove that it is complex, hence finding the giant component of the kernel would be a basis for a complex giant component in $G_g(n,m)$. Moreover, we would like this giant component to have genus g. The following result from [**45**] provides us with a component of genus g in a cubic kernel.

Lemma 3.4.3 ([**45**]). *If $g \geq 1$, then $K_g(2l, 3l)$ whp has one component of genus g and all its other components are planar.*

Intuitively, the non-planar component provided by Lemma 3.4.3 should be the largest component of the kernel, ideally even large enough to be the giant component. The following result shows that this component indeed covers almost all vertices in the kernel.

Lemma 3.4.4. *Let $g \geq 1$. Denote by $\mathrm{pl}(G)$ the subgraph of $G = K_g(2l, 3l)$ consisting of all planar components. Then $|\mathrm{pl}(G)| = O_p(1)$. Furthermore, $|\mathrm{pl}(G)|$ is even and there exist constants $c^+, c^- \in \mathbb{R}^+$ such that for every fixed integer $i \geq 1$ and sufficiently large l,*

$$c^- i^{-7/2}\left(1-\frac{i}{l}\right)^{5g/2-7/2} \leq \mathbb{P}\left[|\mathrm{pl}(G)| = 2i\right] \leq c^+ i^{-7/2}\left(1-\frac{i}{l}\right)^{5g/2-7/2}. \tag{3.15}$$

For the case $g = 0$, [**66**, Lemma 2] provides an analogous statement to (3.15) for the number of vertices outside the giant component of $K_0(2l, 3l)$.

Let us now look at general (not necessarily cubic) kernels. For such kernels, we are not able to give a precise formula for their number, but we can bound their number by comparing them to cubic kernels via a double counting argument.

Lemma 3.4.5. *Let $k \in \mathbb{N}$ be fixed. For $K \in \mathcal{K}_g$, denote by*

(i) $\mathcal{P}_1$ the property that K has precisely k components;

(ii) $\mathcal{P}_2$ the property that, if $g \geq 1$, then each component of K has genus strictly smaller than g.

For $i = 1, 2$, denote by $\mathcal{K}_g(n_K, m_K; \mathcal{P}_i)$ the subclass of $\mathcal{K}_g(n_K, m_K)$ of kernels that have property $\mathcal{P}_i$. Then

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \leq \frac{6^d}{d!} \quad \text{and} \quad \frac{|\mathcal{K}_g(2l-d, 3l-d; \mathcal{P}_i)|}{|\mathcal{K}_g(2l, 3l; \mathcal{P}_i)|} \leq \frac{6^d}{d!} \text{ for } i = 1, 2.$$

If in addition $d \leq \frac{2}{7}l$, then also

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \geq \frac{1}{216^d d!} \quad \text{and} \quad \frac{|\mathcal{K}_g(2l-d, 3l-d; \mathcal{P}_i)|}{|\mathcal{K}_g(2l, 3l; \mathcal{P}_i)|} \geq \frac{1}{216^d d!} \text{ for } i = 1, 2.$$

Lemma 3.4.5 has two main applications. On one hand, together with Theorem 3.4.1, Lemma 3.4.5 provides a way to bound the value $|\mathcal{K}_g(2l-d, 3l-d)|$ in (3.14). On the other hand, Lemma 3.4.5 will also enable us to extend the structural results from Lemmas 3.4.3 and 3.4.4 to kernels with a fixed constant deficiency d (see Theorem 3.5.4).

3.4.2. **Core and deficiency.** We first determine the main contributions to the sums in (3.13) and (3.14). By definition, $|\mathcal{Q}_g(0,0)| = 1$. Thus, throughout this section we will assume that both $n_Q \geq 1$ and $l \geq 1$ (recall that $l = 0$ if and only if $n_Q = 0$). Observe that (3.13), (3.14), and the identity

$$\binom{n_Q}{n_C}\binom{n_C}{2l-d} = \frac{(n_Q)_{n_C}}{(2l-d)!(n_C-2l+d)!}$$

imply that

$$|\mathcal{Q}_g(n_Q, n_Q + l)| = \sum_{n_C, d} \frac{(n_Q)_{n_C} |\mathcal{K}_g(2l-d, 3l-d)| \varphi_{n_C, l, d}\, n_C n_Q^{n_Q - n_C - 1}}{(2l-d)!(n_C - 2l + d)!}. \tag{3.16}$$

The factor $|\mathcal{K}_g(2l-d,3l-d)|$ in (3.16) can be bounded using Theorem 3.4.1 and Lemma 3.4.5. The term $\varphi_{n_C,l,d}$, however, is still unknown. Recall that this value denotes the average number, over all $K \in \mathcal{K}_g(2l-d,3l-d)$, of different ways to subdivide the edges of K that result in a core with n_C vertices and n_C+l edges.

Lemma 3.4.6. *There exists a function $\nu = \nu(n_C,l,d)$ such that*

$$\varphi_{n_C,l,d} = (n_C - 2l + d)! \binom{n_C + \nu l - 1}{3l - d - 1}$$

and $-5 \le \nu \le 1$.

Let us now determine the value of the sum in (3.16) over n_C, as well as its main contribution. To this end, we apply Lemmas 3.4.5 and 3.4.6 to (3.16), gather all parts of the equation that depend on n_C, and denote the sum over these values by Σ_C.

Lemma 3.4.7. *There exists a function $\tau = \tau(l,d)$ such that*

(i) $\frac{1}{216} \le \tau \le 6$ for all $0 \le d \le \lfloor \frac{2l}{7} \rfloor$;

(ii) $0 \le \tau \le 6$ for all $\lfloor \frac{2l}{7} \rfloor < d \le 2l$;

and

$$|\mathcal{Q}_g(n_Q, n_Q + l)| = n_Q^{n_Q - 1} \frac{|\mathcal{K}_g(2l,3l)|}{(2l)!} \sum_{d=0}^{2l} \binom{2l}{d} \frac{\tau^d}{(3l-d-1)!} \Sigma_C, \tag{3.17}$$

where

$$\Sigma_C = \Sigma_C(n_Q,l,d) := \sum_{n_C} \frac{(n_Q)_{n_C}}{n_Q^{n_C}} n_C (n_C + \nu l - 1)_{3l-d-1}. \tag{3.18}$$

The strategy to determine the main contribution to Σ_C is roughly as follows. Using inequalities from Section 3.2.3, we bound $\Sigma_C(n_Q,l,d)$ from above by a sum of the type

$$\sum_{n_C} \exp(A(n_Q,n_C,l,d)).$$

The derivative of $A(n_Q,n_C,l,d)$ with respect to n_C will show to have a zero at $n_C = (1+o(1))\overline{n}_C$, where

$$\overline{n}_C = \sqrt{n_Q(3l-d)}.$$

We then substitute $n_C = \overline{n}_C + r$ and prove that the resulting sum—up to a scaling factor—corresponds to a normally distributed random variable to which the Chernoff bound (3.9) applies. Finally, for n_C from the range of the main contribution to the upper bound, we derive a similar lower bound, which will enable us to derive the main contribution to Σ_C.

Lemma 3.4.8. *Let $f_C = f_C(n_Q,l,d)$ be such that*

$$\Sigma_C(n_Q,l,d) = \sqrt{n_Q} \left(\frac{n_Q(3l-d)}{e} \right)^{(3l-d)/2} \exp(f_C). \tag{3.19}$$

(i) There exist constants $a_C^+, b_C^+ \in \mathbb{R}$ such that

$$f_C \le a_C^+ + b_C^+ \sqrt{\frac{l^3}{n_Q}}.$$

(ii) For every function $\epsilon(n_Q) = o(1)$, there exist constants $N_Q \in \mathbb{N}$, $a_C^-, b_C^- \in \mathbb{R}$ such that whenever $n_Q \ge N_Q$ and $\frac{7}{2}d \le l \le \epsilon n_Q$, then

$$f_C \ge a_C^- + b_C^- \sqrt{\frac{l^3}{n_Q}}.$$

(iii) For every $0 < \delta < \frac{1}{2}$, whenever $n_Q, l \to \infty$ and $\frac{7}{2}d \le l \le \epsilon n_Q$, where $\epsilon = \epsilon(n_Q) = o(1)$ is given, the main contribution to Σ_C is provided by

$$I_C^\delta(n_Q, l, d) := \left\{k \in \mathbb{N} \,\middle|\, |k - \overline{n}_C| < \delta \overline{n}_C\right\}.$$

Our next aim is to analyse the sum over d in (3.17). To this end, observe that for

$$\Sigma_d = \Sigma_d(n_Q, l) := \sum_d \binom{2l}{d} \frac{(3l-d)^{(3l-d+2)/2} e^{d/2} \tau^d}{(3l-d)! n_Q^{d/2}} \exp(f_C), \tag{3.20}$$

(3.17) and (3.19) yield

$$|\mathcal{Q}_g(n_Q, n_Q + l)| = \frac{n_Q^{n_Q + 3l/2 - 1/2} |\mathcal{K}_g(2l, 3l)|}{e^{3l/2}(2l)!} \Sigma_d. \tag{3.21}$$

We determine the value of Σ_d, as well as its main contribution, in a similar fashion as for Σ_C.

Lemma 3.4.9. *Let $f_d = f_d(n_Q, l)$ be such that*

$$\Sigma_d = (3l)^{-(3l-1)/2} e^{3l} \exp(f_d). \tag{3.22}$$

(i) There exist constants $a_d^+ \in \mathbb{R}$ and $b_d^+ \in \mathbb{R}$ such that

$$f_d \le a_d^+ + b_d^+ \sqrt{\frac{l^3}{n_Q}}.$$

(ii) For every function $\epsilon(n_Q) = o(1)$, there exist constants $N_Q \in \mathbb{N}$ and $a_d^-, b_d^- \in \mathbb{R}$ such that

$$f_d \ge a_d^- + b_d^- \sqrt{\frac{l^3}{n_Q}},$$

whenever $n_Q \ge N_Q$ and $l \le \epsilon n_Q$.

(iii) There exists a constant $\beta_d^+ \in \mathbb{R}^+$ such that for $n_Q, l \to \infty$ and $l = o(n_Q)$, the main contribution to Σ_d is provided by

(a) $I_d(n_Q, l) := \{0\}$ if $l = o(n_Q^{1/3})$;

(b) $I_d^h(n_Q, l) := \{k \in \mathbb{N} \mid k \le h(n_Q)\}$ for any fixed function $h = h(n_Q) = \omega(1)$ if $l = \Theta(n_Q^{1/3})$;

(c) $I_d(n_Q, l) := \left\{k \in \mathbb{N} \mid k \le \beta_d^+ \sqrt{\frac{l^3}{n_Q}}\right\}$ if $l = \omega(n_Q^{1/3})$.

Interpreted in a probabilistic sense, Lemmas 3.4.8 and 3.4.9 immediately yield the typical order of a core of a complex graph, as well as the typical deficiency.

Corollary 3.4.10. *For every function $\epsilon(n_Q) = o(1)$, if $n_Q, l \to \infty$ and $l \le \epsilon n_Q$, then whp $Q = Q_g(n_Q, n_Q + l)$ has a core with $\sqrt{3 n_Q l}(1 + o(1))$ vertices. Furthermore, the deficiency of Q is given by*

$$\mathrm{d}(Q) = \begin{cases} 0 \text{ whp} & \text{if } l = o(n_Q^{1/3}), \\ O_p(1) & \text{if } l = \Theta(n_Q^{1/3}), \\ O\left(\sqrt{\frac{l^3}{n_Q}}\right) \text{whp} & \text{if } l = \omega(n_Q^{1/3}). \end{cases}$$

Observe that Corollary 3.4.10 requires n_Q and l to be growing and l to be of smaller order than n_Q. We shall later see that this will whp be the case for the complex part of $G_g(n, m)$.

In addition to Corollary 3.4.10, which tells us the deficiency and the order of the core of $Q_g(n_Q, n_Q + l)$, Lemma 3.4.9 also enables us to express the number of complex graphs that are embeddable on $\mathbb{S}_g$.

Corollary 3.4.11. *For all positive admissible values n_Q, l, we have*

$$|\mathcal{Q}_g(n_Q, n_Q+l)| = \frac{n_Q^{n_Q+3l/2-1/2}|\mathcal{K}_g(2l,3l)|e^{3l/2}}{(3l)^{(3l-1)/2}(2l)!}\exp(f_d).$$

This finalises our analysis of (3.13) and (3.14).

3.4.3. **Complex part and excess.** In this section we derive the main contribution to the double sum (3.12) (with respect to the summation indices n_Q and l). In the previous section, we had to distinguish the cases $n_Q = 0$ and $n_Q > 0$ in order to determine the number of complex graphs. Similarly, it will turn out that our asymptotic formulas will be quite different depending on whether the number $m_U = m - n_Q - l$ of edges *outside* the complex part is zero or not. In order to keep expressions simple, we will deal with the special cases $n_Q = 0$ and $m_U = 0$ separately.

To this end, define $\mathcal{G}_g^*(n,m)$ to be the subclass of $\mathcal{G}_g(n,m)$ consisting of all graphs for which the complex part is non-empty and the non-complex part has at least one edge. After bounding $|\mathcal{G}_g^*(n,m)|$, we shall see that the two special cases $n_Q = 0$ and $m_U = 0$ are 'rare' in the sense that almost all graphs in $\mathcal{G}_g(n,m)$ are also in $\mathcal{G}_g^*(n,m)$.

Lemma 3.4.12. *For every $m = m(n)$ as in Theorem 3.1.5(iii), Theorem 3.1.6, or Theorem 3.1.7 we have*

$$|\mathcal{G}_g(n,m) \setminus \mathcal{G}_g^*(n,m)| = o\big(|\mathcal{G}_g^*(n,m)|\big).$$

By Lemma 3.4.12, we can determine the main contribution to (3.12) by deriving the main contribution to the corresponding sum for $|\mathcal{G}_g^*(n,m)|$, namely

$$|\mathcal{G}_g^*(n,m)| = \sum_{n_Q,l}\binom{n}{n_Q}|\mathcal{Q}_g(n_Q,n_Q+l)|\cdot|\mathcal{U}(n_U,m_U)|, \tag{3.23}$$

where n_Q and l take all admissible values with $n_Q > 0$ and $m_U > 0$.

In order to analyse (3.23), we derive an upper bound for the sum over n_Q and subsequently also for the sum over l. These upper bounds indicate which intervals $I_Q(n)$ and $I_l(n)$ for n_Q and l, respectively, 'should' provide the main contribution to (3.23). For n_Q and l from these intervals, we then derive lower bounds and prove that the lower bound for the sum over $n_Q \in I_Q(n)$ and $l \in I_l(n)$ is much larger than the tails of the upper bound, thus proving that the main contribution to (3.23) is indeed provided by $I_Q(n)$ and $I_l(n)$.

Applying (3.6), Corollary 3.4.11, Lemma 3.3.1, and Theorem 3.4.1 to (3.23), we have

$$|\mathcal{G}_g^*(n,m)| = \Theta(1)n^{n+\frac{1}{2}}\left(\frac{e}{2}\right)^m \sum_l l^{\frac{5g}{2}-3-\frac{3l}{2}}\phi^l \sum_{n_Q}\rho(n_U,m_U)\psi(n_Q,l), \tag{3.24}$$

where $\phi = 2\sqrt{e}\gamma_K^2 3^{-\frac{3}{2}}$ and

$$\psi(n_Q,l) = \left(\frac{2}{e}\right)^{n_Q} n_Q^{\frac{3l}{2}-1} n_U^{2m_U-n_U-\frac{1}{2}} m_U^{-m_U-\frac{1}{2}}\exp(f_d). \tag{3.25}$$

Consider the sum

$$\Sigma_Q = \Sigma_Q(n,m,l) := \sum_{n_Q}\rho(n_U,m_U)\psi(n_Q,l),$$

where we sum over all values of n_Q that are admissible in $\mathcal{G}_g^*(n,m)$. We shall see in Lemma 3.4.19 that for fixed $l > 0$, the main contribution to Σ_Q is centred around

$$\overline{n}_Q = 2m - n - 2l.$$

The corresponding numbers of vertices and edges in the non-complex components are given by

$$\overline{n}_U = 2(n-m+l) \qquad \text{and} \qquad \overline{m}_U = n-m+l.$$

The bounds for Σ_Q will depend on whether l is 'small' or 'large', more precisely, whether

$$9\overline{m}_U^2\left(\frac{3l}{2}-1\right) \le \overline{n}_Q^3 \tag{3.26}$$

is satisfied (if so, l is considered small) or not (if so, l is large).

Lemma 3.4.13. *Define $M_Q = M_Q(n,m,l)$ by*

$$M_Q = \begin{cases} \left(\frac{2}{e}\right)^{2m-n} \overline{n}_Q^{\frac{3l}{2}-1}\overline{m}_U^{-\overline{m}_U-1} & \text{if (3.26) holds,} \\ \left(\frac{2}{e}\right)^{2m-n} l^{\frac{l}{2}-\frac{1}{3}}\overline{m}_U^{-\overline{m}_U+l-\frac{5}{3}} & \text{otherwise.} \end{cases}$$

Then

$$\Sigma_Q \le n^{\frac{3}{2}}\exp(O(l))M_Q. \tag{3.27}$$

Furthermore, for every fixed positive valued function $\epsilon = \epsilon(n) = o(1)$ and every $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$

$$\Sigma_Q \le \Theta(1)n^{\frac{3}{2}}\left(\frac{e}{2}\right)^{2l}(1+\delta)^l M_Q, \tag{3.28}$$

whenever

$$9\overline{m}_U^2\left(\frac{3l}{2}-1\right) \le \epsilon\overline{n}_Q^3. \tag{3.29}$$

For the case that m is larger than $\frac{n}{2}$ by only a small margin, we prove a stronger bound with the help of Lemma 3.3.1(iii) and a more careful analysis of the sums involved.

Lemma 3.4.14. *Let $m = (1+\lambda n^{-1/3})\frac{n}{2}$ with $\lambda = o(n^{1/12})$ and $\lambda \to \infty$. Then we have*

$$\Sigma_Q \le \lambda n^{\frac{2}{3}}\exp(O(l))M_Q. \tag{3.30}$$

In Lemmas 3.4.13 and 3.4.14, the exact bound depends on whether (3.26) is satisfied or violated. Correspondingly, we set

$$\Sigma_l := \sum_l l^{\frac{5g}{2}-3-\frac{3l}{2}}\phi^l\Sigma_Q(n,m,l),$$

where l takes all admissible values for which (3.26) holds, and

$$\tilde{\Sigma}_l := \sum_l l^{\frac{5g}{2}-3-\frac{3l}{2}}\phi^l\Sigma_Q(n,m,l),$$

where l takes all admissible values for which (3.26) is violated. Heuristically, Σ_l should be the larger of the two sums, because $l^{-\frac{3l}{2}}$ should be the dominating term and this term is small when l is large (which is the case when (3.26) is violated). We shall see in Lemma 3.4.17 that $\tilde{\Sigma}_l$ is indeed negligible.

Accordingly, we focus on Σ_l for the moment. Applying the bound (3.27), we have $\Sigma_l \le \Sigma_l^+$, where

$$\Sigma_l^+ = \left(\frac{2}{e}\right)^{2m-n}\sum_l l^{\frac{5g}{2}-3-\frac{3l}{2}}\phi^l\overline{n}_Q^{\frac{3l}{2}-1}\overline{m}_U^{-\overline{m}_U}\exp(O(l)).$$

The main contribution to Σ_l^+ should be centred around its largest summand. We approximate the largest summand by ignoring polynomial terms and replacing the term $\exp(O(l))$ by $(e/2)^{2l}$ (which we saw in Lemma 3.4.13 to be a good approximation when (3.29) holds). The remaining terms attain their largest value at the unique solution l_0 of the equation

$$l_0 = \frac{\phi^{2/3}(2m-n-2l_0)}{e^{1/3}2^{4/3}(n-m+l_0)^{2/3}}\,, \qquad m-n < l_0 < m-\frac{n}{2}\,. \tag{3.31}$$

Before we proceed to prove that the main contribution to $|\mathcal{G}_g^*(n,m)|$ is indeed provided by l 'close to' l_0 (and thus the 'typical excess' of a graph in $\mathcal{G}_g^*(n,m)$ is close to l_0), let us take a closer look at the value l_0. We introduce the following notation for the seven different cases of $m(n)$ from our main results.

1Sub: $m(n) = (1+\lambda n^{-1/3})\frac{n}{2}$ with $\lambda = \lambda(n) = o(n^{1/3})$ and $\lambda \to -\infty$, the *first subcritical regime*;

1Crit: $m(n) = (1+\lambda n^{-1/3})\frac{n}{2}$ with $\lambda \to c_\lambda \in \mathbb{R}$, the *first critical regime*;

1Sup: $m(n) = (1+\lambda n^{-1/3})\frac{n}{2}$ with $\lambda = o(n^{1/3})$ and $\lambda \to \infty$, the *first supercritical regime*;

Int: $m(n) = \mu\frac{n}{2}$ with $\mu = \mu(n) \to c_\mu \in (1,2)$, the *intermediate regime*;

2Sub: $m(n) = (2+\zeta n^{-2/5})\frac{n}{2}$ with $\zeta = \zeta(n) = o(n^{2/5})$ and $\zeta \to -\infty$, the *second subcritical regime*;

2Crit: $m(n) = (2+\zeta n^{-2/5})\frac{n}{2}$ with $\zeta \to c_\zeta \in \mathbb{R}$, the *second critical regime*;

2Sup: $m(n) = (2+\zeta n^{-2/5})\frac{n}{2}$ with $\zeta = o((\log n)^{-2/3}n^{2/5})$ and $\zeta \to \infty$, the *second supercritical regime*.

The union of the first three cases will also be referred to as *the first phase transition*, while the union of the last three cases is called *the second phase transition*. In 1Sub and 1Crit, our main results will follow from well-known results. Thus, for the rest of this section, we assume that we are in one of the other five cases.

The definition of l_0 immediately yields its asymptotic order.

Lemma 3.4.15. *The value l_0 defined in* (3.31) *is positive and whp satisfies*

$$l_0 = \begin{cases} \Theta(\lambda) & \text{in 1Sup,} \\ \Theta(n^{1/3}) & \text{in Int,} \\ \Theta\big(|\zeta|^{-2/3}n^{3/5}\big) & \text{in 2Sub,} \\ \Theta(n^{3/5}) & \text{in 2Crit,} \\ \frac{1}{2}\zeta n^{3/5} + \Theta\Big(\zeta^{-3/2}n^{3/5}\Big) & \text{in 2Sup.} \end{cases}$$

Furthermore, in 2Crit, we have $0 < l_0 - \frac{1}{2}\zeta n^{3/5} = \Theta\big(n^{3/5}\big)$.

In general, l_0 will not be an integer and thus in particular not admissible. Set

$$l_1 := \lceil l_0 \rceil\,.$$

Now (3.31) and Lemma 3.4.15 yield

$$l_1 = (1+o(1))\frac{\phi^{2/3}(2m-n-2l_1)}{e^{1/3}2^{4/3}(n-m+l_1)^{2/3}}\,. \tag{3.32}$$

From Lemma 3.4.15 we deduce that all l 'close to' l_1 are admissible and use this fact to derive a lower bound on $|\mathcal{G}_g^*(n,m)|$.

Lemma 3.4.16. *Let $c > 1$ be given and suppose that $l \in \mathbb{N}$ with $\frac{l_0}{c} \le l \le c\,l_0$ and*

$$0 < \overline{m}_U = \begin{cases} \Theta(n^{3/5}) & \text{in 2Crit,} \\ \Theta\Big(\zeta^{-3/2}n^{3/5}\Big) & \text{in 2Sup.} \end{cases}$$

Then l is admissible. Furthermore, there exists

$$\tilde{n}_Q = \overline{n}_Q + O\Big(\overline{m}_U^{2/3}\Big)$$

such that

$$\Sigma_Q(n,m,l) \geq \Theta(1)\Big(\frac{e}{2}\Big)^{2l}\overline{m}_U^{\frac{2}{3}}\exp(f_d(\tilde{n}_Q,l))M_Q(n,m,l).$$

In particular, for every $\delta > 0$ and n large enough,

$$|\mathcal{G}_g^*(n,m)| \geq \Theta(1)n^{n+\frac{1}{2}}\Big(\frac{e}{2}\Big)^{m+2l_1}l_1^{-\frac{3l_1}{2}}\phi^{l_1}(n-m+l_1)^{\frac{2}{3}}(1-\delta)^{l_1}M_Q(l_1,n,m).$$

The bound in Lemma 3.4.16 enables us to show that $\tilde{\Sigma}_l$ is negligible.

Lemma 3.4.17. *For $n \to \infty$, we have*

$$n^{n+\frac{1}{2}}\Big(\frac{e}{2}\Big)^{m}\tilde{\Sigma}_l = o\big(|\mathcal{G}_g^*(n,m)|\big).$$

Lemma 3.4.17 implies that the main contribution to $|\mathcal{G}_g^*(n,m)|$ is provided by the same intervals that provide the main contribution to Σ_l. After determining lower bounds for the summands in (3.23), our aim is to determine the 'optimal' intervals in view of Definition 3.3.3. In other words, we are looking for intervals $I_Q(n)$ and $I_l(n)$ such that a) the lower bound, summed over $I_Q(n)$ and $I_l(n)$, is much larger than the 'tail' of the upper bound and b) $I_Q(n)$ and $I_l(n)$ are as small as possible. To that end, in the second phase transition, we need an auxiliary result that tells us that f_d (defined in Lemma 3.4.9) does not change 'too much' if we fix l and change n_Q by a small fraction.

Lemma 3.4.18. *Suppose that $m(n)$ lies in* 2Sub, 2Crit, *or* 2Sup. *Let positive valued functions $h = h(n) = \omega(1)$ and $\epsilon = \epsilon(n) = o(1)$ satisfying $h\epsilon = \omega(1)$ be given. Then for all $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $n_Q = (1+o(1))n$, and $h \leq l \leq \frac{n_Q}{h}$, we have*

$$|f_d((1-\epsilon)n_Q,l) - f_d(n_Q,l)| \leq \delta\epsilon l.$$

With this auxiliary result, we can now determine the desired intervals $I_Q(n)$ and $I_l(n)$ that provide the main contribution to $|\mathcal{G}_g^*(n,m)|$.

Lemma 3.4.19. *There exist constants $\beta_l^+, \beta_l^- \in \mathbb{R}^+$ and functions $\vartheta_l^+, \vartheta_l^- : \mathbb{R} \to \mathbb{R}^+$, and $\eta_l^+, \eta_l^- : (1,2) \to \mathbb{R}^+$ with*

$$\beta_l^+ > \beta_l^-, \qquad \eta_l^+(x) > \eta_l^-(x), \qquad \vartheta_l^+(x) > \vartheta_l^-(x) > \frac{x}{2}$$

for all $x \in \mathbb{R}$ such that the following holds.

For every fixed function $h = h(n) = \omega(1)$, the main contribution to (3.24) is provided by $I_l(n)$ (for the sum over l) and $I_Q^h(n,l)$ (for the sum over n_Q), where

$$I_l(n) := \begin{cases} \big\{k \in \mathbb{N} \mid \beta_l^-\,\lambda \leq k \leq \beta_l^+\,\lambda\big\} & \text{in 1Sup},\\ \big\{k \in \mathbb{N} \mid \eta_l^-(c_\mu)n^{1/3} \leq k \leq \eta_l^+(c_\mu)n^{1/3}\big\} & \text{in Int},\\ \big\{k \in \mathbb{N} \mid \beta_l^-|\zeta|^{-2/3}n^{3/5} \leq k \leq \beta_l^+|\zeta|^{-2/3}n^{3/5}\big\} & \text{in 2Sub},\\ \big\{k \in \mathbb{N} \mid \vartheta_l^-(c_\zeta)n^{3/5} \leq k \leq \vartheta_l^+(c_\zeta)n^{3/5}\big\} & \text{in 2Crit},\\ \Big\{k \in \mathbb{N} \mid \beta_l^-\,\zeta^{-3/2}\,n^{3/5} \leq k - \frac{1}{2}\,\zeta\,n^{3/5} \leq \beta_l^+\,\zeta^{-3/2}\,n^{3/5}\Big\} & \text{in 2Sup},\end{cases}$$

and

$$I_Q^h(n,l) := \Big\{k \in \mathbb{N} \mid |k - \overline{n}_Q| \leq h\overline{m}_U^{2/3}\Big\}.$$

3.5. Internal structure

In the Section 3.4, we have determined the main contributions to $|\mathcal{G}_g^*(n,m)|$ and thus, by Lemma 3.4.12, also the main contributions to $|\mathcal{G}_g(n,m)|$. Interpreting these results in a probabilistic sense, we deduce the typical orders n_Q, n_C of the complex part and the core of $G = G_g(n,m)$, respectively, as well as its typical excess $\mathrm{ex}(G)$ and deficiency $\mathrm{d}(G)$. All results in this section are proved in Section 3.6.

The complex part, for instance, grows from order $\lambda\, n^{2/3}$ in the first supercritical regime to linear order in the intermediate regime. The number m_U of edges *outside* the complex part is about half the number n_U of *vertices* outside the complex part.

Theorem 3.5.1. *Let $G = G_g(n,m)$. Then n_Q, n_C, $\mathrm{ex}(G)$, and $\mathrm{d}(G)$ whp lie in the following ranges.*

	n_Q	n_C	$\mathrm{ex}(G)$	$\mathrm{d}(G)$
1Sup	$\lambda\, n^{2/3} + O_p(n^{2/3})$	$\Theta(\lambda\, n^{1/3})$	$\Theta(\lambda)$	0
Int	$(\mu-1)n + O_p(n^{2/3})$	$\Theta(n^{2/3})$	$\Theta(n^{1/3})$	$O_p(1)$

Furthermore,

$$m_U = \frac{n_U}{2} + O_p(n_U^{2/3}).$$

Observe that the ranges for n_Q, n_C, and $\mathrm{ex}(G)$ in 1Sup can be translated to the ones in Int by the substitution $\lambda = (\mu-1)n^{1/3}$ (or, equivalently, $\mu = 1 + \lambda\, n^{-1/3}$).

In the second phase transition, the complex part covers almost all vertices and thus, it is more convenient to consider the parameter $n_U = n - n_Q$ instead of n_Q.

Theorem 3.5.2. *Let $G = G_g(n,m)$. Then n_C, $\mathrm{ex}(G)$, and $\mathrm{d}(G)$ whp lie in the following ranges.*

	n_C	$\mathrm{ex}(G)$	$\mathrm{d}(G)$
2Sub	$\Theta(\lvert\zeta\rvert^{-1/3}n^{4/5})$	$\Theta(\lvert\zeta\rvert^{-2/3}n^{3/5})$	$O(\lvert\zeta\rvert^{-1}n^{2/5})$
2Crit	$\Theta(n^{4/5})$	$\Theta(n^{3/5})$	$O(n^{2/5})$
2Sup	$\Theta\left(\zeta^{1/2}\, n^{4/5}\right)$	$\frac{1}{2}\,\zeta\, n^{3/5} + \Theta\left(\zeta^{-3/2}\, n^{3/5}\right)$	$O\left(\zeta^{3/2}\, n^{2/5}\right)$

Furthermore, we have

$$\begin{aligned} n_U &= 2\,\mathrm{ex}(G) - \zeta\, n^{3/5} + O_p\left(\left(2\,\mathrm{ex}(G) - \zeta\, n^{3/5}\right)^{2/3}\right) \\ &= \begin{cases} \lvert\zeta\rvert n^{3/5} + \Theta\left(\lvert\zeta\rvert^{-2/3}n^{3/5}\right) + O_p\left(\lvert\zeta\rvert^{2/3}n^{2/5}\right) & \text{in 2Sub,} \\ \Theta\left(n^{3/5}\right) & \text{in 2Crit,} \\ \Theta\left(\zeta^{-3/2}\, n^{3/5}\right) & \text{in 2Sup,} \end{cases} \end{aligned}$$

and

$$m_U = \frac{n_U}{2} + O_p(n_U^{2/3}).$$

More generally than stated in Theorem 3.5.2, the formula

$$n_U = 2(n - m + \mathrm{ex}(G)) + O_p\left((n - m + \mathrm{ex}(G))^{2/3}\right),$$

which corresponds to the first expression for n_U in Theorem 3.5.2, holds for *all* ranges of m. In 1Sup, Int, and 2Sub, this formula for n_U consists of the main term $2(n-m)$, a shift by a lower order term $2\,\mathrm{ex}(G)$, and an error term. In 1Sup and Int, the excess is much smaller than the error term, which is why we could omit the corresponding summand in Theorem 3.5.1. In 2Sub, the excess becomes larger than the error term when $\zeta = o\left(n^{3/20}\right)$ (or, equivalently, for $m = n - o\left(n^{3/4}\right)$). For the same reason, the error term is negligible in 2Crit and 2Sup.

Observe that if we substitute ζ by a constant in the cases 2Sub and 2Sup of Theorem 3.5.2, we obtain the corresponding ranges in 2Crit. Moreover, the ranges in the case 2Sub can be translated to those in Int (see Theorem 3.5.1) by the substitution $\zeta = (\mu - 2)n^{2/5}$ (or, equivalently, $\mu = 2 + \zeta\, n^{-2/5}$).

As an immediate corollary of Theorems 3.5.1 and 3.5.2, we deduce the typical order and size of the kernel of $G = G_g(n, m)$.

Corollary 3.5.3. *The number n_K of vertices and $m_K = \frac{3}{2}n_K + \frac{1}{2}\,\mathrm{d}(G)$ of edges of the kernel of $G = G_g(n, m)$ lie in the following ranges whp.*

	n_K	$\mathrm{d}(G)$
1Sup	$\Theta(\lambda)$	0
Int	$\Theta(n^{1/3})$	$O_p(1)$
2Sub	$\Theta\big(\lvert\zeta\rvert^{-2/3} n^{3/5}\big)$	$O\big(\lvert\zeta\rvert^{-1} n^{2/5}\big)$
2Crit	$\Theta\big(n^{3/5}\big)$	$O\big(n^{2/5}\big)$
2Sup	$\zeta\, n^{3/5} + \Theta\Big(\zeta^{-3/2}\, n^{3/5}\Big) + O\Big(\zeta^{3/2}\, n^{2/5}\Big)$	$O\Big(\zeta^{3/2}\, n^{2/5}\Big)$

Theorems 3.5.1 and 3.5.2 and Corollary 3.5.3 tell us the orders of the complex part, the core, and the kernel. What we are ultimately looking for, however, are orders of components. Lemmas 3.4.3 and 3.4.4 cover the case of cubic kernels, which are precisely the kernels of $G_g(n, m)$ in 1Sup. However, we are not interested in the properties a kernel has if we pick it uniformly at random from the class of all kernels. We are rather looking for properties of the *kernel of* $G_g(n, m)$, where the randomness lies in $G_g(n, m)$. Clearly, we cannot expect the probability distribution on the class of kernels given by this construction to be uniform.

However, by a double counting argument, we prove that the aforementioned probability distribution does not differ 'too much' from the uniform distribution if we are in 1Sup or Int. From this, we use Markov's inequality (3.8) to deduce that in these regimes, the kernel K_G, the core C_G, and the complex part Q_G of $G = G_g(n, m)$ have a component of genus g that covers almost all vertices, while all other components are planar. Recall that $H_i(G')$ denotes the i-th largest component of a graph G'. Denote by $R(G')$ the graph $G' \setminus H_1(G')$.

Theorem 3.5.4. *Let $G = G_g(n, m)$, where $m = m(n)$ lies in* 1Sup *or* Int.

- *(i) K_G, C_G, and Q_G have the same number $k = O_p(1)$ of components;*
- *(ii) for every $i \geq 2$, the probability that K_G, C_G, and Q_G have at least i components is bounded away both from 0 and 1;*
- *(iii) whp $H_1(K_G)$ is the kernel of $H_1(C_G)$, which in turn is the core of $H_1(Q_G)$, and all three have genus g;*
- *(iv) whp $R(K_G)$, $R(C_G)$, and $R(Q_G)$ are planar;*
- *(v) $\lvert R(K_G)\rvert = O_p(1)$;*
- *(vi) $\lvert R(C_G)\rvert = O_p(n^{1/3})$;*
- *(vii) $\lvert R(Q_G)\rvert = O_p(n^{2/3})$.*

From Theorem 3.5.4, we deduce the typical order of the largest components of the complex part, the core, and the kernel of $G_g(n, m)$, respectively.

Corollary 3.5.5. *For $G = G_g(n, m)$, the largest components of the complex part Q_G, the core C_G, and the kernel K_G, respectively, have the following order.*

	$\lvert H_1(Q_G)\rvert$	$\lvert H_1(C_G)\rvert$	$\lvert H_1(K_G)\rvert$
1Sup	$\lambda\, n^{2/3} + O_p\big(n^{2/3}\big)$	$\Theta\big(\lambda\, n^{1/3}\big)$	$\Theta(\lambda)$
Int	$(\mu - 1)n + O_p\big(n^{2/3}\big)$	$\Theta\big(n^{2/3}\big)$	$\Theta\big(n^{1/3}\big)$

For the second phase transition, the proof method of Theorem 3.5.4 fails. For these cases, we prove the existence of the giant component by using double counting arguments.

Theorem 3.5.6. *Let $G = G_g(n, m)$, where $m = m(n)$ lies in* 2Sub, 2Crit, *or* 2Sup.

(i) The number of vertices in $H_1(Q_G)$ is

$$|H_1(Q_G)| = n_Q - O_p\Big(n_U^{2/3}\Big) = n_Q - \begin{cases} O_p\big(|\zeta|^{2/3} n^{2/5}\big) & \text{in } 2\textsc{Sub}, \\ O_p\big(n^{2/5}\big) & \text{in } 2\textsc{Crit}, \\ O_p\big(\zeta^{-1}\, n^{2/5}\big) & \text{in } 2\textsc{Sup}; \end{cases}$$

(ii) whp $H_1(Q_G)$ has genus g, while all other complex components are planar.

In addition to the order of $H_1(Q_G)$, we can also determine its size, which plays a key role in the proof of Theorem 3.1.8. In the first phase transition and in the intermediate regime, this is an easy corollary of Theorems 3.5.1 and 3.5.4. In the second phase transition, we need an additional double counting argument.

Corollary 3.5.7. *For $G = G_g(n, m)$, the size of the largest complex component $H_1(Q_G)$ is*

$$|H_1(Q_G)| + \mathrm{ex}(G) - r(G),$$

where $r(G) \geq 0$ and whp

$$r(G) = o(\mathrm{ex}(G)) \qquad \text{and} \qquad r(G) = O_p\Big(n_U^{2/3}\Big).$$

By definition, the excess of a graph is the difference between the size and the order of its complex part. Consequently, one would expect the excess $r(G)$ of $R(Q_G)$ to be around

$$\frac{|R(Q_G)|}{n_Q}\,\mathrm{ex}(G),$$

which is $O_p(1)$ in all regimes. In 1Sup and Int, Theorem 3.5.4 indeed yields this expected bound $r(G) = O_p(1)$. In the second phase transition, our proof method only provides the weaker bounds from Corollary 3.5.7. However, together with the fact that $H_1(Q_G)$ is indeed the giant component of G (which will follow from Theorems 3.1.5 to 3.1.7), these weaker bounds suffice to derive the size of $H_1(G)$.

Corollary 3.5.8. *For $G = G_g(n, m)$, the size of the giant component $H_1(G)$ is*

$$|H_1(G)| + \begin{cases} \Theta(\lambda) & \text{in } 1\textsc{Sup}, \\ \Theta\big(n^{1/3}\big) & \text{in } \textsc{Int}, \\ \Theta\big(|\zeta|^{-2/3} n^{3/5}\big) & \text{in } 2\textsc{Sub}, \\ \Theta\big(n^{3/5}\big) & \text{in } 2\textsc{Crit}, \\ \frac{1}{2}\,\zeta\, n^{3/5} + \Theta\Big(\zeta^{-3/2}\, n^{3/5}\Big) & \text{in } 2\textsc{Sup}. \end{cases}$$

3.6. Proofs of main results

In this section, we prove the main results (Theorems 3.1.5 to 3.1.8) of this chapter, as well as the structural results from Section 3.5.

3.6.1. **Proof of Theorem 3.1.5.** In 1Sub, i.e. $m = (1 + \lambda\, n^{-1/3})\frac{n}{2}$ with $\lambda = o(n^{1/3})$ and $\lambda \to -\infty$, the Erdős–Rényi random graph $G(n, m)$ whp is embeddable on $\mathbb{S}_g$ by Lemma 3.3.1. Thus, Theorem 3.1.5(i) follows immediately from Theorem 3.1.1(i).

In 1Crit, i.e. $\lambda \to c_\lambda \in \mathbb{R}$, Lemma 3.3.1(ii) implies that $G(n, m)$ has no complex components with positive probability. Thus, Theorem 3.1.1(ii) yields the second statement of Theorem 3.1.5(ii). By [**79**, Theorem 5], the probability that $G(n, m)$ is planar, and thus in particular embeddable on $\mathbb{S}_g$, is larger than the probability that $G(n, m)$ has no complex components. Hence the first statement of Theorem 3.1.5(ii) follows as well.

Finally, consider 1Sup, i.e. $\lambda = o(n^{1/3})$ and $\lambda \to \infty$. By Corollary 3.5.5, the largest component $H_1(Q_G)$ of the complex part of $G = G_g(n, m)$ has order

$\lambda n^{2/3} + O_p(n^{2/3})$. Moreover, by Theorem 3.5.4(iii), (iv), and (vii), $H_1(Q_G)$ whp has genus g, while all other complex components are planar and have order $O_p(n^{2/3})$. By Theorem 3.5.4(i) and (ii), it remains to show that for each $k \geq 1$, the k-th largest non-complex component has order $\Theta_p(n^{2/3})$. By Lemma 3.3.1(ii) and the fact that $m_U = \frac{n_U}{2} + O_p\left(n_U^{2/3}\right)$ by Theorem 3.5.1, there is a positive probability that $G(n_U, m_U)$ has no complex component and therefore the claim follows from Theorem 3.1.1(ii). □

3.6.2. **Proof of Theorem 3.1.6.** All statements about H_1 (i.e. that it has genus g, is complex, and has the claimed order), as well as the planarity of all other components, follow directly from Theorems 3.5.2 and 3.5.6. For the order of H_i, $i \geq 2$, observe that

$$m_U = \frac{n_U}{2} + O_p\left(n_U^{2/3}\right)$$

(by Theorem 3.5.2) and Theorem 3.1.1(ii) imply that the k-th largest non-complex component (for fixed $k \geq 1$) has order $\Theta_p\left(n_U^{2/3}\right)$. Now all complex components apart from H_1 have order $O_p\left(n_U^{2/3}\right)$ by Theorem 3.5.6(i) and thus, the claimed order of H_i follows by inserting the value of n_U from Theorem 3.5.2. □

3.6.3. **Proof of Theorem 3.1.7.** Analogously to the proof of the case 1Sup of Theorem 3.1.5, Theorem 3.1.7 follows from Theorem 3.1.1(ii), Lemma 3.3.1(ii), Theorem 3.5.4, and Corollary 3.5.5. □

Proof of Theorem 3.1.8. In 1Crit (that is, $m = \left(1 + \lambda n^{-1/3}\right)\frac{n}{2}$ with $\lambda \to c_\lambda \in \mathbb{R}$), the claim follows from the fact that

$$m_R = \frac{n_R}{2} + O_p\left(n_R^{2/3}\right)$$

holds for $G(n, m)$, because $G(n, m)$ is embeddable on $\mathbb{S}_g$ with positive probability (see Theorem 3.1.1(ii)).

For all other regimes, Theorem 3.1.8 follows from the order of $H_1(G)$ given in Theorems 3.1.5 to 3.1.7, the bound for its size in Corollary 3.5.7, and the values of $\mathrm{ex}(G)$ and n_U from Theorems 3.5.1 and 3.5.2. □

3.6.4. **Proof of Theorems 3.5.1 and 3.5.2.** The results on the excess and the order of the complex part follow from Lemma 3.4.19. Observe that $\mathrm{ex}(G) = o(n_Q)$ for all regimes and thus Corollary 3.4.10 is applicable, yielding the order n_C of the core and the deficiency $\mathrm{d}(G)$. Finally, by Lemma 3.4.19 we know that

$$n_U = 2(n - m + \mathrm{ex}(G)) + O_p\left((n - m + \mathrm{ex}(G))^{2/3}\right)$$

and

$$m_U = n - m + \mathrm{ex}(G) + O_p\left((n - m + \mathrm{ex}(G))^{2/3}\right),$$

which yields the last statements of Theorems 3.5.1 and 3.5.2. □

Proof of Corollary 3.5.3. Corollary 3.5.3 follows directly from Lemma 3.2.1 and the values of $\mathrm{ex}(G)$ and $\mathrm{d}(G)$ stated in Theorems 3.5.1 and 3.5.2. □

Proof of Theorem 3.5.4. Given a fixed kernel K, call a subdivision of K *good* if it is a simple graph (and thus a valid core). We first prove that the fraction of good subdivisions among all subdivisions of K is bounded away from zero.

To this end, suppose that K is a kernel with $2l-d$ vertices and $3l-d$ edges and that we want to subdivide its edges k times (with $k \geq 6l-2d$) in order to construct a core C with $k+2l-d$ vertices. We subdivide K in the following way. First, decide which labels the vertices of K should have in C; there are $\binom{k+2l-d}{2l-d}$ choices for this. Let I be the set of the remaining k labels. We recursively subdivide edges of K and assign the smallest remaining label in I to the new vertex. The number of choices increases by one in each recursion step and thus we have $(k+3l-d-1)_k$ choices in total. This way, we construct each subdivision precisely once. Hence the total number of subdivisions of K is

$$\binom{2l-d+k}{2l-d}(k+3l-d-1)_k\,.$$

In order to give a lower bound on the number of *good* subdivisions, we change our construction slightly by introducing a preliminary step. After choosing the labels for the vertices in K, we subdivide each edge of K twice and then choose labels from I for the new vertices; there are $(k)_{6l-2d}$ choices for this. After this step, we proceed as before, with the additional rule that an edge may only be subdivided if none of its end vertices is a vertex of K. Similar to our first construction, there are $(k-3l+d-1)_{k-6l+2d}$ choices for this part of the construction. Every graph obtained by this type of subdivision is simple and no graph is constructed more than once. Thus, the total number of good subdivisions is at least

$$\binom{2l-d+k}{2l-d}(k)_{6l-2d}(k-3l+d-1)_{k-6l+2d}\,.$$

The fraction of good subdivisions among all subdivisions of K is thus at least

$$\frac{(k)_{6l-2d}(k-3l+d-1)_{k-6l+2d}}{(k+3l-d-1)_k} \geq \left(\frac{k-3l+d}{k-6l+2d}\right)^{-6l+2d} \overset{(3.2)}{\geq} \exp\left(-\frac{2(3l-d)^2}{k-6l+2d}\right).$$

Substituting $l = \mathrm{ex}(G)$, $d = \mathrm{d}(G)$, and $k = n_C - 3l + d$ from Theorem 3.5.1 (and observing that these values satisfy $k \geq 6l-2d$ whp) yields that the fraction of good subdivisions is bounded away from zero.

To make this more precise, denote by $\mathrm{s}(K_G)$ the proportion of subdivisions of K_G that lie in $\mathcal{C}_g(n_C, n_C+l)$. We have shown that for every $\delta > 0$ there exists an $\varepsilon > 0$ such that

$$\begin{aligned} 1-\delta \leq \mathrm{s}(K_G) \leq 1 &\text{ whp in 1}\textsc{Sup}, \\ \varepsilon \leq \mathrm{s}(K_G) \leq 1 &\text{ with probability at least } 1-\delta \text{ in } \textsc{Int}. \end{aligned} \tag{3.33}$$

Recall the construction steps (C1)–(C3): the core C_G is constructed from K_G by subdividing edges; the complex part Q_G is obtained from C_G by attaching rooted trees to all vertices; adding trees and unicyclic components to Q_G yields G. Let X be an event that depends on the choice of $K \in \mathcal{K}_g$. From the above construction, (3.33), and the fact that the kernel of $G = G_g(n,m)$ has a growing number of vertices by Theorem 3.5.1, we deduce that

$$\varepsilon \leq \frac{\mathbb{P}\left[X \text{ holds for } K = K_G\right]}{\mathbb{P}\left[X \text{ holds for } K = K_g(2l-d, 3l-d)\right]} \leq \frac{1}{\varepsilon}, \tag{3.34}$$

provided that the denominator is non-zero.

To prove (i), observe that the kernel, the core, and the complex part of a graph have the same number k of components by construction. Lemmas 3.4.3 and 3.4.4 (for $g \geq 1$) and [**66**, Lemma 2] (for $g = 0$) tell us that the cubic kernel $K_g(2l, 3l)$ has $O_p(1)$ components. Thus by (3.34), we have $k = O_p(1)$ if the kernel is cubic, which is the case whp in 1Sup by Theorem 3.5.1. In Int, we have $\mathrm{d}(G) = O_p(1)$. Thus, we apply Lemma 3.4.5 and deduce that $k = O_p(1)$. By analogous arguments, we deduce (ii), (v), and the genus statements about K_G from (iii) and (iv).

The observation that subdividing edges (when constructing C_G) and attaching trees (constructing Q_G) does not change the genus of any component proves the remaining statements of (iv).

In order to prove (iii), (vi) and (vii), let A_K be any fixed component of K_G. Denote by A_C and A_Q the corresponding components of C_G and Q_G, respectively. Observe that

- in a random (not necessarily good) subdivision of the kernel, the expected number of subdivisions of any given edge e is $\frac{n_C}{n_K} - 1$;
- if we attach a rooted forest to the core in order to construct the complex part, the expected order of the tree attached to any given vertex v is $\frac{n_Q}{n_C}$.

By Theorem 3.5.1, we have $\frac{n_C}{n_K} = \Theta(n^{1/3})$ and $\frac{n_Q}{n_C} = \Theta(n^{1/3})$ whp. Therefore, (3.33) and Markov's inequality (3.8), applied to the random variables $|A_C|$ and $|A_Q|$, imply that

$$|A_C| = O_p(n^{1/3})|A_K| \qquad \text{and} \qquad |A_Q| = O_p(n^{1/3})|A_C|$$

for every *fixed* component A_K. On the other hand, there are $O_p(1)$ components, which proves (vi) and (vii). Together with the observation that A_K, A_C, and A_Q have the same genus, (iii) follows as well. □

Proof of Corollary 3.5.5. Corollary 3.5.5 is an immediate consequence of Theorems 3.5.1 and 3.5.4. □

Proof of Theorem 3.5.6. Let $m(n)$ be a function from the second phase transition, that is, $m(n) = (2 + \zeta n^{-2/5})\frac{n}{2}$ with $\zeta = \zeta(n) = o(n^{2/5})$. As usual, we denote the number $n - n_Q$ of vertices outside the complex part of a given graph $G \in \mathcal{G}_g(n,m)$ by n_U and the number of edges outside the complex part by m_U.

We first prove (i). To that end, for every $\delta > 0$, we need to find a constant c_δ so that $n_Q - |H_1(Q_G)| \leq c_\delta n_U^{2/3}$ with probability greater than $1 - \delta$ for sufficiently large n. Fix $\delta > 0$ and denote by $\mathcal{E}_1(n,m)$ the subclass of $\mathcal{G}_g(n,m)$ of those graphs G for which $n_Q - |H_1(Q_G)| > c_\delta n_U^{2/3}$ with some (sufficiently large) c_δ. We have to prove that $|\mathcal{E}_1(n,m)| < \delta|\mathcal{G}_g(n,m)|$ for sufficiently large n.

Suppose that there exists an infinite set $I \subset \mathbb{N}$ such that $|\mathcal{E}_1(n,m)| \geq \delta|\mathcal{G}_g(n,m)|$ for all $n \in I$. We use double counting in order to derive a contradiction from this assumption. Let $n \in I$ be fixed and pick a graph $G \in \mathcal{E}_1(n,m)$. Theorem 3.5.2 together with the assumption $|\mathcal{E}_1(n,m)| \geq \delta|\mathcal{G}_g(n,m)|$ yields that

$$m_U = m - m_Q = \frac{n_U}{2} + O_p\left(n_U^{2/3}\right). \tag{3.35}$$

By definition, $|H_1(Q_G)| < n_Q - c_\delta n_U^{2/3}$. Thus, there is a partition (A,B) of the vertices in Q_G such that each component is contained either in A or in B and that $|A| \geq \frac{n_Q}{2}$ and $|B| \geq c_\delta n_U^{2/3}$. Now we perform the following operation. We delete one edge from the non-complex components and instead add an edge between some vertex $a \in A$ and a vertex $b \in B$. The resulting graph G' is still embeddable on $\mathbb{S}_g$ and thus lies in $\mathcal{G}_g(n,m)$. The number of choices for this construction is therefore

$$m_U|A| \cdot |B| \geq (1 + o(1))\frac{c_\delta}{4} n_U^{5/3} n_Q.$$

Observe that the core $C_{G'}$ of G' is obtained from the core C_G of G by adding ab and paths from a and b to C_G, respectively. In order to determine the number of vertices added to C_G, recall that the complex part of G is obtained from the core of G by attaching a rooted forest with n_Q vertices and n_C components. By [**92**, Theorem 3], which is applicable since $n_C = o(n_Q)$ and $n_Q = o(n_C^2)$ by Theorem 3.5.2, we know that a random forest with this many vertices and components has height (that is, the maximum distance of a vertex from the root of the tree it is contained in)

$$\eta = O_p\left(\frac{\log\left(\frac{2n_C^2}{n_Q - n_C}\right)}{\log\left(1 + \frac{n_C}{n_Q - n_C}\right)}\right) = O_p\left(\frac{n_Q}{n_C}\log\left(\frac{n_C^2}{n_Q}\right)\right). \tag{3.36}$$

From Theorem 3.5.2, we deduce that in particular $\eta = o(n_C)$ whp. Therefore, we have $|C_{G'}| = n_C + o(n_C)$.

The reverse construction is to delete the edge ab (which lies in the core) and add an edge outside the complex part (not creating any new complex components). There are at most $(1+o(1))n_C$ choices for ab. Thus, there are less than $n_U^2 n_C$ possibilities for the reverse direction, yielding

$$(1+o(1))\frac{c_\delta}{4}n_U^{5/3}n_Q|\mathcal{E}_1(n,m)| < n_U^2 n_C|\mathcal{G}_g(n,m)|$$

and thus

$$|\mathcal{E}_1(n,m)| < (1+o(1))\frac{4n_U^{1/3}n_C}{c_\delta n_Q}|\mathcal{G}_g(n,m)|.$$

Now $\frac{n_U^{1/3}n_C}{n_Q} = \Theta(1)$ by Theorem 3.5.2 and thus we can choose c_δ so that

$$|\mathcal{E}_1(n,m)| < \delta|\mathcal{G}_g(n,m)|$$

for sufficiently large $n \in I$, a contradiction. This finishes the proof of (i).

In order to prove (ii), we first show that $Q_G - H_1(Q_G)$ is planar whp. We already know that G whp satisfies (i) as well as (3.35). In particular, we have $|H_1(Q_G)| > \frac{n}{2}$ and $m_U > \frac{n_U}{3}$ whp. Denote by $\mathcal{E}_2(n,m)$ the class of graphs G with these two properties for which $G - H_1(Q_G)$ is not planar. We use a double counting argument to show that $|\mathcal{E}_2(n,m)| = o(|\mathcal{G}_g(n,m|)$.

To that end, suppose that $G \in \mathcal{E}_2(n,m)$ and let $H \neq H_1(Q_G)$ be a non-planar (and thus in particular complex) component of G. We construct a new graph from G by deleting an edge outside the complex part and inserting an edge between an arbitrary vertex u in $H_1(Q_G)$ and a vertex v in a maximal non-planar 2-connected subgraph H' of H. The number of choices for this construction is larger than

$$m_U|H_1(Q_G)| > \frac{nn_U}{6}.$$

The reverse construction is to delete an edge uv from the largest complex component that separates u and v and add an edge outside the complex part (not creating any new complex components). Moreover, the edge uv has to be chosen in such a way so that a) v lies in a maximal non-planar 2-connected subgraph H' and b) the component (after deleting uv) containing u has more than $\frac{n}{2}$ vertices. It is well known that the genus of a graph is the sum of genera of its maximal 2-connected subgraphs and thus, there are at most g choices for H'. Given H', there might be several edges uv with $v \in H'$ and $u \notin H'$, and each such edge will separate u and v. However, for at most one such edge, its deletion can result in the component containing u having more than $\frac{n}{2}$ vertices, because these components are pairwise disjoint for different choices for uv. Thus, there are at most g edges that satisfy both a) and b) above. In total, there are at most gn_U^2 choices for the reverse construction. Therefore, we conclude that

$$|\mathcal{E}_2(n,m)| \le \frac{6gn_U^2}{nn_U}|\mathcal{G}_g(n,m)|$$

and thus $|\mathcal{E}_2(n,m)| = o(|\mathcal{G}_g(n,m)|)$, since $n_U = o(n)$.

We thus know that $Q_G - H_1(Q_G)$ is planar whp. Now let $\mathcal{E}_3(n,m)$ denote the class of graphs $G \in \mathcal{G}_g(n,m)$ for which $|H_1(Q_G)| > \frac{n}{2}$ and $m_U > \frac{n_U}{3}$ hold, $Q_G - H_1(Q_G)$ is planar, but $H_1(Q_G)$ has genus smaller than g. Given a graph $G \in \mathcal{E}_3(n,m)$, delete an edge outside the complex part and add an edge between two non-adjacent vertices in $H_1(Q_G)$ to obtain a graph G'. Observe that adding this edge cannot increase the genus by more than one and thus, $G' \in \mathcal{G}_g(n,m)$. The number of choices for this construction is at least

$$m_U\left(\binom{|H_1(Q_G)|}{2} - m\right) = \Theta(n_U n^2).$$

For the reverse direction, we delete an edge from the largest component (at most $m = \Theta(n)$ possibilities) and add an edge outside the complex part (less than n_U^2 possibilities) without creating any new complex components. Therefore,

$$|\mathcal{E}_3(n,m)| = O\left(\frac{n_U}{n}\right)|\mathcal{G}_g(n,m)|,$$

implying that $H_1(Q_G)$ has genus g whp. This finishes the proof of Theorem 3.5.6. $\square$

Proof of Corollary 3.5.7. The size of the complex part is $n_Q + \mathrm{ex}(G)$ and thus $H_1(Q_G)$ has size

$$|H_1(Q_G)| + \mathrm{ex}(G) - r(G),$$

where $r(G)$ is the difference between the size and the order of $R(Q_G) = Q_G - H_1(Q_G)$, which is non-negative by the definition of a complex graph (and zero only if $R(Q_G)$ is empty).

The property

$$r(G) = O_p\left(n_U^{2/3}\right)$$

follows immediately from Theorem 3.5.4 and the fact that $n_U = \Theta(n)$ in 1Sup and Int, and from Theorem 3.5.6(i) in 2Sub, 2Crit, and 2Sup.

Denote by K_R the kernel of $R(Q_G) = Q_G - H_1(Q_G)$. By construction, $r(G)$ equals the difference between the size and the order of K_R and thus by the Euler formula

$$r(G) \leq 2|K_R| + 2(g-1).$$

In 1Sup and Int, we have $|K_R| = O_p(1)$ by Theorem 3.5.4(iii) and (v), while $\mathrm{ex}(G) = \omega(1)$ by Theorem 3.5.1. Thus, $r(G) = o(\mathrm{ex}(G))$ in these regimes.

It remains to show that $r(G) = o(\mathrm{ex}(G))$ also holds in the second phase transition. In 2Crit and 2Sup, we have $n_U^{2/3} = o(\mathrm{ex}(G))$ by Theorem 3.5.2 and thus $r(G) = o(\mathrm{ex}(G))$ follows from $r(G) = O_p\left(n_U^{2/3}\right)$. We may thus assume that m lies in 2Sub. For fixed $\varepsilon > 0$, let $\mathcal{F}_\varepsilon(n,m)$ denote the class of all $G \in \mathcal{G}_g(n,m)$ for which $r(G) \geq \varepsilon\,\mathrm{ex}(G)$. We shall prove by a double counting argument that $|\mathcal{F}_\varepsilon(n,m)| = o(|\mathcal{G}_g(n,m)|)$.

Suppose, for contradiction, that $|\mathcal{F}_\varepsilon(n,m)| \geq \delta|\mathcal{G}_g(n,m)|$ for some $\delta > 0$ and arbitrarily large n. Let $G \in \mathcal{F}_\varepsilon(n,m)$ be fixed. First observe that we have $|R(Q_G)| = O_p\left(n_U^{2/3}\right)$ by Theorem 3.5.6(i) and $m_U = \frac{n_U}{2} + O_p\left(n_U^{2/3}\right)$ by Theorem 3.5.2 and hence there exists a constant $c > 0$ such that

$$|R(Q_G)| \leq c n_U^{2/3} \qquad \text{and} \qquad |m_U - \frac{n_U}{2}| \leq c n_U^{2/3} \tag{3.37}$$

with probability at least $\frac{3}{4}$. Suppose that m_U lies in that range. Then $G(n_U, m_U)$ has no complex components with positive probability. On the other hand, the largest unicyclic component of $G(n_U, m_U)$ has order $\Theta_p\left(n_U^{2/3}\right)$ (see e.g. [**78**]). Thus, we may assume that the constant c above is large enough such that G satisfies

(3.37) and its largest unicyclic component has order between $\frac{1}{c}n_U^{2/3}$ and $cn_U^{2/3}$ with probability at least $\frac{1}{2}$.

Delete an edge from $R(Q_G)$ that does not separate its component; there are at least $r(G) \geq \varepsilon \operatorname{ex}(G)$ choices for this edge. Then we add an edge between $H_1(Q_G)$ and a component in $R(Q_G)$ or a unicyclic component. Let $k \geq \frac{1}{c}n_U^{2/3}$ be the number of vertices in $R(Q_G)$ and all unicyclic components. Then the number of possibilities for this construction is at least

$$(1 + o(1))\varepsilon \operatorname{ex}(G) k n_Q,$$

because $|H_1(Q_G)| = (1 + o(1))n_Q$ by Theorem 3.5.6.

Observe that by Theorem 3.1.1(ii) and Theorem 3.5.2, the largest non-complex component has order $\Theta_p\left(n_U^{2/3}\right)$. Analogously to the proof of Theorem 3.5.6, using $n_U^{2/3} = o(n_C)$, we see that the core $C_{G'}$ of the resulting graph G' contains the edge uv and has order $(1 + o(1))n_C$.

For the reverse direction, we delete a separating edge from the core of the largest complex component, cutting off a component of order at most $cn_U^{2/3}$ (at most $(1 + o(1))n_C$ possibilities), and add an edge uv between two vertices in a complex or unicyclic component (but not in the largest complex component). In order to bound the number of possibilities for uv, observe that the total number of vertices in such components is at most

$$k + cn_U^{2/3} \leq (c^2 + 1)k.$$

We thus have at most $(c^2 + 1)k$ possibilities to choose u. The component that contains u has at most $cn_U^{2/3}$ vertices, implying that we have at most

$$(1 + o(1))(c^2 + 1)cn_U^{2/3} k n_C$$

possibilities for the reverse direction. In total, this yields that

$$|\mathcal{F}_\varepsilon(n, m)| \leq (1 + o(1)) \frac{2(c^2 + 1)cn_U^{2/3} n_C}{\varepsilon \operatorname{ex}(G) n_Q} |\mathcal{G}_g(n, m)|.$$

By Theorem 3.5.2, we have

$$\frac{n_U^{2/3} n_C}{\operatorname{ex}(G) n_Q} = \Theta\left(\frac{|\zeta|}{n^{2/5}}\right) = o(1),$$

contracting $|\mathcal{F}_\varepsilon(n, m)| \geq \delta |\mathcal{G}_g(n, m)|$, as desired. □

Proof of Corollary 3.5.8. By Theorems 3.1.5 to 3.1.7, the largest component $H_1(Q_G)$ of the complex part of G is the giant component $H_1(G)$ of G. Now the statement follows directly from Theorems 3.5.1 and 3.5.2, and Corollary 3.5.7. □

3.7. Proofs of auxiliary results

In this section we prove all results from Sections 3.3 and 3.4.

3.7.1. **Proof of Lemma 3.3.1.** It remains to prove (iv). From Lemma 3, (10.11), and (10.12) in [**62**], we deduce that

$$\rho(n, m) = \frac{2^{2m-n} e^n m! n!}{(n - m)! n^{2m} 2\pi i} \oint \sqrt{1 - z} \exp(nk(z)) \frac{\mathrm{d}z}{z},$$

where the contour of the integral is a closed curve around the origin with $|z| \leq 1$ and

$$k(z) = z - 1 - \frac{m}{n} \log(z) + \left(1 - \frac{m}{n}\right) \log(2 - z).$$

We use the contour consisting of a) the line segment from 1 to i, b) the semicircle of radius one with negative real value, and c) the line segment from $-i$ to 1. Along this contour we have $|\exp(k(z))| \leq 1$ and thus

$$\begin{aligned}\rho(n,m) &\leq \frac{2^{2m-n}e^n m!n!}{(n-m)!n^{2m}2\pi}\left|\oint \frac{\sqrt{1-z}}{z}\,\mathrm{d}z\right| \\ &\overset{(3.6)}{\leq} \frac{e^2(\pi+2\sqrt{2})}{\sqrt{2}\pi^{\frac{3}{2}}}\left(\frac{2}{e}\right)^{2m-n}\frac{m^{m+1/2}n^{n-2m+1/2}}{(n-m)^{n-m+1/2}},\end{aligned}$$

proving the lemma. □

Proof of Lemma 3.4.4. We abbreviate the class of cubic kernels embeddable on $\mathbb{S}_g$ by $\mathcal{A}_g$ and the subclass of $\mathcal{A}_g$ of *connected* cubic kernels by $\mathcal{B}_g$. Clearly, every graph in $\mathcal{A}_g$ has an even number of vertices. We first prove (3.15).

By Theorems 3.4.1 and 3.4.2 there exist positive constants $a_g^-, a_g^+, b_g^-, b_g^+$ such that for all l

$$a_g^- \leq \frac{|\mathcal{A}_g(2l)|}{(2l)^{5g/2-7/2}\gamma_K^{2l}(2l)!} \leq a_g^+ \qquad \text{and} \qquad b_g^- \leq \frac{|\mathcal{B}_g(2l)|}{(2l)^{5g/2-7/2}\gamma_K^{2l}(2l)!} \leq b_g^+.$$

By Lemma 3.4.3, the elements of $\mathcal{A}_g(2l)$ whp have a unique non-planar component. Therefore the probability that $\mathrm{pl}(G)$ has exactly $2i$ vertices is given by

$$\mathbb{P}\left[|\mathrm{pl}(G)| = 2i\right] = (1+o(1))\binom{2l}{2i}\frac{|\mathcal{B}_g(2l-2i)|\cdot|\mathcal{A}_0(2i)|}{|\mathcal{A}_g(2l)|}$$

and we can therefore conclude that (3.15) holds.

It remains to show that for every $\delta > 0$ there exists a constant c_δ such that $\mathbb{P}\left[|\mathrm{pl}(G)| > 2c_\delta\right] < \delta$ for sufficiently large l. By Lemma 3.4.3, (3.15), and the fact that $g \geq 1$, we have for any $c_\delta \in \mathbb{N}_{>0}$

$$\mathbb{P}\left[|\mathrm{pl}(G)| > 2c_\delta\right] \leq (1+o(1))\sum_{i=c_\delta+1}^{l-3} c_g^+ i^{-7/2}\left(1-\frac{i}{l}\right)^{-1}.$$

The summand (as a function in i) has a unique minimum at $i = \frac{7l}{9}$. Therefore,

$$\begin{aligned}\mathbb{P}\left[|\mathrm{pl}(G)| \geq 2c_\delta\right] &\leq (1+o(1))c_g^+\int_{c_\delta}^{l-2} x^{-7/2}\left(1-\frac{x}{l}\right)^{-1}\mathrm{d}x \\ &= \left(\frac{2}{5}+o(1)\right)c_g^+ c_\delta^{-5/2}(1+O(l^{-1/2})) < \delta\end{aligned}$$

for c_δ and l large enough, as desired. □

Proof of Lemma 3.4.5. For $K \in \mathcal{K}_g(2l,3l)$ and $\overline{K} \in \mathcal{K}_g(2l-d,3l-d)$, we say that K *contracts* to $\overline{K}$ if for each vertex in K with label $i \in \{2l-d+1,\ldots,2l\}$ we can choose an edge $e_i = \{i, v_i\}$ so that contracting these edges results in $\overline{K}$ (the contracted vertices obtain the smaller of the two labels). We say that $e_{2l-d+1},\ldots,e_{2l}$ are the *contracted edges*. Denote by $\mathcal{K}_g^{\Delta=4}(2l-d,3l-d)$ the subclass of $\mathcal{K}_g(2l-d,3l-d)$ consisting of multigraphs with maximum degree four. We say that a contraction of K to $\overline{K}$ has *degree four* if $\overline{K} \in \mathcal{K}_g^{\Delta=4}(2l-d,3l-d)$.

If K contracts to $\overline{K}$, then the compensation factor defined in (2.1) satisfies

$$w(\overline{K}) \leq w(K) \leq 6^d w(\overline{K}). \tag{3.38}$$

Each $K \in \mathcal{K}_g(2l,3l)$ contracts in at most 3^d ways, because K is cubic and hence there are at most 3^d choices for the edges $e_{2l-d+1},\ldots,e_{2l}$. Vice versa, we claim that every fixed $\overline{K} \in \mathcal{K}_g(2l-d,3l-d)$ is obtained by at least $d!2^{-d}$ different contractions from graphs in $\mathcal{K}_g(2l,3l)$. By recursively splitting vertices of $\overline{K}$ of degree at least four into two new adjacent vertices of degree at least three each, not increasing the genus throughout the process, we obtain a weighted multigraph $K \in \mathcal{K}_g(2l, 3l)$ that contracts to $\overline{K}$. The new vertices can be labelled in $d!$ ways, of which at least $2^{-d}d!$ result in distinct multigraphs in $\mathcal{K}_g(2l, 3l)$. Together with (3.38), this proves the upper bound

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \leq \frac{6^d}{d!}.$$

The corresponding bound for $|\mathcal{K}_g(2l-d, 3l-d; \mathcal{P}_i)|$ follows analogously observing that the two constructions above do neither change the number of components nor increase the genus of any component.

For the lower bound, we claim that the elements of $\mathcal{K}_g(2l, 3l)$ have at least 6^{-d} contractions of degree four *on average*. Indeed, first observe that $K \in \mathcal{K}_g(2l, 3l)$ contracts to $\overline{K} \in \mathcal{K}_g^{\Delta=4}(2l-d, 3l-d)$ if and only if the contracted edges form a matching in K. By choosing the edges of the matching recursively, we see that K contains at least $\left(2^d d!\right)^{-1} \prod_{j=0}^{d-1}(2l-6j)$ matchings of size d.

Denote by $\mathcal{A}(K)$ the class of all weighted multigraphs that are isomorphic to K. If we choose $A \in \mathcal{A}(K)$ and a matching M of size d in A uniformly at random, then the probability that every edge in M has precisely one end vertex with label in $\{2l-d+1, \ldots, 2l\}$ is $\frac{2^d}{\binom{2l}{d}}$. Therefore, the average number of contractions of degree four of graphs in $\mathcal{A}(K)$ is at least

$$\frac{\prod_{j=0}^{d-1}(2l-6j)}{2^d d!} \cdot \frac{2^d}{\binom{2l}{d}} \geq \left(\frac{2l-6d}{2l-d}\right)^d \geq 6^{-d},$$

where the last inequality uses the fact that $d \leq \frac{2l-d}{6}$. The fact that the classes $\mathcal{A}(K)$ partition $\mathcal{K}_g(2l, 3l)$ proves that $K \in \mathcal{K}_g(2l, 3l)$ has at least 6^{-d} contractions of degree four on average.

Vice versa, let $\overline{K} \in \mathcal{K}_g^{\Delta=4}(2l-d, 3l-d)$. By recursively splitting the d vertices of degree four in $\overline{K}$, we see that $\overline{K}$ can be obtained by at most $\binom{4}{2}^d d! = 6^d d!$ contractions of degree four. Together with (3.38), we deduce that

$$\frac{|\mathcal{K}_g(2l-d, 3l-d)|}{|\mathcal{K}_g(2l, 3l)|} \geq \frac{1}{216^d d!}.$$

The corresponding bound for $|\mathcal{K}_g(2l-d, 3l-d; \mathcal{P}_i)|$ follows analogously. $\square$

3.7.2. **Remark.** Observe that the proof of Lemma 3.4.5 applies to any class $\mathcal{F}$ of (multi-) graphs that is a) closed under taking minors and b) *weakly addable*, that is, if G is obtained by adding an edge between two distinct components of $F \in \mathcal{F}$, then also $G \in \mathcal{F}$. For more details, see Section 3.8.

Proof of Lemma 3.4.6. Let $K \in \mathcal{K}_g(2l-d, 3l-d)$. We subdivide the edges of K by inserting $n_C - 2l + d$ vertices and then assign labels to these new vertices in one of $(n_C - 2l + d)!$ possible ways so as to obtain a core with n_C vertices.

Call a distribution of $n_C - 2l + d$ new vertices to the edges of K *feasible* if the resulting graph has no loops or multiple edges. The number $\binom{n_C+l-1}{3l-d-1}$ of *all* distributions is clearly an upper bound for the number of feasible distributions. On the other hand, a distribution is feasible if and only if each loop is subdivided at least twice and for every multiple edge, at most one of its edges is not subdivided. Denote by s_K the minimal number of times that we need to subdivide the edges of K in order to obtain a simple graph. Then $\binom{n_C+l-s_K-1}{3l-d-1}$ is a lower bound on the number of feasible distributions.

By construction, $s_K \le 2(3l-d) \le 6l$ and we thus deduce that

$$\min_{-5\le\nu\le1}\left((n_C-2l+d)!\binom{n_C+\nu l-1}{3l-d-1}\right) \le \varphi_{n_C,l,d} \le (n_C-2l+d)!\binom{n_C+l-1}{3l-d-1}.$$

Now the lemma follows from the intermediate value theorem and the fact that the function $\binom{x}{k}$ for fixed $k\in\mathbb{N}$ is continuous for $x\in\mathbb{R}$. $\square$

Proof of Lemma 3.4.7. Lemma 3.4.7 follows directly from (3.16), Lemmas 3.4.5 and 3.4.6, the intermediate value theorem, and the fact that x^d is continuous. $\square$

Proof of Lemma 3.4.8. We first derive an upper bound for Σ_C, as well as the main contribution to this upper bound. We substitute $n_C = \overline{n}_C + r$ (recall that $\overline{n}_C = \sqrt{n_Q(3l-d)}$). Applying (3.5) to (3.18), and then using (3.2) and (3.7) we deduce that

$$\Sigma_C \le \Sigma_C^+ := \sum_r \exp\left(-\frac{r^2}{2n_Q}+rA_1+A_2\right), \tag{3.39}$$

where

$$\begin{aligned} A_1 =& \frac{1-2\overline{n}_C}{2n_Q}+\frac{3l-d-1}{\overline{n}_C}+\frac{(3l-d-1)(3l-d-2)}{2(\overline{n}_C+l-1)^2}, \\ A_2 =& (3l-d)\left(\log(\overline{n}_C)-\frac{1}{2}\right)+\sqrt{\frac{3l-d}{4n_Q}}+(3l-d-1)\left(\frac{l-1}{\overline{n}_C}-\frac{3l-d-2}{2(\overline{n}_C+l-1)}\right). \end{aligned}$$

Evaluating the 'Gaussian' sum in (3.39) we obtain

$$\Sigma_C^+ \le \sqrt{2\pi n_Q}\exp\left(A_2+\frac{n_Q A_1^2}{2}\right).$$

The existence of the constants a_C^+, b_C^+ from (i) now follows from

$$\exp(A_2) \le \left(\frac{n_Q(3l-d)}{e}\right)^{(3l-d)/2}\exp\left(O\left(\sqrt{\frac{l^3}{n_Q}}\right)\right)$$

and the observation that $n_Q A_1^2 = O\left(\frac{l^2}{n_Q}\right)$, which is $O\left(\sqrt{\frac{l^3}{n_Q}}\right)$, because $l = O(n_Q)$.

In order to prove (ii), suppose that $\frac{7}{2}d \le l \le \epsilon n_Q = o(n_Q)$; then also $l = o(\overline{n}_C)$. In (3.18), we set $n_C = \overline{n}_C - \nu l + 1 + s$. If we let the parameter $s = r+\nu l-1$ take only values for which $n_C \in I_C^\delta(n_Q,l,d)$ with fixed $0<\delta<\frac{1}{2}$, then

$$\Sigma_C \ge \sum_s \frac{(n_Q)_{n_C}}{n_Q^{n_C}} n_C(\overline{n}_C+s)_{3l-d-1}.$$

The interval $I_C^\delta(n_Q,l,d)$ has length $2\delta\overline{n}_C > 2\delta\sqrt{n_Q}$ and hence we can choose for s an interval I_s of length $\delta\sqrt{n_Q}$ in which $|s|<\delta\overline{n}_C$ holds.

We use (3.5) for both falling factorials and obtain

$$\Sigma_C \ge \overline{n}_C^{3l-d}\sum_s\left(1+\frac{s}{\overline{n}_C}\right)^{3l-d-1}\left(1+\frac{s+1+\nu l}{\overline{n}_C}\right)\exp(B_1), \tag{3.40}$$

where

$$B_1 = -\frac{(\overline{n}_C-\nu l+1+s)^2}{2(n_Q-\overline{n}_C+\nu l-1-s)}-\frac{(3l-d-1)^2}{2(\overline{n}_C-3l+d+1+s)}.$$

Observe that $1+\frac{s}{\overline{n}_C}=\Theta(1)$ and $1+\frac{s+1+\nu l}{\overline{n}_C}=\Theta(1)$. Using (3.3), we deduce that

$$\left(1+\frac{s}{\overline{n}_C}\right)^{3l-d}\exp(B_1) \ge \exp\left(-\frac{3l-d}{2}+O\left(\sqrt{\frac{l^3}{n_Q}}\right)+O(1)\right). \tag{3.41}$$

Now (3.40) and (3.41), together with $|I_s| = \delta\sqrt{n_Q}$ prove (ii).

It remains to prove (iii). First observe that if we take the sum (3.39) over all $r \in \mathbb{Z}$ and normalise, we obtain a Gaussian random variable X with mean $n_Q A_1 = O(l)$ and variance n_Q. Applying the Chernoff bound (3.9) to X, we deduce

$$\sum_{|r-n_Q A_1|>\frac{\delta}{2}\overline{n}_C} \exp\left(-\frac{r^2}{2n_Q} + rA_1 + A_2\right) \leq 2\exp\left(-\delta^2 \frac{3l-d}{8}\right)\Sigma_C^+.$$

Note that $|r - n_Q A_1| < \frac{\delta}{2}\overline{n}_C$ implies that $n_C \in I_C^\delta(n_Q, l, d)$ for sufficiently large n_Q and $l = o(n_Q)$, because then $n_Q A_1 = o(n_Q)$. Therefore,

$$\frac{\displaystyle\sum_{n_C \notin I_C^\delta} \frac{(n_Q)_{n_C}}{n_Q^{n_C}} n_C (n_C + \nu l - 1)_{3l-d-1}}{\displaystyle\sum_{n_C \in I_C^\delta} \frac{(n_Q)_{n_C}}{n_Q^{n_C}} n_C (n_C + \nu l - 1)_{3l-d-1}} \leq \exp\left(-\delta^2 \frac{3l-d}{8} + \Theta(1) + \Theta\left(\sqrt{\frac{l^3}{n_Q}}\right)\right).$$

Now $\delta^2 \frac{3l-d}{8} = \Theta(l)$, $\sqrt{\frac{l^3}{n_Q}} = o(l)$, and the fact that $l \to \infty$ finish the proof of (iii). □

Proof of Lemma 3.4.9. We start by proving (i). We apply

$$\frac{(3l-d)^{(3l-d+2)/2}}{(3l-d)!} \overset{(3.4)}{\leq} \frac{e^{3l-d}}{\sqrt{2\pi}}(3l-d)^{-\frac{3l-d-1}{2}} \overset{(3.2)}{\leq} \frac{e^{3l-\frac{d}{2}}}{\sqrt{2\pi}}(3l)^{-\frac{3l-d-1}{2}}$$

and Lemmas 3.4.7 and 3.4.8 to deduce that

$$\Sigma_d \leq \frac{\exp\left(a_C^+ + b_C^+\sqrt{\frac{l^3}{n_Q}}\right)}{\sqrt{2\pi}} e^{3l}(3l)^{-\frac{3l-1}{2}} \sum_{d=0}^{2l} \binom{2l}{d}\left(\frac{108l}{n_Q}\right)^{\frac{d}{2}},$$

proving (i) with $a_d^+ = a_C^+ - \frac{1}{2}\log(2\pi)$ and $b_d^+ = b_C^+ + 2\sqrt{108}$.

For (ii), first note that we have a lower bound for Σ_d if we restrict the sum (3.20) to $0 \leq d \leq \lfloor \frac{2l}{7} \rfloor$. By analogous arguments as for the upper bound, we deduce that

$$\Sigma_d \geq \frac{\exp\left(a_C^- + b_C^-\sqrt{\frac{l^3}{n_Q}}\right)}{e} e^{3l}(3l)^{-\frac{3l-1}{2}} \sum_{d=0}^{\frac{2l}{7}} \binom{2l}{d}\left(\frac{3l}{216^2 e n_Q}\right)^{\frac{d}{2}}.$$

The sum above can be extended to a sum $Y = \sum_{d=0}^{2l} \binom{2l}{d} y^d$ with $y = o(1)$. Normalising this sum results in a binomial random variable $X = \mathrm{Bi}(2l, p)$ with $p = \frac{y}{1+y}$ and $\mathbb{E}[X] = \Theta\left(\sqrt{l^3/n_Q}\right)$. If $\mathbb{E}[X] \to 0$, then the main contribution to Y is provided by the index $d = 0$. Otherwise, the Chernoff bound (3.10) yields that the main contribution to X—and thus also to Y—is provided by an interval contained in the range $0 \leq d \leq \frac{2l}{7}$. Thus, with (3.3) we deduce that

$$\Sigma_d \geq \frac{\exp\left(a_C^- + b_C^-\sqrt{\frac{l^3}{n_Q}}\right)}{e(1+o(1))} e^{3l}(3l)^{-\frac{3l-1}{2}} \exp\left(\frac{\sqrt{3}}{108\sqrt{e}}\sqrt{\frac{l^3}{n_Q}} - \frac{\sqrt{3}l^2}{216\sqrt{e}n_Q}\right).$$

Observing that $l^2/n_Q = o(\sqrt{l^3/n_Q})$, we have thus proved (ii) for any choice of $a_d^- < a_C^- - 1$ and $b_d^- < b_C^- + \frac{\sqrt{3}}{108\sqrt{e}}$.

In order to prove (iii), it remains to show that the tail of Σ_d has smaller order than its total value, that is

$$e^{b_d^+\sqrt{\frac{l^3}{n_Q}}} \sum_{d \notin I_d} \binom{2l}{d}\left(\frac{6\sqrt{3l}}{\sqrt{n_Q}}\right)^d = o\left(e^{b_d^-\sqrt{\frac{l^3}{n_Q}}}\right). \tag{3.42}$$

Write

$$Z = \sum_{d=0}^{2l} \binom{2l}{d} \left(\frac{6\sqrt{3l}}{\sqrt{n_Q}} \right)^d.$$

For $\sqrt{\frac{l^3}{n_Q}} \to 0$, the exponential terms in (3.42) are both $1 + o(1)$ and the sum on the left hand side is $o(1)$, because its range does not include the main contribution to the binomial sum Z, which is located at $d = 0$.

If $\sqrt{\frac{l^3}{n_Q}} \to c \in \mathbb{R}^+$, then both exponential terms in (3.42) are $\Theta(1)$. For any fixed $h = h(n_Q) = \omega(1)$, we deduce from (3.10), applied to the normalised sum Z,

$$\sum_{d>h} \binom{2l}{d} \left(\frac{6\sqrt{3l}}{\sqrt{n_Q}} \right)^d \leq \exp(-ch)$$

for some constant $c > 0$, which proves (3.42).

Finally, if $\sqrt{\frac{l^3}{n_Q}} \to \infty$, we can choose β_d^+ sufficiently large so that (3.10) yields

$$\sum_{d>\beta_d^+\sqrt{\frac{l^3}{n_Q}}} \binom{2l}{d} \left(\frac{6\sqrt{3l}}{\sqrt{n_Q}} \right)^d \leq \exp\left(-(b_d^+ - b_d^- + 1)\sqrt{\frac{l^3}{n_Q}} \right),$$

which proves (3.42) also in this last case. □

Proof of Corollary 3.4.10. The typical range for $d(G)$ follows directly from Lemma 3.4.9(iii). Substituting this deficiency in the formulas for the main contribution for n_C from Lemma 3.4.8 yields the typical order of the core. □

Proof of Corollary 3.4.11. This follows directly from (3.21) and (3.22). □

Proof of Lemma 3.4.12. We prove Lemma 3.4.12 using the lower bound on $|\mathcal{G}_g^*(n,m)|$ from Lemma 3.4.16. It is important to note that vice versa, the proof of Lemma 3.4.16 does *not* rely on Lemma 3.4.12.

Suppose first $n_Q = 0$, i.e. the complex part is empty and the graph only consists of trees and unicyclic components. In this case Lemma 3.3.1(iv) implies that the number of such graphs satisfies

$$|\mathcal{U}(n,m)| \leq \Theta(1) n^m 2^{m-n} e^{n-\frac{m^2}{n}}.$$

Comparing this to the lower bound from Lemma 3.4.16 shows that

$$\frac{|\mathcal{U}(n,m)|}{|\mathcal{G}_g^*(n,m)|} \leq e^{-l_1} = o(1).$$

The remaining case is $m_U = 0$, i.e. $m = n_Q + l \geq n_Q + 1$ (recall that $n_Q > 0$ implies $l > 0$). The number of such graphs is given by

$$\sum_{n_Q \leq m-1} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m)|.$$

The case $n_Q = m - 1$ in the sum above is of smaller order than the lower bound for $|\mathcal{G}_g^*(n,m)|$ from Lemma 3.4.16. For every $n_Q < m - 1$, Corollary 3.4.11 implies that

$$\frac{\binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m)|}{\binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m-1)| \binom{n-n_Q}{2}} = \Theta(1) n_Q^{\frac{3}{2}} (m - n_Q)^{-\frac{3}{2}} (n - n_Q)^{-2}. \tag{3.43}$$

In 1SUP and INT, the right hand side of (3.43) is $O(n^{-1/2})$. Observing that the denominator is a summand of $|\mathcal{G}_g^*(n,m)|$, we deduce that

$$\sum_{n_Q \le m-1} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m)| = o\big(|\mathcal{G}_g^*(n,m)|\big) \qquad \text{in 1SUP and INT.}$$

Suppose now that we are in the second phase transition and write $I_l = [p_l, q_l]$. For $n_Q < m - p_l$, the right hand side of (3.43) is $o(1)$ and thus

$$\sum_{n_Q < m-p_l} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m)| = o\big(|\mathcal{G}_g^*(n,m)|\big).$$

For $n_Q \ge m - p_l$, or equivalently $l \le p_l$, we have

$$\sum_{n_Q=m-p_l}^{m-2} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m-1)| \binom{n-n_Q}{2} \le \exp(-f(n)) |\mathcal{G}_g^*(n,m)|,$$

where $f = \omega(\log n)$ is a positive valued function. From this, we deduce that

$$\sum_{n_Q=m-p_l}^{m-2} \binom{n}{n_Q} |\mathcal{Q}_g(n_Q, m)| \overset{(3.43)}{\le} \Theta\Big(n^{\frac{3}{2}}\Big) \exp(-f(n)) |\mathcal{G}_g^*(n,m)| = o\big(|\mathcal{G}_g^*(n,m)|\big).$$

This concludes the proof of Lemma 3.4.12. □

Proof of Lemma 3.4.13. In $\Sigma_Q = \sum_{n_Q} \rho\psi$ (see (3.25) for the definition of ψ), we substitute $n_Q = \overline{n}_Q + r$. We then have $n_U = n - n_Q = \overline{n}_U - r$ and $m_U = m - n_Q - l = \overline{m}_U - r$.

With this substitution, we obtain

$$\psi = \left(\frac{2}{e}\right)^{\overline{n}_Q + r} (\overline{n}_Q + r)^{\frac{3l}{2}-1} (\overline{n}_U - r)^{-r-\frac{1}{2}} (\overline{m}_U - r)^{-\overline{m}_U + r - \frac{1}{2}} \exp(f_d).$$

Because n_Q, l are admissible, we have $l = O(n_Q)$ and thus

$$f_d \le a_d^+ + b_d^+ \sqrt{\frac{l^3}{n_Q}} = O(l). \tag{3.44}$$

If in addition (3.29) holds, then $l = o(\overline{n}_Q)$ and thus, for every fixed $h(n) = \omega(1)$,

$$f_d \le a_d^+ + o(1) l, \tag{3.45}$$

whenever $r \ge -\overline{n}_Q + hl$. In either case, we distinguish whether $r > 0$ or $r \le 0$.

Let $\Sigma_{r>0}$ be the part of Σ_Q consisting of the summands with $r > 0$. We bound $\rho(n_U, m_U)$ from above by 1. Additionally we claim that

$$\left(\frac{2}{e}\right)^r (\overline{n}_U - r)^{-r} (\overline{m}_U - r)^{-\overline{m}_U + r} < \overline{m}_U^{-\overline{m}_U} \exp\left(-\frac{r^3}{24\overline{m}_U^2}\right). \tag{3.46}$$

Indeed, for $r \ge 0$, the quotient of the two sides in (3.46) has a unique maximum at $r = 0$, where we have equality. Furthermore, there exists a constant $c > 0$ with

$$(\overline{n}_U - r)^{-\frac{1}{2}} (\overline{m}_U - r)^{-\frac{1}{2}} \le c \overline{m}_U^{-1} \exp\left(\frac{r^3}{216\overline{m}_U^2}\right). \tag{3.47}$$

Now (3.44), (3.46), and (3.47) yield

$$\Sigma_{r>0} \le \left(\frac{2}{e}\right)^{\overline{n}_Q} \overline{m}_U^{-\overline{m}_U - 1} \exp(O(l)) \sum_r (\overline{n}_Q + r)^{\frac{3l}{2}-1} \exp\left(-\frac{r^3}{27\overline{m}_U^2}\right). \tag{3.48}$$

If in addition (3.29) holds, we can replace $\exp(O(l))$ by $(1+o(1))^l$. The summand above is maximised at the (not necessarily integral) unique positive solution r_0 of

$$r_0^3 + r_0^2 \overline{n}_Q = 9\overline{m}_U^2\left(\frac{3l}{2} - 1\right).$$

Suppose first that (3.26) holds, that is, $\overline{n}_Q^3 \geq 9\overline{m}_U^2\left(\frac{3l}{2}-1\right)$. Then

$$\frac{1}{2}\sqrt{\frac{9\overline{m}_U^2\left(\frac{3l}{2}-1\right)}{\overline{n}_Q}} \leq r_0 \leq \sqrt{\frac{9\overline{m}_U^2\left(\frac{3l}{2}-1\right)}{\overline{n}_Q}} \tag{3.49}$$

and thus

$$\begin{aligned}(\overline{n}_Q + r_0)^{\frac{3l}{2}-1}\exp\left(-\frac{r_0^3}{27\overline{m}_U^2}\right) &\overset{(3.2)}{\leq} \overline{n}_Q^{\frac{3l}{2}-1}\exp\left(\frac{r_0\left(\frac{3l}{2}-1\right)}{\overline{n}_Q} - \frac{r_0^3}{27\overline{m}_U^2}\right)\\ &\overset{(3.26),(3.49)}{\leq} \overline{n}_Q^{\frac{3l}{2}-1}\exp(O(l)).\end{aligned}$$

Summing over $1 \leq r \leq \overline{m}_U - 1$, we deduce that

$$\Sigma_{r>0} \leq \left(\frac{2}{e}\right)^{\overline{n}_Q}\overline{n}_Q^{\frac{3l}{2}-1}\overline{m}_U^{-\overline{m}_U}\exp(O(l)),$$

which proves (3.27) for $\Sigma_{r>0}$ if (3.26) holds. If the stronger condition (3.29) is satisfied, the factor $\exp(O(l))$ improves to $\exp(O(\sqrt{\epsilon}l)) = \exp(o(1)l)$, proving (3.28) for $\Sigma_{r>0}$.

Now consider the case $\overline{n}_Q^3 < 9\overline{m}_U^2\left(\frac{3l}{2}-1\right)$. Then

$$\frac{1}{2}\sqrt[3]{9\overline{m}_U^2\left(\frac{3l}{2}-1\right)} \leq r_0 \leq 2\sqrt[3]{9\overline{m}_U^2\left(\frac{3l}{2}-1\right)} \tag{3.50}$$

and hence

$$(\overline{n}_Q + r_0)^{\frac{3l}{2}-1}\exp\left(-\frac{r_0^3}{27\overline{m}_U^2}\right) \leq (3r_0)^{\frac{3l}{2}-1}\exp(O(l)).$$

Summing over less than $\overline{m}_U$ values for r, we deduce that

$$\Sigma_{r>0} \leq \left(\frac{2}{e}\right)^{\overline{n}_Q} r_0^{\frac{3l}{2}-1}\overline{m}_U^{-\overline{m}_U}\exp(O(l)).$$

Together with (3.50), this proves (3.27) for $\Sigma_{r>0}$ in the case that (3.26) is violated.

Finally, consider the part $\Sigma_{r\leq 0}$ of Σ_Q consisting of the summands with $r \leq 0$. Observe that $-\overline{n}_Q + 1 \leq r \leq 0$; in particular, the case $r \leq 0$ only occurs if $\overline{n}_Q > 0$. We use Lemma 3.3.1(iv) as an upper bound for $\rho = \rho(\overline{n}_U - r, \overline{m}_U - r)$ to deduce

$$\rho\psi \leq c\left(\frac{2}{e}\right)^{\overline{n}_Q}(\overline{n}_Q + r)^{\frac{3l}{2}-1}\overline{m}_U^{-\overline{m}_U - \frac{1}{2}}\exp(f_d). \tag{3.51}$$

We bound the factor $\exp(f_d)$ by (3.44). Furthermore, $(\overline{n}_Q + r)^{\frac{3l}{2}-1} \leq \overline{n}_Q^{\frac{3l}{2}-1}$, because $r \leq 0$. Summing over r, we deduce that

$$\Sigma_{r\leq 0} \leq c\left(\frac{2}{e}\right)^{\overline{n}_Q}\overline{n}_Q^{\frac{3l}{2}}\overline{m}_U^{-\overline{m}_U - \frac{1}{2}}\exp(O(l)).$$

This proves (3.27) for $\Sigma_{r\leq 0}$, independent of whether (3.26) is satisfied.

Finally, suppose that (3.29) holds. Then in (3.51), we bound the factor $\exp(f_d)$ by (3.45) for $r \geq r_1 := -\overline{n}_Q + hl$ and deduce by analogous arguments as above that

$$\sum_{r=r_1}^{0}\rho\psi \leq \Theta(1)\left(\frac{2}{e}\right)^{\overline{n}_Q}\overline{n}_Q^{\frac{3l}{2}}\overline{m}_U^{-\overline{m}_U - \frac{1}{2}}(1+o(1))^l.$$

For $r < r_1$, observe that Euler's formula yields $r \geq r_2 := -\overline{n}_Q + \Theta(l)$. In this range, the summand $\rho\psi$ is maximised at the upper bound $r = r_1 - 1$; this yields

$$\sum_{r=r_2}^{r_1-1} \rho\psi \leq \Theta(1)\left(\frac{2}{e}\right)^{\overline{n}_Q} (hl)^{\frac{3l}{2}} \overline{m}_U^{-\overline{m}_U - \frac{1}{2}} (1+o(1))^l.$$

If we choose h to be growing slowly enough so that $hl = o(\overline{n}_Q)$, then this proves (3.28) for $\Sigma_{r<0}$.

The trivial observation $\Sigma_Q = \Sigma_{r>0} + \Sigma_{r\leq 0}$ finishes the proof. □

Proof of Lemma 3.4.14. Like in the proof of Lemma 3.4.13, we distinguish the cases $r > 0$ and $r \leq 0$ as well as whether (3.26) holds or not.

First consider $\Sigma_{r>0}$ when (3.26) holds. Then (3.49) implies $r_0 \leq \overline{n}_Q$, which yields

$$\begin{aligned}\sum_{r=1}^{\overline{n}_Q} (\overline{n}_Q + r)^{\frac{3l}{2}-1} \exp\left(-\frac{r^3}{27\overline{m}_U^2}\right) &\leq \overline{n}_Q^{\frac{3l}{2}} \left(1 + \frac{r_0}{\overline{n}_Q}\right)^{\frac{3l}{2}-1} \exp\left(-\frac{r_0^3}{27\overline{m}_U^2}\right) \\ &\leq \overline{n}_Q^{\frac{3l}{2}} \exp(O(l)).\end{aligned}$$

The sum over the remaining values for r is bounded by the integral

$$\int_{\overline{n}_Q}^{\infty} (2r)^{\frac{3l}{2}-1} \exp\left(-\frac{r^3}{27\overline{m}_U^2}\right) \mathrm{d}r \leq \overline{m}_U^l \Gamma\left(\frac{l}{2}\right) \exp(O(l)) = \overline{m}_U^l l^{\frac{l}{2}} \exp(O(l)).$$

Now (3.26), (3.48), and the fact that $\overline{n}_Q = 2m - n - 2l < \lambda n^{2/3}$ prove (3.30) for $\Sigma_{r>0}$.

If (3.26) is violated, we split $\Sigma_{r>0}$ into the sums for $1 \leq r \leq r_0$ and $r_0 < r$. Observe that (3.50) implies $\overline{n}_Q < 2r_0$. Thus, the sum for $1 \leq r \leq r_0$ is smaller than $\overline{m}_U^l l^{\frac{l}{2}} \exp(O(l))$, while the sum for $r_0 < r$ is bounded by the integral

$$\int_{r_0}^{\infty} (3r)^{\frac{3l}{2}-1} \exp\left(-\frac{r^3}{27\overline{m}_U^2}\right) \mathrm{d}r \leq \overline{m}_U^l l^{\frac{l}{2}} \exp(O(l)) < \overline{m}_U^l l^{\frac{l}{2}-\frac{1}{3}} \exp(O(l)).$$

Now (3.30) for $\Sigma_{r>0}$ follows from (3.48) and the trivial fact that $\overline{m}_U = O(n)$.

For $r \leq 0$, observe that $m_U = \frac{n_U}{2} - \frac{r}{2}$. Furthermore, we have $n_Q \leq \overline{n}_Q = O(\lambda n^{2/3})$ and thus

$$r = O(\lambda n^{2/3}) \qquad \text{and} \qquad n_U = (1+o(1))n.$$

By the assumption $\lambda = o(n^{1/12})$, Lemma 3.3.1(iii) applies to $\rho(n_U, m_U)$ and summing over $-\overline{n}_Q + 1 \leq r \leq 0$ yields

$$\Sigma_{r\leq 0} \leq c\left(\frac{2}{e}\right)^{\overline{n}_Q} \overline{n}_Q^{\frac{3l}{2}} \overline{n}_U^{-\frac{1}{2}} \overline{m}_U^{-\overline{m}_U - \frac{1}{2}} \exp(O(l)).$$

Now (3.30) follows for $\Sigma_{r\leq 0}$ analogously to the proof of Lemma 3.4.13, with the additional fact $\overline{n}_Q = O(\lambda n^{2/3})$. □

Proof of Lemma 3.4.15. By (3.31), l_0 is positive. We prove the order of l_0 separately for each of the five regimes.

1Sup: In this regime, we have

$$l_0 = \frac{\phi^{2/3}(\lambda n^{2/3} - 2l_0)}{e^{1/3} 2^{4/3} \left(\frac{n}{2} - \lambda n^{2/3} + l_0\right)^{2/3}}.$$

The denominator is of order $\Theta(n^{2/3})$. Thus, in order for the equality to be true, the numerator must be of order $\lambda n^{2/3}$ and thus $l_0 = \Theta(\lambda)$.

Int: Here, the denominator is still of order $n^{2/3}$ and the numerator is of order $\Theta(n)$ and thus $l_0 = \Theta(n^{1/3})$.

2Sub: The numerator is of order $\Theta(n)$ and thus

$$l_0 = \frac{\Theta(n)}{(l_0 - \frac{1}{2}\,\zeta\, n^{3/5})^{2/3}}.$$

If $l_0 = \Omega(|\zeta| n^{3/5})$, then we have $l_0 = \Theta\Bigl(\frac{n}{l_0^{2/3}}\Bigr)$ and thus $l_0 = \Theta(n^{3/5}) = o(|\zeta| n^{3/5})$, a contradiction. Therefore, $l_0 = o(|\zeta| n^{3/5})$ and

$$l_0 = \Theta\biggl(\frac{n}{(|\zeta| n^{3/5})^{2/3}}\biggr) = \Theta\Bigl(|\zeta|^{-2/3} n^{3/5}\Bigr).$$

2Crit: The numerator has order $\Theta(n)$. For the denominator we have a contradiction similar to the previous case if l_0 is not $\Theta(n^{3/5})$. Furthermore, the denominator has order $\Theta\bigl(n^{3/5}\bigr)$.

2Sup: The numerator is $\Theta(n)$ and we obtain a contradiction if there is no cancellation in the denominator. Thus we set $l_0 = \frac{1}{2}\,\zeta\, n^{3/5} + r$ with $r = o(\zeta\, n^{3/5})$ and deduce that $r = \Theta\Bigl(\zeta^{-3/2}\, n^{3/5}\Bigr)$. ☐

Proof of Lemma 3.4.16. By Lemma 3.4.15, we have $0 < l = o(\overline{n}_Q)$ and $0 < \overline{n}_Q < n$. Thus, $\overline{n}_U$ and $\overline{m}_U$ are also positive. Therefore, we have $\mathcal{Q}_g(\overline{n}_Q, \overline{n}_Q + l) \neq \emptyset$ and $\mathcal{U}(\overline{n}_U, \overline{m}_U) \neq \emptyset$, showing that the given value l and $n_Q = \overline{n}_Q$ are admissible. Recall that

$$\Sigma_Q = \sum_{n_Q} \rho(n_U, m_U)\psi(n_Q, l).$$

Observe that (at least) all n_Q with $\overline{n}_Q \leq n_Q \leq \overline{n}_Q + \overline{m}_U - 1$ are admissible in this sum. For each such n_Q, we have $m_U \leq \frac{n_U}{2}$ and thus Lemma 3.3.1(ii) yields

$$\Sigma_Q \geq \Theta(1) \sum_{n_Q = \overline{n}_Q}^{\overline{n}_Q + \overline{m}_U - 1} \psi(n_Q, l).$$

Set $n_Q = \overline{n}_Q + r$. There exists a $c > 0$ such that

$$\psi(\overline{n}_Q + r, l) \geq \left(\frac{2}{e}\right)^{\overline{n}_Q} (\overline{n}_Q + r)^{\frac{3l}{2} - 1} \overline{m}_U^{-\overline{m}_U - 1} \exp\biggl(f_d - \frac{r^3}{12\overline{m}_U^2}\biggr)$$

holds for $0 \leq r \leq c\overline{m}_U$. The factor $(\overline{n}_Q + r)^{\frac{3l}{2} - 1} \exp\Bigl(-\frac{r^3}{12\overline{m}_U^2}\Bigr)$ is increasing until the unique positive solution r_0 of

$$r_0^3 + r_0^2 \overline{n}_Q = 4\overline{m}_U^2 \biggl(\frac{3l}{2} - 1\biggr).$$

The assumptions on the size of l imply that l satisfies (3.26), which in turn yields $r_0 = \Theta\Bigl(\overline{m}_U^{2/3}\Bigr)$. Therefore, for $1 \leq r \leq r_0$, we have

$$\psi(\overline{n}_Q + r, l) \geq \left(\frac{2}{e}\right)^{\overline{n}_Q} \overline{n}_Q^{\frac{3l}{2} - 1} \overline{m}_U^{-\overline{m}_U - 1} \exp\biggl(f_d(\overline{n}_Q + r, l) - \frac{1}{12\overline{m}_U^2}\biggr).$$

Let $\tilde{n}_Q = \overline{n}_Q + r$ be the value that minimises $f_d(\overline{n}_Q + r, l)$ for $1 \leq r \leq r_0$; then $\tilde{n}_Q = \overline{n}_Q + O(\overline{m}_U^{2/3})$, since $r \leq r_0$. This proves the lower bound for Σ_Q. The lower bound for $|\mathcal{G}_g^*(n, m)|$ follows directly from (3.24), the bound for Σ_Q, and the fact that l_1 is admissible. ☐

Proof of Lemma 3.4.17. First observe that there exists $l_b > 0$ such that (3.26) is violated precisely when $l \geq l_b$. In the first supercritical regime, we have $l_b = \Theta(\lambda^3)$, in all other regimes $l_b = \Theta(n)$.

By Lemma 3.4.13, we have

$$\tilde{\Sigma}_l \leq \sum_l n^{\frac{3}{2}} l^{-l+\frac{5g}{2}-\frac{10}{3}} (n-m+l)^{m-n-\frac{5}{3}} \exp(O(l)),$$

where the sum is taken over all $l \geq l_b$. The sum on the right hand side is bounded from above by a geometric sum $\sum_l \exp(-cl)$ with $c > 0$ and thus

$$\tilde{\Sigma}_l \leq (1+o(1)) n^{\frac{3}{2}} l_b^{-l_b+\frac{5g}{2}-\frac{10}{3}} (n-m+l_b)^{m-n-\frac{5}{3}} \exp(O(l_b)).$$

Comparing this with the lower bound for $|\mathcal{G}_g^*(n,m)|$ from Lemma 3.4.16 and implementing (3.32), we deduce that

$$\frac{n^{n+1/2}\left(\frac{e}{2}\right)^m \tilde{\Sigma}_l}{|\mathcal{G}_g^*(n,m)|} \leq (2m-n-2l_1) n^{\frac{1}{6}} l_b^{-l_b+\frac{5g}{2}-\frac{10}{3}} \left(\frac{n-m+l_b}{n-m+l_1}\right)^{m-n} \exp(O(l_b)).$$

The right hand side is $o(1)$, unless we are in the first supercritical regime and λ (and thus also l_b) is too small for the term $l_b^{-l_b}$ to compensate the polynomial terms in n. For this to be the case, we would in particular have $\lambda = o(n^{1/12})$. For such λ, we have the stronger upper bound for $\tilde{\Sigma}_l$ provided by Lemma 3.4.14, which is smaller than the one from Lemma 3.4.13 by a factor of $\lambda^{-1} n^{5/6}$. Thus, for these λ, we have

$$\begin{aligned}\frac{n^{n+1/2}\left(\frac{e}{2}\right)^m \tilde{\Sigma}_l}{|\mathcal{G}_g^*(n,m)|} &\leq \lambda(2m-n-2l_1) n^{-\frac{2}{3}} l_b^{-l_b+\frac{5g}{2}-\frac{10}{3}} \left(\frac{n-m+l_b}{n-m+l_1}\right)^{m-n} \exp(O(l_b)) \\ &\leq \lambda^2 l_b^{-l_b} \exp(O(l_b)),\end{aligned}$$

which is $o(1)$, because $l_b = \Theta(\lambda^3)$. $\square$

Proof of Lemma 3.4.18. We first show that for $d = o(l)$ we have

$$|f_C(d,(1-\epsilon)n_Q,l) - f_C(n_Q,l,d)| = o(\epsilon l). \tag{3.52}$$

By (3.19), we have

$$\frac{\Sigma_C(d,(1-\epsilon)n_Q,l)}{\Sigma_C(n_Q,l,d)} = (1-\epsilon)^{\frac{3l-d+1}{2}} \exp(f_C(d,(1-\epsilon)n_Q,l) - f_C(n_Q,l,d)). \tag{3.53}$$

We can also compare the summands of the two terms $\Sigma_C(d,(1-\epsilon)n_Q,l)$ and $\Sigma_C(n_Q,l,d)$ separately. Denote the summands by

$$s(n_C,d,n_Q,l) = \frac{(n_Q)_{n_C}}{n_Q^{n_C}} n_C (n_C + \nu l - 1)_{3l-d-1}.$$

Then we have for $1 \leq n_C = o(n_Q)$

$$\begin{aligned}\frac{s(n_C,d,(1-\epsilon)n_Q,l)}{s(n_C,d,n_Q,l)} &= \frac{\binom{(1-\epsilon)n_Q}{n_C}}{(1-\epsilon)^{n_C}\binom{n_Q}{n_C}} \\ &\overset{(3.6)}{=} \Theta(1)\left(1+\frac{\epsilon n_C}{(1-\epsilon)n_Q - n_C}\right)^{(1-\epsilon)n_Q - n_C}\left(1-\frac{n_C}{n_Q}\right)^{\epsilon n_Q} \\ &\overset{(3.1)}{=} \Theta(1)\exp\left(-(1+o(1))\frac{\epsilon n_C^2}{2n_Q}\right).\end{aligned}$$

There exists an interval I that contains the ranges of the main contribution to both $\Sigma_C(n_Q,l,d)$ and $\Sigma_C(d,(1-\epsilon)n_Q,l)$, such that $n_C=(1+o(1))\sqrt{n_Q(3l-d)}$, and thus in particular $n_C=o(n_Q)$, for all $n_C\in I$. Then for $n_C\in I$ and $d=o(l)$,

$$\frac{s(n_C,d,(1-\epsilon)n_Q,l)}{s(n_C,d,n_Q,l)}=\Theta(1)\exp\left(-\left(\frac{3}{2}+o(1)\right)\epsilon l\right).$$

Summing over $n_C\in I$, we deduce that

$$\Sigma_C(d,(1-\epsilon)n_Q,l)=\Theta(1)\exp\left(-\left(\frac{3}{2}+o(1)\right)\epsilon l\right)\Sigma_C(n_Q,l,d).$$

Combining this with (3.53) and the condition $\epsilon l=\omega(1)$ yields (3.52).

Lemma 3.4.9 yields

$$\frac{\Sigma_d((1-\epsilon)n_Q,l)}{\Sigma_d(n_Q,l)}=\exp(f_d((1-\epsilon)n_Q,l)-f_d(n_Q,l)). \tag{3.54}$$

Suppose that J is an interval that contains the ranges of the main contributions to both $\Sigma_d(n_Q,l)$ and $\Sigma_d((1-\epsilon)n_Q,l)$, such that $d\le d_0=o(l)$ for all $d\in J$. Denote the summands of $\Sigma_d(n_Q,l)$ by

$$s_d(n_Q,l)=\binom{2l}{d}\frac{(3l-d)^{(3l-d+2)/2}e^{d/2}\tau^d}{(3l-d)!n_Q^{d/2}}\exp(f_C(n_Q,l,d)).$$

Recall that $\tau=\tau(d,l)$ does *not* depend on n_Q. With (3.52), we have

$$\frac{s_d((1-\epsilon)n_Q,l)}{s_d(n_Q,l)}=(1-\epsilon)^{-\frac{d}{2}}\exp(o(\epsilon l))$$

for $d\in J$. Summing over J and comparing with (3.54) proves the lemma. □

Proof of Lemma 3.4.19. Let us write $I_l(n)=[p_l(n),q_l(n)]$ and $I_Q^h(n,l)=[p_Q(n,l),q_Q(n,l)]$. Without loss of generality $p_l<l_1<q_l$. We first prove that the main contribution to the sum over l is provided by $I_l(n)$. To that end, we bound the tail of the sum (the part with $l\notin I_l(n)$) from above and prove that this upper bound has smaller order than the lower bound from Lemma 3.4.16.

Observe that for $l\in I_l(n)$, we have

$$\frac{l\overline{m}_U^{2/3}}{\overline{n}_Q}=\Theta(1). \tag{3.55}$$

For this proof, let $s_l(n,m)$ be the summand of the sum Σ_l, i.e. $\Sigma_l=\sum_l s_l(n,m)$, and $s_Q(n,m,l)$ be the summand of $\Sigma_Q=\sum_{n_Q}s_Q(n,m,l)$. We need to show that

$$T:=n^{n+\frac{1}{2}}\left(\frac{e}{2}\right)^m\sum_{l\notin I_l(n)}s_l(n,m)=o\left(|\mathcal{G}_g^*(n,m)|\right).$$

By Lemma 3.4.17, we may take our sum only over l that satisfy (3.26). If l_2 denotes the index where $s_l(n,m)$ takes its maximal value outside $I_l(n)$, then

$$\sum_{l\notin I_l(n)}s_l(n,m)\le ns_{l_2}(n,m). \tag{3.56}$$

In 1Sup, (3.26) is violated for all $l\ge l_b=\Theta(\lambda^3)$ and thus we have the stronger bound

$$\sum_{l\notin I_l(n)}s_l(n,m)\le\Theta(\lambda^3)s_{l_2}(n,m). \tag{3.57}$$

By Lemma 3.4.13, there exists a constant $\alpha>1$ such that

$$s_l(n,m)\le n^{\frac{3}{2}}\left(\frac{2}{e}\right)^{2m-n}M(l,n,m;\alpha), \tag{3.58}$$

where

$$M(l,n,m;\alpha) = l^{-\frac{3l}{2}} \left(\frac{e^2\phi}{4}\right)^l (2m-n+2l)^{\frac{3l}{2}-1}(n-m+l)^{m-n-l-1}\alpha^l.$$

By choosing β_l^- (respectively η_l^- or ϑ_l^-) small enough and β_l^+ (respectively η_l^+ or ϑ_l^+) large enough, we may assume that $M(l,n,m;\alpha)$ is strictly increasing (with respect to l) for $l \le p_l$ and strictly decreasing for $l \ge q_l$. For $l \le p_l$, (3.29) is satisfied and thus (3.58) holds for every $\alpha = 1+\delta$, where $\delta > 0$ is any given constant. Thus,

$$s_{l_2}(n,m) \le n^{\frac{3}{2}} \left(\frac{2}{e}\right)^{2m-n} \max\{M(p_l,n,m;1+\delta), M(q_l,n,m;\alpha)\}. \tag{3.59}$$

In 1SUP, when $\lambda = o\big(n^{1/12}\big)$, Lemma 3.4.14 together with analogous arguments gives us an upper bound

$$s_{l_2}(n,m) \le \lambda\, n^{\frac{2}{3}} \left(\frac{2}{e}\right)^{2m-n} \max\{M(p_l,n,m;\alpha), M(q_l,n,m;\alpha)\}. \tag{3.60}$$

If m is such that (3.60) applies and if the maximum in (3.60) is $M(q_l,n,m;\alpha)$, then (3.57) and (3.60) yield (for large enough β_l^+)

$$\frac{T}{|\mathcal{G}_g^*(n,m)|} \le \lambda^4\, e^{-l_1},$$

which is $o(1)$ by Lemma 3.4.15 and the fact that $\lambda \to \infty$. If (3.60) does *not* apply and the maximum in (3.59) is $M(q_l,n,m,\alpha)$, then (3.56), (3.59), and Lemma 3.4.16 imply that if we choose β_l^+, η_l^+, or ϑ_l^+ large enough, respectively, then

$$\frac{T}{|\mathcal{G}_g^*(n,m)|} \le n^{\frac{5}{2}} e^{-l_1},$$

which is $o(1)$.

If the maximum in (3.59) or (3.60) is $M(p_l,n,m,1+\delta)$ or $M(p_l,n,m,\alpha)$, respectively, then analogous considerations show that we can choose β_l^-, η_l^-, and ϑ_l^- so that for every $m = m(n)$ there exists a constant $c > 0$ such that

$$\frac{T}{|\mathcal{G}_g^*(n,m)|} \le \begin{cases} \lambda^4 \exp(-cl_1) & \text{in 1S\textsc{up} for } \lambda = o(n^{1/12}),\\ n^{\frac{5}{2}} \exp\Big(-c\,\zeta^{-3/2}\, n^{3/5}\Big) & \text{in 2S\textsc{up}},\\ n^{\frac{5}{2}} \exp(-cl_1) & \text{otherwise.} \end{cases}$$

In 2CRIT and 2SUP, the fact that we have $\alpha = 1+\delta$ is essential for deducing the above bound. In all regimes—using that $\zeta = o\big((\log n)^{-2/3} n^{3/5}\big)$ in 2SUP—we deduce that this upper bound is $o(1)$. This proves that the main contribution to Σ_l is indeed provided by $I_l(n)$.

It remains to prove that for each $l \in I_l(n)$, the main contribution to Σ_Q is provided by $I_Q^h(n,m,l)$. We substitute $n_Q = \overline{n}_Q + r$.

First consider the case $n_Q < p_Q = \overline{n}_Q - h\overline{m}_U^{2/3}$, i.e. $r < -h\overline{m}_U^{2/3}$. We shall split the sum into the three parts $-v\overline{m}_U^{2/3} \le r \le -h\overline{m}_U^{2/3}$, $-w\overline{m}_U^{2/3} \le r \le -v\overline{m}_U^{2/3}$, and $r \le -w\overline{m}_U^{2/3}$, where

$$v := \overline{m}_U^{1/24} \qquad \text{and} \qquad w := \begin{cases} \lambda^{1/2} & \text{in 1S\textsc{up}},\\ l\overline{m}_U^{-2/9} & \text{otherwise.} \end{cases}$$

Observe that the interval $-w\overline{m}_U^{2/3} \leq r \leq -v\overline{m}_U^{2/3}$ is empty in 1SUP if $\lambda < \overline{m}_U^{1/12}$. Furthermore,

$$w = \omega\left(\sqrt{\frac{l^3}{\overline{n}_Q}}\right) \qquad \text{and} \qquad w = o\left(\frac{\overline{n}_Q}{\overline{m}_U^{2/3}}\right) \overset{(3.55)}{=} o(l). \tag{3.61}$$

By (3.51) and Lemma 3.4.16, in each of the three intervals,

$$\frac{\sum \rho\psi}{\Sigma_Q(n,m,l)} \leq \Theta(1)\overline{m}_U^{-\frac{1}{6}} \sum s_r(n,m,l) \tag{3.62}$$

with

$$s_r(n,m,l) = \left(1 + \frac{r}{\overline{n}_Q}\right)^{\frac{3l}{2}-1} \exp(f_d(\overline{n}_Q + r, l) - f_d(\tilde{n}_Q, l)).$$

Recall that for (3.51), Lemma 3.3.1(iv) was used to bound ρ. Observe that for $-v\overline{m}_U^{2/3} \leq r \leq -h\overline{m}_U^{2/3}$, Lemma 3.3.1(iii) is applicable and thus (3.62) holds with a factor of $\overline{m}_U^{-\frac{2}{3}}$ instead of $\overline{m}_U^{-\frac{1}{6}}$. Furthermore, we claim that $f_d(\overline{n}_Q+r,l)-f_d(\tilde{n}_Q,l) = o\left(\frac{rl}{\overline{n}_Q}\right)$. Indeed, in 1SUP and INT, the left hand side is $O(1)$ and the claim follows by observing that $\frac{rl}{\overline{n}_Q} = \Omega(h)$ by (3.55). In the second phase transition, such r satisfy the conditions of Lemma 3.4.18 with $\epsilon = \Theta\left(\frac{r}{\overline{n}_Q}\right)$ and thus the claim follows. Therefore, there exists a constant $c > 0$ such that

$$\begin{aligned}\frac{\sum_{r=-v\overline{m}_U^{2/3}}^{-h\overline{m}_U^{2/3}} \rho\psi}{\Sigma_Q(n,m,l)} &\leq \Theta(1)\overline{m}_U^{-\frac{2}{3}} \sum_{r=-v\overline{m}_U^{2/3}}^{-h\overline{m}_U^{2/3}} \exp\left(\left(\frac{3}{2} - o(1)\right)\frac{rl}{\overline{n}_Q}\right) \\ &\overset{(3.55)}{\leq} \Theta(1)\int_h^\infty e^{-cx}\,\mathrm{d}x = \Theta(1)\exp(-ch) = o(1).\end{aligned}$$

Observe that in 1SUP, if $\lambda = o(n^{1/24})$, then $r > -\overline{n}_Q > -v\overline{m}_U^{2/3}$ and thus the interval $-v\overline{m}_U^{2/3} \leq r \leq -h\overline{m}_U^{2/3}$ covers all cases for negative r. From now on, we may thus assume that $\lambda = \Omega(n^{1/24})$, which implies $w = \Omega(n^{1/48})$.

Now consider the interval $-w\overline{m}_U^{2/3} \leq r \leq -v\overline{m}_U^{2/3}$. In this regime, we still have $f_d(\overline{n}_Q + r, l) - f_d(\tilde{n}_Q, l) = o\left(\frac{rl}{\overline{n}_Q}\right)$ and thus

$$\frac{\sum_{r=-w\overline{m}_U^{2/3}}^{-v\overline{m}_U^{2/3}} \rho\psi}{\Sigma_Q(n,m,l)} \leq \Theta(1)\overline{m}_U^{1/2}\exp(-cv) = o(1).$$

Finally, suppose that $r \leq -w\overline{m}_U^{2/3}$. In this regime,

$$s_r \leq \exp\left(\left(\frac{3l}{2} - 1\right)\log\left(1 + \frac{r}{\overline{n}_Q}\right) + c_1\sqrt{\frac{l^3}{\overline{n}_Q + r}} - c_2\sqrt{\frac{l^3}{\tilde{n}_Q}}\right).$$

The right hand side has its maximum (with respect to r) at $r = -w\overline{m}_U^{2/3}$. For this r, the first summand is negative and has order w by (3.55). The other two summands are $o(w)$ by (3.61). Thus, there exists a constant $c > 0$ such that

$$\frac{\sum_{r \leq -w\overline{m}_U^{2/3}} \rho\psi}{\Sigma_Q(n,m,l)} \leq \Theta(1) n \exp(-cw),$$

which is $o(1)$, because $w = \Omega(n^{1/48})$. This finishes the proof for $r < 0$.

Suppose now that $n_Q > q_Q = \overline{n_Q} + h\overline{m}_U^{2/3}$, i.e. $r > h\overline{m}_U^{2/3}$. By (3.46), (3.47), and Lemma 3.4.16 we conclude that

$$\frac{\sum \rho\psi}{\Sigma_Q(n,m,l)} \le \Theta(1)\overline{m}_U^{-\frac{2}{3}} \sum_{r > h\overline{m}_U^{2/3}} \exp\left(\frac{3lr}{2\overline{n}_Q} + f_d(\overline{n}_Q + r, l) - f_d(\tilde{n}_Q, l) - \frac{r^3}{27\overline{m}_U^2}\right).$$

Note that for all r in this sum, $\frac{rl}{\overline{n}_Q} = o\left(\frac{r^3}{\overline{m}_U^2}\right)$. We claim that additionally

$$f_d(\overline{n}_Q + r, l) - f_d(\tilde{n}_Q, l) = o\left(\frac{r^3}{\overline{m}_U^2}\right).$$

Indeed, this difference is $O(1)$ in 1Sup and Int, while $\frac{r^3}{\overline{m}_U^2} \ge h^3 = \omega(1)$. In the second phase transition, the claim follows immediately if $r \ge \sqrt{l}\overline{m}_U^{2/3}$. If $h\overline{m}_U^{2/3} < r < \sqrt{l}\overline{m}_U^{2/3}$, the conditions of Lemma 3.4.18 are satisfied with $\epsilon = \Theta\left(\frac{r}{\overline{n}_Q}\right)$ and thus $f_d(\overline{n}_Q + r, l) - f_d(\tilde{n}_Q, l) = o\left(\frac{rl}{\overline{n}_Q}\right) = o\left(\frac{r^3}{\overline{m}_U^2}\right)$. Therefore, we deduce that

$$\begin{aligned}\frac{\sum \rho\psi}{\Sigma_Q(n,m,l)} &\le \Theta(1)\overline{m}_U^{-\frac{2}{3}} \sum_{r > h\overline{m}_U^{2/3}} \exp\left(-\frac{r^3}{30\overline{m}_U^2}\right) \\ &\le \Theta(1)\int_h^\infty \exp\left(-x^3\right) \mathrm{d}x \le \Theta(1)\exp(-h).\end{aligned}$$

This finishes the proof also for $r > 0$. □

3.8. Discussion and open problems

Comparing the range for m that we cover in Theorems 3.1.5–3.1.8 with the 'dense' regime $m = \lfloor \mu n \rfloor$ for $1 < \mu < 3$ considered in [**30, 59**], a gap of order $(\log n)^{2/3}$ becomes apparent—a significant improvement of [**66**], where the gap had order $n^{1/3}$. The order term $\zeta^{-3/2} n^{3/5}$ in Theorems 3.1.6 and 3.1.8 becomes constant when $\zeta = \Theta(n^{2/5})$, which matches the results from [**30, 59**] that the giant component covers all but finitely many vertices in the dense regime. Therefore, we expect Theorems 3.1.6 and 3.1.8 to hold for all $m = (1 + o(1))n$.

The gap of order $(\log n)^{2/3}$ originates from the fact that we can only determine the number of cores up to an exponential error term in the second phase transition. This error term has two causes. On one hand, the bounds for the number of kernels differ by an exponential factor (see Lemma 3.4.5). On the other hand, a second exponential error term appears when we construct the core from the kernel by subdividing edges (see Lemma 3.4.6). We thus believe that the key to closing the gap would be to determine the number of cores more precisely.

Problem 3.8.1. *Find the exact value of $|\mathcal{C}_g(n_C, n_C + l)|$ for any $n_C, l \in \mathbb{N}$.*

Solving Problem 3.8.1 would pave the way to prove Theorem 3.1.6 for all $m = (1+o(1))n$. Moreover, it might open the possibility to prove an analogous version of Theorem 3.5.4 in the second phase transition, thus rendering the additional double counting argument in the proof of Theorem 3.1.6 unnecessary.

Conjecture 3.8.2. *Let $m = \left(2 + \zeta\, n^{-2/5}\right)\frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$ tends to ∞ with n. Then whp the largest component H_1 of $G_g(n,m)$ is complex, has genus g, and satisfies*

$$n - |H_1| = \Theta\left(\zeta^{-3/2}\, n^{3/5}\right).$$

It is important to note that the results in this chapter apply to more general graph classes than $\mathcal{G}_g(n,m)$. Indeed, the constructive decomposition that yields (3.12), (3.13), and (3.14) relies on the fact that a graph is in $\mathcal{G}_g$ if and only if its kernel is in the corresponding class $\mathcal{K}_g$ of multigraphs. The only other ingredients of the proof that are specifically tailored for graphs on $\mathbb{S}_g$ are Theorems 3.4.1 and 3.4.2, and Lemmas 3.4.3 to 3.4.5. Recall that we saw in Section 3.7.2 that Lemma 3.4.5 holds for any class of multigraphs that is weakly addable (that is, closed under adding an edge between two components; see e.g. [**83**] for some results) and closed under taking minors.

Remark 3.8.3. *Let $\mathcal{X}$ be a graph class and $\mathcal{Y}$ be a class of (weighted) multigraphs of minimum degree at least three. Suppose that*

(i) a graph lies in $\mathcal{X}$ if and only if its kernel is in $\mathcal{Y}$;

(ii) there are constants $c,\gamma > 0$ and $k \in \mathbb{R}$ such that

$$|\mathcal{Y}(2l,3l)| = (1+o(1))c\,l^k\gamma^{2l}(2l)!;$$

(iii) there is a constant $0 < q \leq 1$ with

$$\mathbb{P}\left[Y(2l,3l) \text{ is connected}\right] \overset{l\to\infty}{\longrightarrow} q;$$

(iv) $|H_1(Y(2l,3l))| = 2l - O_p(1)$ and for each fixed $i \in \mathbb{N}\setminus\{0\}$, the probability that $|H_1(Y(2l,3l))| = 2l-2i$ is bounded away from both 0 and 1;

(v) $\mathcal{Y}$ is weakly addable and closed under taking minors.

Then analogous statements to Theorems 3.1.5–3.1.8 hold for $\mathcal{X}$.

Obvious candidates for the classes $\mathcal{X}$ and $\mathcal{Y}$ would be (multi)graphs on non-orientable surfaces. For such classes, (i) and (v) in Remark 3.8.3 are automatically satisfied, (ii) and (iii) would follow if Theorems 3.4.1 and 3.4.2 also hold for non-orientable surfaces, and (iv) holds if Lemma 3.4.3 is true for non-orientable surfaces.

Problem 3.8.4. *Prove analogous versions of Theorems 3.4.1 and 3.4.2 as well as Lemma 3.4.3 for non-orientable surfaces.*

One striking difference between $G_g(n,m)$ and $G(n,m)$ is the order and the structure of the i-th largest component for $i \geq 2$. In $G_g(n,m)$, the second largest component is much larger than in $G(n,m)$; in 1Sup for instance, the order is $\Theta_p(n^{2/3})$ versus $o(n^{2/3})$. Moreover, the i-th largest component of $G(n,m)$ is a tree whp. In contrast, the largest tree components of $G_g(n,m)$ have order $\Theta_p(n^{2/3})$, and it can also contain complex components of that order. It would thus be interesting to know whether there is a hierarchy in the size of the largest tree component and the second largest complex component.

Question 3.8.5. *Given $i \geq 2$, what is the probability that the i-th largest component of $G_g(n,m)$ is a tree?*

For $G(n,m)$, the giant component is in fact far better understood than it is stated in Theorem 3.1.1. Central and local limit theorems provide much stronger concentration results about the order (i.e. the number of vertices) and the size (i.e. the number of edges) of the giant component [**4, 5, 24, 25, 94, 101**] and give more insight into the global and local structure of the giant component and its core.

Problem 3.8.6. *Derive central and local limit theorems for the giant component of $G_g(n,m)$.*

As mentioned in Section 3.1, the component structure of $G(n,m)$ is closely related to a Galton-Watson branching process. More precisely, the local structure of $G(n,\alpha\frac{n}{2})$ converges to that of a Galton-Watson tree with offspring distribution $\mathrm{Po}(\alpha)$ in the sense of Benjamini-Schramm local weak convergence [**14, 71**]. For

$G_g(n,m)$, the additional constraint of the graph being embeddable on $\mathbb{S}_g$, exploration via a simple Galton-Watson type process is not possible. This naturally raises the question if the local structure of $G_g(n,m)$ can be described in terms of the Benjamini-Schramm local weak convergence.

Question 3.8.7. *What is the limit of the local structure of $G_g(n,m)$ in the sense of the Benjamini-Schramm local weak convergence?*

The core, which plays a central role in our constructive decomposition, is also known as the 2-*core*. More generally, given $k \geq 2$, the k-*core* of a graph G is the largest subgraph of G of minimum degree at least k. Like the core, the k-core can be constructed by a *peeling process* that recursively removes vertices of degree less than k. The order and size of the k-core of $G(n,m)$ has been determined in a seminal paper by Pittel, Spencer, and Wormald [**93**]. Following Pittel, Spencer, and Wormald, the k-core has been extensively studied [**31, 32, 63, 73, 76, 98**]. The most striking results in this area are the astonishing theorem by Łuczak [**76**] that the k-core for $k \geq 3$ jumps to linear order at the very moment it becomes non-empty, the central limit theorem by Janson and Luczak [**63**], and the local limit theorem by Coja-Oghlan, Cooley, Kang, and Skubch [**31**] that described—in addition to the order and size—several other parameters of the k-core of $G(n,m)$. In [**32**], the same authors used a 5-type branching process in order to determine the local structure of the k-core. In terms of *global* structure, [**31**] provides a randomised algorithm that constructs a random graph with given order and size of the k-core.

Question 3.8.8. *What are the local and global structure of the k-core of $G_g(n,m)$?*

One of the main difficulties regarding $G_g(n,m)$ is that while graph properties such as having a component of a certain order are monotone for $G(n,m)$ (that is, for every fixed n, the probability that $G(n,m)$ has this property is monotone for $0 \leq m \leq \binom{n}{2}$), this is not necessarily the case for $G_g(n,m)$. Indeed, monotonicity of graph properties in $G(n,m)$ usually follows immediately from the equivalence between $G(n,m)$ and the *random graph process*, where we add one random edge at a time. For graphs on surfaces, however, not all edges are allowed to be added in the corresponding process. Thus, the process is fundamentally different from $G_g(n,m)$. For instance, in the dense regime $m = \lfloor \mu n \rfloor$ with $\mu > 1$, we know by [**59**] that the probability that $P(n,m)$ is connected is bounded away from both 0 and 1. The planar graph *process*, however, is connected whp in that regime [**58**]. Knowing which graph properties are monotone for $G_g(n,m)$ would yield a significant improvement to the complexity of the arguments.

Question 3.8.9. *Which graph properties are monotone for $G_g(n,m)$?*

The constructive decomposition and generating functions of cubic planar graphs and their relation to the core of sparse planar graphs by Kang and Łuczak [**66**] have been strengthened by Noy, Ravelomanana, and Rué [**87**] to yield an answer to a challenging open question of Erdős and Rényi [**41**] about the limiting probability of $G(n,m)$ being planar at the critical phase 1Crit, that is, for every constant $\lambda \in \mathbb{R}$, the limit $p(\lambda)$ of the probability that $G\big(n, \big(1 + \lambda\, n^{-1/3}\big)\frac{n}{2}\big)$ is planar. For graphs embeddable on a surface of positive genus, they gave a general strategy of how to determine the corresponding probability. However, determining the *exact* limiting probability for $g \geq 1$ is still an open problem.

Furthermore, for m beyond 1Crit, we know that $G(n,m)$ whp is not embeddable on any surface of fixed genus. This immediately raises the question what genus g we need in order to embed $G(n,m)$ on $\mathbb{S}_g$.

Question 3.8.10. *Let $m = m(n)$ and $g = g(n)$ be given.*

(i) When is the limiting probability of $G(n,m)$ being embeddable on $\mathbb{S}_g$ positive?
(ii) When is $G(n,m)$ embeddable on $\mathbb{S}_g$ whp?
(iii) What is the expected genus of $G(n,m)$?

Another interesting direction, which might provide insight into the answer of Question 3.8.10, is to consider $G_g(n,m)$ for genus $g = g(n)$ that tends to infinity with n. If g grows 'fast enough' (e.g. as $\binom{n}{2}$), then $G_g(n,m)$ will coincide with $G(n,m)$ and will thus exhibit the emergence of the giant component, but not the second phase transition described in Theorem 3.1.6. For 'slowly' growing g, on the other hand, it is to be expected that the second phase transition does take place.

Question 3.8.11. *For which functions $g = g(n)$ does $G_g(n,m)$ feature two phase transitions analogous to Theorems 3.1.5 and 3.1.6?*

CHAPTER 4

Cubic graphs with constant genus

4.1. Introduction

Determining the numbers of maps and graphs *embeddable* on surfaces have been one of the main objectives of enumerative combinatorics for the last 50 years. Starting from the enumeration of *planar* maps by Tutte [**104**] various types of maps on the sphere were counted, e.g. planar *cubic* maps by Gao and Wormald [**54**]. Furthermore, Tutte's methods were generalised to enumerate maps on surfaces of higher genus [**7, 8, 13**].

An important subclass of maps are *triangulations*. Brown [**29**] determined the number of triangulations of a disc, and Tutte [**103**] enumerated planar triangulations. Later, Rathie [**95**] enumerated *simple* planar triangulations. Triangulations on other surfaces have been considered as well. Gao enumerated 2-connected triangulations on the projective plane [**50**] as well as connected [**51**], 2-connected [**52**] and 3-connected [**53**] triangulations on surfaces of arbitrary genus.

Frieze [**70**] was arguably the first to ask about properties of random planar *graphs*. McDiarmid, Steger, and Welsh [**82**] showed the existence of an exponential growth constant for the number of vertex-labelled planar graphs with n vertices. This growth constant and the asymptotic number of planar graphs were determined by Giménezand Noy [**59**], while the corresponding results for the higher genus case were derived by Chapuy, Fusy, Giménez, Mohar and Noy [**30**] and independently by Bender and Gao [**9**]. Since then various other classes of planar graphs were counted [**12, 19, 20, 64, 66, 69, 89**].

An interesting subclass of planar graphs is the class of *cubic* planar graphs, which have been counted by Bodirsky, Kang, Löffler and McDiarmid [**20**]. Cubic planar graphs occur as substructures of sparse planar graphs and have thus been one of the essential ingredients in the study of sparse random planar graphs [**66**]. For surfaces of higher genus, the number of embeddable cubic graphs has not been studied.

Throughout the paper, let g be a fixed non-negative integer and let $\mathbb{S}_g$ be the orientable[1] surface of genus g. In this paper, we study cubic graphs embeddable on $\mathbb{S}_g$, in particular their asymptotic number. Similar to the case of planar graphs, cubic graphs embeddable on $\mathbb{S}_g$ appear as essential substructures of sparse graphs embeddable on $\mathbb{S}_g$. Therefore, the results of this paper pave the way to the study of sparse random graphs embeddable on $\mathbb{S}_g$ [**67**].

4.1.1. **Main results.** The main contributions of this paper are fourfold. We determine the asymptotic number of cubic multigraphs embeddable on $\mathbb{S}_g$. We also determine the asymptotic number of *weighted cubic multigraphs and cubic* simple graphs embeddable on $\mathbb{S}_g$. Finally we prove that almost all (multi)graphs from either of the three classes have exactly one non-planar component.

The first main result provides the exact asymptotic expression of the number of cubic multigraphs embeddable on $\mathbb{S}_g$.

[1]We believe that one can also prove the main results for multigraphs embeddable on non-orientable surfaces, but with considerably more effort and case distinctions.

THEOREM 4.1.1. *The number $m_g(n)$ of vertex-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $2n$ vertices is given by*

$$m_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) d_g n^{5g/2-7/2} \gamma_1^{2n} (2n)!\,,$$

where γ_1 is an algebraic constant independent of the genus g and d_g is a constant depending only on g. The first digits of γ_1 are 3.986.

Our next main result concerns multigraphs weighted by the so-called *compensation factor* introduced by Janson, Knuth, Łuczak and Pittel [**62**]. This factor is defined as the number of ways to orient and order all edges of the multigraph divided by $2^r r!$, which is equal to the number of such oriented orderings if all edges were distinguishable. For example, a double edge results in a factor $\frac{1}{2}$ and simple graphs are the only multigraphs with compensation factor one.

THEOREM 4.1.2. *The number $w_g(n)$ of vertex-labelled cubic multigraphs embeddable on $\mathbb{S}_g$ with $2n$ vertices* weighted by their compensation factor is given by

$$w_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) e_g n^{5g/2-7/2} \gamma_2^{2n} (2n)!\,,$$

where $\gamma_2 = \frac{79^{3/4}}{54^{1/2}}$ and e_g is a constant depending only on the genus g. The first digits of γ_2 are 3.606

Theorem 4.1.2 can be used to derive the asymptotic number and structural properties of graphs embeddable on $\mathbb{S}_g$ [**67**]. *Planar* cubic multigraphs weighted by the compensation factor were counted by Kang and Łuczak [**66**]. The discrepancy to their exponential growth constant $\gamma \approx 3.38$ is due to incorrect initial conditions in [**66**], as pointed out by Noy, Ravelomanana and Rué [**87**]. While the explicit value of the correct exponential growth constant γ was not determined in [**87**], the implicit equations given there yield the same exponential growth constant γ_2 as in Theorem 4.1.2.

Our methods also allow us to count cubic *simple* graphs (graphs without loops and multi-edges) embeddable on $\mathbb{S}_g$.

THEOREM 4.1.3. *The number $s_g(n)$ of vertex-labelled cubic* simple graphs embeddable on $\mathbb{S}_g$ with $2n$ vertices is given by

$$s_g(n) = \left(1 + O\left(n^{-1/4}\right)\right) f_g n^{5g/2-7/2} \gamma_3^{2n} (2n)!\,,$$

where γ_3 is an algebraic constant independent of the genus g and f_g is a constant depending only on g. The first digits of γ_3 are 3.133.

The exponential growth constant γ_3 coincides with the growth constant calculated for vertex-labelled cubic simple planar graphs by Bodirsky, Kang, Löffler and McDiarmid [**20**].

The final result describes the structure of cubic multigraphs embeddable on $\mathbb{S}_g$.

THEOREM 4.1.4. *Let $g \geq 1$ and let G be a graph chosen uniformly at random from the class of vertex-labelled cubic multigraphs, cubic weighted multigraphs, or cubic simple graphs embeddable on $\mathbb{S}_g$ with n vertices, respectively. Then with probability $1 - O(n^{-2})$, G has one component that is embeddable on $\mathbb{S}_g$, but not on $\mathbb{S}_{g-1}$, while all other components of G are planar.*

4.1.2. **Proof techniques.** To derive our results we will use topological manipulations of surfaces called *surgeryies*, constructive decomposition of graphs along connectivity, and singularity analysis of generating functions.

More precisely, in order to enumerate cubic multigraphs we apply constructive decompositions along connectivity. The basic building blocks in the decomposition are 3-connected cubic graphs, which we will then relate to their corresponding cubic *maps*. Note that, due to Whitney's Theorem [**106**], 3-connected *planar* graphs have a unique embedding on the sphere. Therefore, we can directly relate 3-connected planar graphs to the corresponding maps. For surfaces of positive genus, however, embeddings of 3-connected graphs are *not* unique. Following an idea from [**30**], we circumvent this problem by using the concept of the *facewidth* of a graph and by applying results of Robertson and Vitray [**99**] which relate 3-connected graphs and maps.

Counting 3-connected cubic maps on $\mathbb{S}_g$ is a challenging task. We shall use the dual of cubic maps, triangulations, in order to overcome this challenge. In fact, Gao [**50, 52, 53**] enumerated triangulations on $\mathbb{S}_g$ with various restrictions on the existence of loops and multi-edges. However, it turns out that the duals of 3-connected cubic maps on $\mathbb{S}_g$ have very specific constraints that have not been considered by Gao. In this paper we therefore investigate such triangulations by relating them to simple triangulations counted by Gao [**52**] (see Propositions 4.3.2 to 4.3.4). We strengthen Gao's result and derive very precise singular expansions of generating functions. These expansions are obtained from recursive formulas for the generating functions, which we derive by applying surgeries to the surfaces on which the respective triangulations are embedded. This enables us to apply singularity analysis to the generating functions of these triangulations, as well as to the generating functions of all other classes of maps and graphs considered in this paper.

This paper is organised as follows. In Section 4.2 we introduce some basic notions and notations. In Section 4.3 we enumerate the triangulations that are duals of 3-connected cubic maps and in Section 4.4 we prove the main results (Theorems 4.1.1 to 4.1.4) after giving a constructive decomposition along connectivity. Our strengthening of Gao's results and proofs, as well as other proofs for similar theorems from Section 4.3, are given in the appendix.

4.2. Preliminaries

A graph G is *simple* if it does not contain loops or multi-edges. If in a multigraph there are more than two edges connecting the same pair of vertices, we call each pair of those edges a *double edge*. Therefore, every multi-edge consisting of r edges between the same two vertices contains $\binom{r}{2}$ double edges. If e is a loop and incident to a vertex v, we say that v is the *base* of e. Similarly, we say that e is *based at* its base. An edge that is neither a loop nor part of a double edge is a *single edge*. An edge e of a connected multigraph G is called a *bridge* if deleting e disconnects G.

A multigraph is called *cubic* if each vertex has degree three. We adopt the convention that a loop counts as two in the degree of its base. By Φ we denote the cubic multigraph with two distinguished vertices u, v and three edges between u and v (i.e. a triple edge). Given a connected cubic multigraph G, let k and l denote the number of double edges and loops of G, respectively. We define the *weight* of G to be

$$W(G) = \begin{cases} \frac{1}{6} & \text{if } G = \Phi, \\ 2^{-(k+l)} & \text{otherwise.} \end{cases}$$

If G is not connected, we define $W(G)$ as the product of weights of its components. For cubic multigraphs, this weight coincides with the compensation factor introduced in [**62**]. Throughout this paper, when we refer to a *weighted cubic multigraph* G, the weight in consideration will always be $W(G)$.

Definition 4.2.1. An *embedding* of a multigraph G on $\mathbb{S}_g$ is a drawing of G on $\mathbb{S}_g$ without crossing edges. We consider G as a subset of $\mathbb{S}_g$, and therefore $\mathbb{S}_g \setminus G$ consists of connected components called *faces*. An embedding where additionally all faces are homeomorphic to open discs, or equivalently, where all faces are simply connected, is called a 2-*cell embedding*. Multigraphs that have an embedding are called *embeddable* on $\mathbb{S}_g$ and multigraphs that have a 2-cell embedding are called *strongly embeddable*.

A 2-cell embedding of a strongly embeddable multigraph is also called *map*. A *triangulation* is a map where each face is bounded by a triangle. These triangles might be degenerated, i.e., being three loops with the same base, or a double edge and a loop based at one of the end vertices of the double edge, or a loop and an edge from the base of the loop to a vertex of degree one.

If S is the disjoint union of $\mathbb{S}_{g_1}, \dots, \mathbb{S}_{g_r}$ for non-negative integers $g_1, \dots, g_r$ and M_i is a 2-cell embedding of a graph G_i on $\mathbb{S}_{g_i}$ for each $i = 1, \dots, r$, then the induced function $N : (G_1 \cup \dots \cup G_n) \to S$ is called a *map* on S. Triangulations on S are defined analogously. We denote by $V(M)$, $E(M)$, and $F(M)$ the set of all vertices, edges, and faces of an embedding M, respectively.

We call a set $E' \subseteq E(M)$ *separating*, if the map $M' = (V(M), E')$ has at least two faces, i.e. if M' separates the surface.

From results of Mohar and Thomassen [**86**] we obtain some initial properties of embeddable graphs.

Proposition 4.2.2. [**86**] *Let G be a multigraph.*

(i) If G is connected and g is minimal such that G is embeddable on $\mathbb{S}_g$, then every embedding of G on $\mathbb{S}_g$ is a 2-cell embedding. In particular, G is strongly embeddable on $\mathbb{S}_g$.

(ii) G is embeddable on $\mathbb{S}_g$ if and only if each connected component C_i of G is strongly embeddable on a surface $\mathbb{S}_{g_i}$ such that $\sum_i g_i \leq g$.

Let M be a map on a surface $\mathbb{S}$. We construct the *dual map* of M by first putting a vertex in each face of M, then for each edge e in M, we draw an edge between the two (possibly coincident) vertices inside the faces on both side of e while crossing e exactly once. The newly drawn edges should only intersect at their end points. Note that the dual map has multi-edges if two faces of the original (*primal*) map have more than one edge in common. It is well known that the dual of a map is again a map, see e.g. [**86**].

For each vertex $v \in V(M)$ of a map M, the edges and faces incident to v have a canonical cyclic order $e_0, f_0, e_1, f_1, \dots, e_{d-1}, f_{d-1}$ by the way they are arranged around v (in counterclockwise direction). Note that faces can appear multiple times here and that a loop based at v will appear twice in this sequence. To avoid ambiguities, we distinguish the two ends of the loop in this sequence (e.g. by using half-edges or by orienting each loop). A triple $(v, e_i, e_{(i+1) \bmod d})$ of a vertex v and two consecutive edges $e_i, e_{i+1 \bmod d}$ in the cyclic sequence is called a *corner* (at v). We also say that $(v, e_i, e_{(i+1) \bmod d})$ is a corner *of the face* f_i. When we enumerate maps, we always work with maps with one distinguished corner, called the *root* of the map. If (v, e_i, e_{i+1}) is the root corner, we will call v the *root vertex*, e_i the *root edge*, and f_i the *root face*.

4.2.1. **Generating functions and singularity analysis.** We will use generating functions to enumerate the various classes of maps, graphs and multigraphs we consider. Unless stated otherwise, the formal variables x and y will always mark vertices and edges respectively. Generating functions for classes of *maps* will be *ordinary* unless stated otherwise. Generating functions for *multigraphs* will be *exponential* in x, because we always consider *vertex-labelled* multigraphs. If $\mathcal{A}$ is a class of maps, we write $\mathcal{A}(m)$ for the subclass of $\mathcal{A}$ containing all maps with exactly m edges. The generating function $\sum_m |\mathcal{A}(m)| y^m$ will be denoted by $A(y)$. If $\mathcal{B}$ is a class of multigraphs, we write $\mathcal{B}(n)$ for the subclass of $\mathcal{B}$ containing all multigraphs with exactly n vertices. The generating function $\sum_n \frac{|\mathcal{B}(n)|}{n!} x^n$ will be denoted by $B(x)$. For an ordinary generating function $F(z) = \sum_n f_n z^n$, we use the notation $[z^n]F(z) := f_n$. For an exponential generating function $G(z) = \sum \frac{g_n}{n!} z^n$, we write $[z^n]G(z) := \frac{g_n}{n!}$.

If two generating functions $F(z), G(z)$ satisfy $0 \leq [z^n]F(z) \leq [z^n]G(z)$ for all n, we say that F is *coefficient-wise smaller* than G, denoted by $F \preceq G$. The singularities of $F(z)$ with the smallest modulus are called *dominant singularities* of $F(z)$. Because every generating function we consider in this paper always has non-negative coefficients $[z^n]F(z)$, there is a dominant singularity located on the positive real axis by Pringsheim's Theorem [**102**, pp. 214 ff.]. We denote this dominant singularity by ρ_F. If an arbitrary function $F : \mathbb{C} \to \mathbb{C}$ has a unique singularity with smallest modulus and this singularity lies on the positive real axis, then we also denote it by ρ_F. The function F converges on the open disc of radius ρ_F and thus corresponds to a holomorphic function on this disc. In many cases, this function can be holomorphically extended to a larger domain. Given $\rho, R \in \mathbb{R}$ with $0 < \rho < R$ and $\theta \in (0, \pi/2)$,

$$\Delta(\rho, R, \theta) := \{z \in \mathbb{C} \mid |z| < R \wedge |\arg(z - \rho)| > \theta\}$$

is called a Δ-*domain*. Here, $\arg(z)$ denotes the *argument* of a complex number, i.e. $\arg(0) := 0$ and $\arg(re^{it}) := t$ for $r > 0$ and $t \in (-\pi, \pi]$. We say that F is Δ-*analytic* if it is holomorphically extendable to some Δ-domain $\Delta(\rho_F, R, \theta)$.

A function F is *subdominant* to a function G if either $\rho_F > \rho_G$ or $\rho_F = \rho_G$ and $\lim_{z \to \rho_G} \frac{F(z)}{G(z)} = 0$. In the latter case, if both F and G are Δ-analytic, then in the above limit, z is taken from some fixed Δ-domain to which both F and G are holomorphically extendable. If F is subdominant to G, we also write $F(z) = o(G(z))$. Analogously we write $F(z) = O(G(z))$ if either $\rho_F > \rho_G$ or $\rho_F = \rho_G$ and $\limsup_{z \to \rho_G} \frac{|F(z)|}{|G(z)|} < \infty$.

Given a function $F(z)$ with a dominant singularity ρ_F, we say that a function $G(z) = c\big(1 - \rho_F^{-1} z\big)^{-\alpha}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}, c \in \mathbb{R} \setminus \{0\}$ or $G(z) = c \log\big(1 - \rho_F^{-1} z\big)$ is the *dominant term* of F if there is a decomposition

$$F(z) = P(z) + G(z) + o(G(z)),$$

where $P(z)$ is a polynomial. The dominant term, if it exists, is uniquely defined and Δ-analytic. If $G(z) = c(1 - \rho_F^{-1} z)^{-\alpha}$, the exponent $-\alpha$ is called the *dominant exponent* of F. If $G(z) = c \log\big(1 - \rho_F^{-1} z\big)$, then we say that F has the dominant exponent 0.

The number of edges in cubic multigraphs and triangulations is always a multiple of three. In terms of generating functions, this is reflected by the existence of three different dominant singularities, all of which differ only by a third root of unity. The corresponding dominant terms will also be the same up to a third root of unity. Analogously, the number of vertices in cubic multigraphs is always even, resulting in two dominant singularities ρ_F and $-\rho_F$. Again, the dominant terms differ only by a factor of -1. In either case, the terms for the coefficients coming from the different dominant singularities will also differ only by the corresponding root of unity. Therefore, we will state our results only for the singularity ρ_F. With a slight abuse of notation, we will also refer to ρ_F as *the* dominant singularity.

Singularity analysis allows us to derive an asymptotic expression for the coefficients of a generating function $F(z)$ with help of the dominant singularity and the dominant term of $F(z)$. We state the well-known 'transfer theorem' for the specific cases we will need.

THEOREM 4.2.3 ([**46**]). *Let $A(z)$ be a Δ-analytic generating function.*

(i) If

$$A(z) = P(z) + c\big(1-\rho_A^{-1}z\big)^{-\alpha} + O\Big(\big(1-\rho_A^{-1}z\big)^{1/4-\alpha}\Big)$$

with a polynomial $P(z)$ and constants $c \neq 0$, $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, then

$$[z^n]A(z) = \Big(1 + O\Big(n^{-1/4}\Big)\Big)\frac{c}{\Gamma(\alpha)}n^{\alpha-1}\rho_A^{-n}.$$

Here, $\Gamma(\alpha) := \int_0^\infty z^{\alpha-1}e^{-z}\,\mathrm{d}z$ is the gamma function.

(ii) If

$$A(z) = P(z) + c\cdot\log\big(1-\rho_A^{-1}z\big) + O\Big(\big(1-\rho_A^{-1}z\big)^{1/4}\Big),$$

then

$$[z^n]A(z) = \Big(1 + O\Big(n^{-1/4}\Big)\Big)(-c)n^{-1}\rho_A^{-n}.$$

We use the standard notation $\gamma_A = \rho_A^{-1}$ for the exponential growth constant of $[z^n]A(z)$. If we are counting rooted maps or multigraphs, the roots will be counted in the generating function unless stated otherwise. We will often mark vertices or edges of multigraphs or maps, which corresponds to applying the differential operator $z\frac{d}{dz}$ to the generating functions (with $z = x$ if vertices are marked and $z = y$ if edges are marked). To simplify notation we write δ_z for $z\frac{d}{dz}$ and δ_z^n for repeatedly applying n times the operator $z\frac{d}{dz}$, which corresponds to marking n vertices or edges, while *allowing multiple marks*. We use the notation $F'(z) = \frac{dF}{dz}$ for the standard differential operator.Vice versa, we say that F is a *primitive* of F'.

The dominant terms of derivatives and primitives of Δ-analytic functions can be determined using Theorems VI.8 and VI.9 from [**47**]. Again, we state these results in a slightly different way tailored for our specific needs.

Lemma 4.2.4 ([**47**]). *Let $A(z)$ be a Δ-analytic generating function with the dominant term $A_d(z)$. Suppose that there exists $\beta \in \mathbb{R}$ with*

$$A(z) = P(z) + A_d(z) + O\Big(\big(1-\rho_A^{-1}z\big)^{-\beta}\Big),$$

where $P(z)$ is a polynomial and $\big(1-\rho_A^{-1}z\big)^{-\beta} = o(A_d(z))$.

(i) We have

$$A'(z) = P'(z) + A_d'(z) + O\Big(\big(1-\rho_A^{-1}z\big)^{-\beta-1}\Big).$$

(ii) If in addition $A_d(z) = c\big(1-\rho_A^{-1}z\big)^{-\alpha}$ for some $\alpha \in \mathbb{R}\setminus\mathbb{Z}_{\leq 0}$, then for any primitive $\mathbf{A}(z)$ of $A(z)$ there exists a primitive $\mathbf{P}(z)$ of $P(z)$ such that

$$\mathbf{A}(z) = \mathbf{P}(z) + \mathbf{A}_d(z) + O(R(z))$$

with

$$\mathbf{A}_d(z) = \begin{cases} \frac{c\rho_A}{\alpha-1}\big(1-\rho_A^{-1}z\big)^{-\alpha+1} & \text{if } \alpha \neq 1,\\ -c\rho_A\log\big(1-\rho_A^{-1}z\big) & \text{if } \alpha = 1,\end{cases}$$

and

$$R(z) = \begin{cases} \big(1-\rho_A^{-1}z\big)^{-\beta+1} & \text{if } \beta \neq 1,\\ \log\big(1-\rho_A^{-1}z\big) & \text{if } \beta = 1.\end{cases}$$

Theorem 4.2.3 and Lemma 4.2.4 are very helpful when the generating functions in question are Δ-analytic. However, for many of our generating functions we will not be able to guarantee Δ-analyticity. In order to utilise the results of this section also for generating functions that are *not necessarily* Δ-analytic, we introduce the following concept and notation.

Definition 4.2.5. Given a generating function $F(z)$ and Δ-analytic functions $A(z)$ and $B(z)$, we say that $F(z)$ is *congruent* to $A(z) + O(B(z))$ and write

$$F(z) \cong A(z) + O(B(z))$$

if there exist Δ-analytic functions $F^+(z), F^-(z)$ and polynomials $P^+(z), P^-(z)$ such that

- $F^- \preceq F \preceq F^+$;
- $F^+(z) = P^+(z) + A(z) + O(B(z))$;
- $F^-(z) = P^-(z) + A(z) + O(B(z))$.

Here we allow $A(z) \equiv 0$.

With Definition 4.2.5, we are able to apply the transfer theorem even if F itself is not Δ-analytic. The following lemma is an immediate consequence of Theorem 4.2.3 and the fact that $F^- \preceq F \preceq F^+$.

Lemma 4.2.6. *If $F(z) \cong A(z)$, where $A(z)$ is as in Theorem 4.2.3, then*

$$[x^n]F(z) = \Big(1 + O\Big(n^{-1/4}\Big)\Big)[z^n]A(z).$$

In this paper we will often encounter sums, products, differentials and integrals of generating functions. The following lemma states that these operations are compatible with the notion of congruence. We will frequently use this lemma without explicitly mentioning it.

Lemma 4.2.7. *Let A, A_1, A_2, B_1, B_2 be Δ-analytic functions with finitely many negative coefficients. Let F_1, F_2 be generating functions such that*

$$F_1(z) \cong A_1(z) + O(B_1(z)) \qquad \text{and} \qquad F_2(z) \cong A_2(z) + O(B_2(z))$$

and let $F(z)$ be a generating function with

$$F(z) \cong A(z) + O\Big(\big(1 - \rho_F^{-1} z\big)^{-\beta}\Big),$$

where $\beta \in \mathbb{R}$ and the dominant term $A_d(z)$ of $A(z)$ satisfies $\big(1 - \rho_F^{-1} z\big)^{-\beta} = o(A_d(z))$. Then the following holds.

$$F_1(z) \pm F_2(z) \cong A_1(z) \pm A_2(z) + O\Big(B_1(z)) + O(B_2(z)\Big),$$

$$F_1(z)F_2(z) \cong A_1(z)A_2(z) + O\Big(A_1(z)B_2(z) + B_1(z)A_2(z) + B_1(z)B_2(z)\Big),$$

$$F'(z) \cong A'(z) + O\Big(\big(1 - \rho_F^{-1} z\big)^{-\beta-1}\Big).$$

Furthermore, if $A_d(z) = c\big(1 - \rho_F^{-1} z\big)^{-\alpha}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, then for any primitive $\mathbf{F}(z)$ of $F(z)$ we have

$$\mathbf{F}(z) \cong \mathbf{A}_d(z) + O(R(z)),$$

where $\mathbf{A}_d$ and R are as in Lemma 4.2.4(ii).

Proof. The first congruence follows immediately from

$$F_1^- + F_2^- \preceq F_1 + F_2 \preceq F_1^+ + F_2^+ \quad \text{and} \quad F_1^- - F_2^+ \preceq F_1 - F_2 \preceq F_1^+ - F_2^-.$$

For the product $F_1(z)\cdot F_2(z)$, we may assume that F_1^- and F_2^- have nonnegative coefficients, since A_1 and A_2 have only finitely many negative coefficients. Hence

$$F_1^- \cdot F_2^- \preceq F_1 \cdot F_2 \preceq F_1^+ \cdot F_2^+$$

and the second congruence follows.

The last two congruences follow from Lemma 4.2.4 and the fact that

$$(F^-)' \preceq F' \preceq (F^+)' \qquad \text{and} \qquad \mathbf{F}^- \preceq \mathbf{F} \preceq \mathbf{F}^+,$$

where $\mathbf{F}^-, \mathbf{F}^+$ are primitives of F^-, F^+, respectively, with $\mathbf{F}^-(0) \leq \mathbf{F}(0) \leq \mathbf{F}^+(0)$. □

4.2.2. **Maps with large facewidth.** An *essential circle* on $\mathbb{S}_g$ is a circle that is not contractible to a point on $\mathbb{S}_g$. Let M be an embedding of a multigraph on $\mathbb{S}_g$. An *essential cycle* of M is a cycle of M that is an essential circle on the surface. The *facewidth* $\mathrm{fw}(M)$ of M is the minimal number of intersections of M with an essential circle on $\mathbb{S}_g$. The *edgewidth* $\mathrm{ew}(M)$ of M is defined as the minimal number of edges of an essential cycle of M. If $g = 0$, there are neither essential circles nor essential cycles and we use the convention $\mathrm{fw}(M) = \mathrm{ew}(M) = \infty$. Observe that if M is connected and *not* a 2-cell embedding, then $\mathrm{fw}(M) = 0$, as an essential circle can be found in any face that is not simply connected. The facewidth $\mathrm{fw}_g(G)$ of a multigraph G that is embeddable on $\mathbb{S}_g$ is defined as the *maximal* facewidth of all its embeddings on $\mathbb{S}_g$. If the genus is clear from the context, we omit it and write $\mathrm{fw}(G)$. When we count multigraphs with restrictions to their facewidth, we indicate the restriction by a superscript to the corresponding generating function, e.g. $G^{\mathrm{fw}\geq 2}(x)$ for the generating function of all multigraphs with facewidth at least two.

Having large facewidth proves to be a very helpful property, because it allows us to derive a constructive decomposition along connectivity as well as the existence of a unique embedding for 3-connected multigraphs. The following lemma was applied in a similar way in [**30**] as later in this paper.

Lemma 4.2.8. [**99**] *Let $g > 0$ and let M be an embedding of a connected multigraph G on $\mathbb{S}_g$.*

- *(i) M has facewidth $\mathrm{fw}(M) = k \geq 2$ if and only if M has a unique 2-connected component embedded on $\mathbb{S}_g$ with facewidth k and all other 2-connected components of M are planar.*
- *(ii) If G is 2-connected, M has facewidth $\mathrm{fw}(M) = k \geq 3$ if and only if M has a unique 3-connected component embedded on $\mathbb{S}_g$ with facewidth k and all other 3-connected components of M are planar.*
- *(iii) Let M_1, M_2 be embeddings of a 3-connected multigraph on $\mathbb{S}_g$ and suppose that $\mathrm{fw}(M_1) \geq 2g + 3$. Then there is a homeomorphism of $\mathbb{S}_g$ that maps M_1 to M_2.*

Lemma 4.2.8(iii) is a generalisation of Whitney's theorem [**106**] that all 3-connected planar multigraphs have a unique embedding up to orientation on the sphere. Because we will need Lemma 4.2.8 for multigraphs rather than for embeddings, we shall use the following easy corollary.

Corollary 4.2.9. *Let $g > 0$ and let G be a non-planar connected multigraph strongly embeddable on $\mathbb{S}_g$.*

- *(i) If $\mathrm{fw}_g(G) \geq 2$, then G has a unique 2-connected non-planar component strongly embeddable on $\mathbb{S}_g$ with facewidth $\mathrm{fw}_g(G)$ and all other 2-connected components are planar.*

(ii) If G is 2-connected and $\mathrm{fw}_g(G) \geq 3$, then G has a unique 3-connected non-planar component strongly embeddable on $\mathbb{S}_g$ with facewidth $\mathrm{fw}_g(G)$ and all other 3-connected components are planar.

(iii) If G is 3-connected and $\mathrm{fw}_g(G) \geq 2g+3$, then G has a unique 2-cell embedding on $\mathbb{S}_g$ up to orientation.

PROOF. By Lemma 4.2.8(i), for any fixed embedding of G all but one components are planar. As G itself is not planar, that exceptional component has to be non-planar. As the component structure is the same for *all* embeddings, the non-planar component I is independent of the embedding. Therefore, I is the unique component described in (i). Part (ii) is proved analogously and (iii) is a direct consequence of Lemma 4.2.8(iii). □

4.3. Maps and triangulations

The goal of this section is to enumerate cubic 3-connected *maps* on $\mathbb{S}_g$. The duals of these maps are triangulations, which are characterised in the following proposition.

Proposition 4.3.1. *Let M be a 2-cell embedding of a cubic multigraph on $\mathbb{S}_g$ and let M^* be its dual map. Then M is 3-connected if and only if M^* is a triangulation with at least six edges and without separating loops, separating double edges, or separating pairs of loops.*

PROOF. For cubic graphs with at least four vertices (and thus at least six edges), 3-connectivity and 3-edge-connectivity coincide. This can be seen by a simple case analysis. We thus use 3-edge-connectivity hereafter. Since a vertex in M corresponds to a face in M^*, deleting edges in the primal M has the same effect as cutting the surface along the dual edges of M^* (for a formal definition of "cutting" see Section 4.5.1), and a set of edges is a separator in M if and only if cutting along the dual edges in M^* separates the surface. Thus, a bridge in M corresponds to a separating loop in M^*. A 2-edge-separator in M corresponds either to a separating double edge or a pair of loops in M^* which together separate the surface. □

In order to enumerate the triangulations described in Proposition 4.3.1, we will relate them to simple triangulations that have been studied by Bender and Canfield [**7**]. To this end we will use the following classes of triangulations.

Let $\mathcal{M}_g$ be the class of triangulations on $\mathbb{S}_g$ without separating loops, separating double edges, and separating pairs of loops and let $M_g(y)$ be its ordinary generating function. Note that these triangulations are either the duals of 3-connected cubic maps on $\mathbb{S}_g$ by Proposition 4.3.1 or a triangulation with exactly three edges. Furthermore, let $\mathcal{S}_g$ be the class of simple triangulations on $\mathbb{S}_g$ (i.e. without loops or double edges) and let $\hat{\mathcal{T}}_g$ be the class of triangulations on $\mathbb{S}_g$ without separating loops or separating double edges. Let $T_g(y)$ and $\hat{T}_g(y)$ be their generating functions, respectively.

The starting point in obtaining an asymptotic expansion for $M_g(y)$ will be results on simple triangulations which were obtained by Gao [**51**] and (in the planar case) by Tutte [**103**]. However, the results obtained by Gao are not strong enough in order to apply the theory of singularity analysis (developed in Section 4.2.1). We obtain more refined versions of their results by following the ideas of Bender and Canfield [**7**].

Proposition 4.3.2. *The dominant singularity of $T_g(y)$ is given by $\rho_T = \frac{3}{2^{8/3}}$. The generating function $T_0(y)$ is Δ-analytic and satisfies*

$$T_0(y) = \frac{1}{8} - \frac{9}{16}\left(1-\rho_T^{-1}y\right) + \frac{3}{2^{5/2}}\left(1-\rho_T^{-1}y\right)^{3/2} + O\Big(\left(1-\rho_T^{-1}y\right)^2\Big). \tag{4.1}$$

For $g \geq 1$ we have

$$T_g(y) \cong c_g\left(1-\rho_T^{-1}y\right)^{-5g/2+3/2}\Big(1+O\Big(\left(1-\rho_T^{-1}y\right)^{1/4}\Big)\Big), \tag{4.2}$$

where c_g is a constant depending only on g.

Furthermore, for $g \geq 0$, the asymptotic number of simple triangulations on $\mathbb{S}_g$ with m edges is given by

$$|\mathcal{T}_g(m)| = \Big(1+O\Big(m^{-1/4}\Big)\Big)\frac{c_g}{\Gamma(5(g-1)/2)}m^{5g/2-5/2}\rho_T^{-m},$$

where $c_0 = \frac{3}{2^{5/2}}$.

The exact values of c_g can be found in [**53**].

Along the same lines we obtain similar results for $\hat{T}_g(y)$.

Proposition 4.3.3. *We have $\hat{T}_0(y) = T_0(y)$ and for $g \geq 1$,*

$$\hat{T}_g(y) \cong c_g\left(1-\rho_T^{-1}y\right)^{-5g/2+3/2}\Big(1+O\Big(\left(1-\rho_T^{-1}y\right)^{1/4}\Big)\Big), \tag{4.3}$$

where c_g is the same constant depending only on g as in Proposition 4.3.2.

Furthermore, for $g \geq 0$, the asymptotic number of triangulations without separating loops or separating double edges on $\mathbb{S}_g$ with m edges is given by

$$\left|\hat{\mathcal{S}}_g(m)\right| = \Big(1+O\Big(m^{-1/4}\Big)\Big)\frac{c_g}{\Gamma(5g/2-5/2))}m^{5g/2-5/2}\rho_T^{-m}.$$

The proofs of Propositions 4.3.2 and 4.3.3 can be found in Section 4.5.

From these two results and the fact that $\mathcal{S}_g \subseteq \mathcal{M}_g \subseteq \hat{\mathcal{T}}_g$ we obtain immediately our results for the number of triangulations in $\mathcal{M}_g$, i.e. triangulations on $\mathbb{S}_g$ without separating loops, separating double edges, and separating pairs of loops.

Proposition 4.3.4. *The dominant singularity of $M_g(y)$ is given by $\rho_M = \rho_T = \frac{3}{2^{8/3}}$. The generating function $M_0(y)$ is Δ-analytic and satisfies*

$$M_0(y) = \frac{1}{8} - \frac{9}{16}\left(1-\rho_M^{-1}y\right) + \frac{3}{2^{5/2}}\left(1-\rho_M^{-1}y\right)^{3/2} + O\Big(\left(1-\rho_M^{-1}y\right)^2\Big). \tag{4.4}$$

For $g \geq 1$ we have

$$M_g(y) \cong c_g\left(1-\rho_M^{-1}y\right)^{-5g/2+3/2}\Big(1+O\Big(\left(1-\rho_M^{-1}y\right)^{1/4}\Big)\Big), \tag{4.5}$$

where c_g is the same constant depending only on g as in Proposition 4.3.2.

Furthermore, for $g \geq 0$, the asymptotic number of triangulations in $\mathcal{M}_g(m)$ is given by

$$|\mathcal{M}_g(m)| = \Big(1+O\Big(m^{-1/4}\Big)\Big)\frac{c_g}{\Gamma(5g/2-5/2))}m^{5g/2-5/2}\rho_M^{-m}.$$

Observe that from Propositions 4.3.2 and 4.3.4 it follows immediately that the dual of a cubic map on $\mathbb{S}_g$ is simple with high probability, i.e. with probability tending to one as m tends to infinity.

4.4. Cubic graphs

Unless stated otherwise, graphs are unrooted. Recall that, in our generating functions, x marks vertices and y marks edges. Additionally, we will distinguish whether edges are single edges, double edges or loops because they will be treated differently when obtaining relations between graph classes. We will use the variable z to mark double edges and the variable w to mark loops. It is easy to see that 3-connected cubic graphs are simple and that 2-connected cubic multigraphs do not contain loops. The generating functions for these classes will only feature the variables of edges that can occur.

In order to derive asymptotic results we shall deal with univariate generating functions $F(v)$. As cubic (multi)graphs always have $2n$ vertices and $3n$ edges, where $n \in \mathbb{N}$, the coefficient $(2n)![v^n]F(v)$ will denote the number of graphs (or multigraphs or weighted multigraphs) in the corresponding class with $2n$ vertices and $3n$ edges. Such a univariate generating function can be obtained by the following substitution.

Definition 4.4.1. Let $\mathcal{F}$ be a class of connected cubic vertex-labelled multigraphs without triple edges and let

$$F(x, y, z, w) = \sum_{n,m,k,l \geq 0} \frac{f_{n,m,k,l}}{n!} x^n y^m z^k w^l$$

be its exponential generating function. We define functions $F(v)$, $F^u(v)$, and $F^s(v)$ as follows.

$$\begin{aligned} F(v) &:= F\left(v^{1/4}, v^{1/6}, \frac{v^{1/3}}{2}, \frac{v^{1/6}}{2}\right), \\ F^u(v) &:= F(v^{1/4}, v^{1/6}, v^{1/3}, v^{1/6}), \\ F^s(v) &:= F(v^{1/4}, v^{1/6}, 0, 0). \end{aligned}$$

If the generating function of $\mathcal{F}$ involves only two or three variables, we define $F(v)$, $F^u(v)$, and $F^s(v)$ analogously, only using the substitutions of those variables that occur.

We claim that $(2n)![v^n]F(v)$ is the number of weighted multigraphs in $\mathcal{F}(2n)$, i.e. the sum of $W(G)$ for all $G \in \mathcal{F}$ with $2n$ vertices (and thus with $3n$ edges). For, if $G \in \mathcal{F}(2n)$ has k double edges, l loops, and m single edges, then there are $2k + l + m = 3n$ edges in total and the substitution transforms the monomial $x^{2n}y^m z^k w^l$ into $2^{-(k+l)}v^{n/2+m/6+k/3+l/6} = W(G)v^n$. Similarly, $(2n)![v^n]F^u(v)$ is the number of (unweighted) multigraphs in $\mathcal{F}(2n)$. Finally, $(2n)![v^n]F^s(v)$ is the number of simple graphs in $\mathcal{F}(2n)$, since replacing z and w by 0 ensures that no graphs with double edges or loops are counted in $F^s(v)$.

4.4.1. **From maps to graphs.** Let $\mathcal{D}_g$ be the class of 3-connected cubic vertex-labelled graphs *strongly embeddable* on $\mathbb{S}_g$ and let $D_g(x, y)$ be its generating function. In this section we provide some necessary properties of $D_g(v)$. We will use the auxiliary classes $\overline{\mathcal{D}}_g$ of 3-connected cubic *edge-labelled* graphs strongly embeddable on $\mathbb{S}_g$, and $\overline{\mathcal{M}}_g$ of *edge-labelled, unrooted* triangulations where the triangulations are in $\mathcal{M}_g$.

Proposition 4.4.2. *The dominant singularity of $D_g(v)$ is $\rho_D = \rho_T^3 = \frac{27}{256}$ and we have the following congruences.*

$$D_0(v) \cong c_0 \left(1-\rho_D^{-1}v\right)^{5/2} + O\Big(\left(1-\rho_D^{-1}v\right)^{3}\Big),$$
$$D_1(v) \cong c_1 \log\left(1-\rho_D^{-1}v\right) + O\Big(\left(1-\rho_D^{-1}v\right)^{1/4}\Big),$$
$$D_g(v) \cong c_g \left(1-\rho_D^{-1}v\right)^{-5g/2+5/2} + O\Big(\left(1-\rho_D^{-1}v\right)^{-5g/2+11/4}\Big) \qquad \text{for } g \geq 2,$$

where c_g is the same constant depending only on g as in Proposition 4.3.2.

Applying Theorem 4.2.3, we immediately obtain the coefficients of $D_g(v)$.

Corollary 4.4.3. *The coefficients of $D_g(v)$ satisfy*

$$[v^n]D_g(v) = \Big(1+O\Big(n^{-1/4}\Big)\Big)\overline{c}_g n^{5(g-1)/2-1}\rho_D^{-n},$$

where $\overline{c}_g$ is a constant depending only on g.

Proof of Proposition 4.4.2. First we compare $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$. For each triangulation $M \in \mathcal{M}_g$ with m edges, there are $m!$ possibilities of labelling its edges. Conversely, for a triangulation $\overline{M} \in \overline{\mathcal{M}}_g$, there are $2m$ possibilities of rooting. Therefore, the exponential generating function $\overline{M}_g(y)$ of $\overline{\mathcal{M}}_g$ satisfies

$$[y^m]M_g(y) = 2m[y^m]\overline{M}_g(y)$$

and thus

$$M_g(y) = 2\delta_y\overline{M}_g(y).$$

Every graph $G \in \overline{\mathcal{D}}_g$ has at least two (edge-labelled) 2-cell embeddings. By Proposition 4.3.1, the maps obtained in this way are precisely the duals of triangulations in $\overline{\mathcal{M}}_g$. As y marks the number of edges in $\overline{M}_g(v^{1/3})$ and v marks a third of the number of edges in $\overline{D}_g(v)$, we obtain

$$2\overline{D}_g(v) \preceq \overline{M}_g(v^{1/3}).$$

We claim that, for a cubic map M on $\mathbb{S}_g$, its facewidth $\mathrm{fw}(M)$ is exactly the edgewidth $\mathrm{ew}(M^*)$ of the triangulation M^* that is the dual of M. To see this, we observe that an essential cycle of M^* witnessing the edgewidth of M^* corresponds to an essential circle on $\mathbb{S}_g$ that meets M in $\mathrm{ew}(M^*)$ edges and no vertices, resulting in $\mathrm{fw}(M) \leq \mathrm{ew}(M^*)$. On the other hand, any two faces of M that share a vertex also share an edge, as M is cubic. Thus, there is an essential circle witnessing the facewidth of M that meets M only at edges. As this circle corresponds to an essential cycle of M^*, we have $\mathrm{fw}(M) \geq \mathrm{ew}(M^*)$.

Since by Lemma 4.2.8(iii) a 3-connected graph embeddable on $\mathbb{S}_g$ with facewidth at least $2g+3$ has exactly two embeddings, we have

$$2\overline{D}_g^{\mathrm{fw}\geq 2g+3}(v) = \overline{M}_g^{\mathrm{ew}\geq 2g+3}(v^{1/3}).$$

As obviously $\overline{D}_g^{\mathrm{fw}\geq 2g+3}(v) \preceq \overline{D}_g(v)$, we obtain the following relations:

$$\overline{M}_g^{\mathrm{ew}\geq 2g+3}(v^{1/3}) = 2\overline{D}_g^{\mathrm{fw}\geq 2g+3}(v) \preceq 2\overline{D}_g(v) \preceq \overline{M}_g(v^{1/3}). \tag{4.6}$$

Since there are no double edges in a 3-connected cubic graph, we know that the two generating functions $\overline{D}_g(v)$ and $D_g(v)$ are closely related. To be precise, $(2n)![v^n]D_g(v)$ is the number of vertex-labelled graphs in $\mathcal{D}_g(2n)$. Since every such graph has $3n$ edges, $(3n)!(2n)![v^n]D_g(v)$ is the number of 3-connected cubic graphs with $2n$ vertices embeddable on $\mathbb{S}_g$ with both vertices and edges labelled. As this number is equal to $(2n)!(3n)![v^n]\overline{D}_g(v)$ by an analogous argument, we have

$$[v^n]D_g(v) = [v^n]\overline{D}_g(v).$$

Therefore, we can replace $\overline{D}_g(v)$ by $D_g(v)$ in (4.6) to obtain

$$D_g(v) \preceq \frac{1}{2}\overline{M}_g(v^{1/3}) = \frac{1}{4}\int t^{-1}M_g(t)\,\mathrm{d}t\bigg|_{t=v^{1/3}}.$$

By Lemma 4.2.4 we obtain an upper bound for $D_g(v)$ as claimed. To finish the proof we will show the following claim.

Claim 1. *The generating functions $M_g(y)^{\mathrm{ew}\geq 2g+3}$ and $M_g(y)$ have the same dominant singularity and*

$$M_g(y) - M_g^{\mathrm{ew}\geq 2g+3}(y) \cong O\Big(\big(1-\rho_S^{-1}y\big)^{-5g/2+7/4}\Big).$$

Before we prove the claim, let us note that Proposition 4.4.2 follows immediately from Claim 1, Lemma 4.2.7, and Proposition 4.3.4.

A statement more general than Claim 1 was proven in [**10**] for a variety of map classes. Although we believe that the proof in [**10**] generalises to $\mathcal{M}_g$, which was not considered in [**10**], we give a slightly different proof here for completeness.

The generating function of $\mathcal{M}_g \setminus \mathcal{T}_g$ is congruent to $O\Big(\big(1-\rho_S^{-1}y\big)^{-5g/2+7/4}\Big)$ by Propositions 4.3.2 to 4.3.4. It thus suffices to show that

$$T_g^{\mathrm{ew}\leq 2g+2} \cong O\Big(\big(1-\rho_S^{-1}y\big)^{-5g/2+7/4}\Big).$$

For $i \geq 3$, let $\mathcal{T}_g^{C=i}$ be the class of triangulations in $\mathcal{T}_g$ where one non-contractible cycle C of length i is marked, and we denote its generating function by $T_g^{C=i}(y)$. Clearly $T_g^{\mathrm{ew}=i}(y) \preceq T_g^{C=i}(y)$. Let $M \in \mathcal{T}_g^{C=i}$ and let C be the marked cycle of M. Consider the surface $\mathbb{S}_g$ on which M embeds. We cut $\mathbb{S}_g$ along the cycle C, and duplicate the vertices and edges of C so that the map structure in the neighbourhood on the two sides of C is preserved (for a precise definition of "cut", see Section 4.5.1 in the appendix). We then close the two resulting holes by inserting disks to them in $\mathbb{S}_g$. This operation is also called "cutting along C on $\mathbb{S}_g$", and a more general and rigorous definition of such topological surgeries can be found in Section 4.5. For each of the two disks, we then add a new vertex inside the disk and use it to triangulate the disk (see Figure 4.1). If C was separating, we mark one of the corners at the inserted vertex in the component that contains the original root face of M. For the other component, we choose one of the corners at the inserted vertex to be its root. If C was not separating, then we mark one corner at each of the two inserted vertices. In total we add $3i$ edges to the map. These operations result in

- two triangulations $M^{(1)}, M^{(2)}$, where $M^{(1)}$ contains the original root face of M and a marked corner;
- or one triangulation M^* with two marked corners.

All resulting triangulations are in $\mathcal{T}_{g'}$ for some $g' < g$, because the surgery does not create loops or double edges. Thus, disregarding markings, in the first case $M^{(1)} \in \mathcal{T}_{g_1}$ and $M^{(2)} \in \mathcal{T}_{g_2}$ with $g_1 + g_2 = g$ and $g_1, g_2 \geq 1$, and then in the second case $M^* \in \mathcal{T}_{g-1}$.

Since a corner (v_0, e, e') is uniquely defined once v_0 and e are given, marking a corner is equivalent to marking an edge and choosing one of its end vertices. In terms of generating functions, this corresponds to applying the operator $2\delta_y = 2y\frac{\partial}{\partial y}$. As in previous proofs, we will mark repeatedly, which will result in overcounting. Since we added $3i$ edges to M by our construction, we have to compensate by a factor of y^{3i}. Therefore, we obtain the relation

$$y^{3i}T_g^{C=i}(y) \preceq 4\delta_y^2(T_{g-1}(y)) + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} 2\delta_y(T_{g_1}(y))T_{g_2}(y).$$

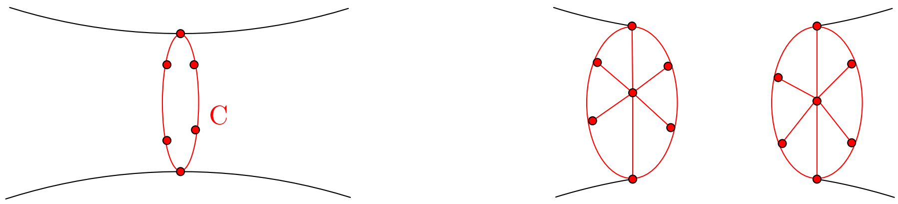

FIGURE 4.1. Cutting along C and triangulating the inserted disc.

By Proposition 4.3.2, we know that

$$4\delta_y^2(T_{g-1}(y)) + \sum_{\substack{g_1+g_2=g\\ g_j\geq 1}} 2\delta_y(T_{g_1}(y))T_{g_2}(y) \cong O\Big(\big(1-\rho_S^{-1}y\big)^{-5g/2+7/4}\Big).$$

Because

$$T_g^{\mathrm{ew}\leq 2g+2}(y) = \sum_{i=3}^{2g+2} T_g^{\mathrm{ew}=i}(y) \preceq \sum_{i=3}^{2g+2} T_g^{C=i}(y),$$

we complete the proof of the claim. □

4.4.2. **From 3-connected graphs to connected multigraphs.** In this section we derive dominance relations between different classes of cubic multigraphs. In the end we will relate connected cubic multigraphs via 2-connected cubic multigraphs to 3-connected cubic graphs enumerated in the previous section.

Denote by $\mathcal{D}_g$, $\mathcal{B}_g$, and $\overline{\mathcal{W}}_g$ the classes of 3-connected, 2-connected, and connected vertex-labelled cubic multigraphs strongly embeddable on $\mathbb{S}_g$, respectively. Let $D_g(x,y) = \sum \frac{d_{n,m}}{n!}x^n y^m$, $B_g(x,y,z) = \sum \frac{b_{n,m,k}}{n!}x^n y^m z^k$, and $\overline{W}_g(x,y,z,w) = \sum \frac{c_{n,m,k,l}}{n!}x^n y^m z^k w^l$, be the corresponding generating functions. In the generating function $\overline{W}_g(x,y,z,w)$, the graph Φ consisting of two vertices connected by three edges will not be taken into account. This graph will be treated separately at the end.

First we give a relation between a subclass of $\mathcal{D}_g$ and a subclass of $\mathcal{B}_g$. To do this we need the class $\mathcal{N}$ of edge-rooted 2-connected vertex-labelled cubic planar multigraphs, called *networks*. In the exponential generating function $N(x,y,z)$ of $\mathcal{N}$ we mark the root always with y as a single edge, and with z marking double edges not including the root edge.

Lemma 4.4.4. *For $g\geq 1$, the generating functions of $\mathcal{D}_g$ and $\mathcal{B}_g$ satisfy*

$$D_g^{\mathrm{fw}\geq 3}(x,\overline{y}) - D_0(x,\overline{y}) \preceq B_g^{\mathrm{fw}\geq 3}(x,y,z) \preceq D_g^{\mathrm{fw}\geq 3}(x,\overline{y}), \tag{4.7}$$

where $\overline{y} = y(1+N(x,y,z))$.

PROOF. Let B be a multigraph in $\mathcal{B}_g^{\mathrm{fw}\geq 3}$. We show that it is counted at least once on the right-hand side and at most once on the left-hand side of (4.7).

First, suppose that B is not planar. Then Corollary 4.2.9(ii) states that B has a unique 3-connected component T strongly embeddable on $\mathbb{S}_g$ with the same facewidth. T is in $\mathcal{D}_g^{\mathrm{fw}\geq 3}$ and therefore counted once in $D_g^{\mathrm{fw}\geq 3}(x,y)$. To get B from T, we have to attach 2-connected components along the edges (see Figure 4.2). That means, either we leave an edge as it is (obtaining a summand of y) or we replace it by two edges (obtaining a factor of y^2) and one multigraph in $\mathcal{N}$ without its root edge (obtaining a factor of $\frac{1}{y}N$). Thus, B is counted exactly once on the right-hand side of (4.7).

If B is planar, then it might be counted more than once on the right-hand side. Indeed, in this case the 2-connected components carrying the facewidth might be

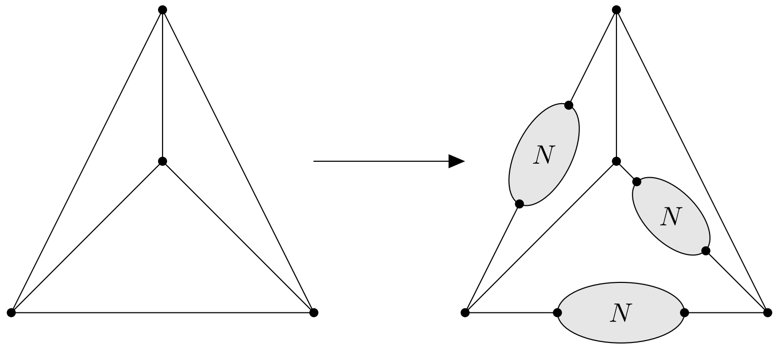

FIGURE 4.2. Obtaining 2-connected graphs from 3-connected graphs by substituting edges with networks.

different for different embeddings. Therefore, $D_g^{\mathrm{fw}\geq 3}(x, y + yN(x, y, z))$ is an upper bound. To get a lower bound we have to subtract all multigraphs we overcounted. This is achieved by subtracting $D_0(x, y + yN(x, y, z))$, as only planar multigraphs are overcounted and each such multigraph is subtracted once for each of its 3-connected components. $\square$

In the same spirit we can relate connected and 2-connected multigraphs, using the auxiliary class $\mathcal{Q}$ of all edge-rooted connected vertex-labelled cubic planar multigraphs whose root edge is a loop. To simplify the formulas later on, the root will be marked by y in the generating function $Q(x, y, z, w)$ and only non-root loops are marked by w.

Lemma 4.4.5. *For $g \geq 1$ the generating functions of $\overline{\mathcal{W}}_g$ and $\mathcal{B}_g$ satisfy the following relation:*

$$B_g^{\mathrm{fw}\geq 2}(x, \overline{y}, \overline{z}) - B_0(x, \overline{y}, \overline{z}) \preceq \overline{W}_g^{\mathrm{fw}\geq 2}(x, y, z, w) \preceq B_g^{\mathrm{fw}\geq 2}(x, \overline{y}, \overline{z}), \tag{4.8}$$

where $\overline{y} = \frac{y}{1-Q(x,y,z,w)}$ and $\overline{z} = \frac{1}{2}(\frac{y}{1-Q(x,y,z,w)})^2 - \frac{y^2}{2} + z$.

PROOF. Let $C \in \overline{\mathcal{W}}_g^{\mathrm{fw}\geq 2}$. We shall show that C is counted at least once on the right-hand side and at most once on the left-hand side of (4.8).

First, suppose that C is not planar. Then Corollary 4.2.9(i) states that C has a unique 2-connected component B strongly embeddable on $\mathbb{S}_g$ with the same facewidth, i.e., $B \in \mathcal{B}_g^{\mathrm{fw}\geq 2}$. To construct C from B we have to replace each edge by a sequence of edges and multigraphs in $\mathcal{Q}$, which means we replace one edge by a sequence of alternating edges and multigraphs in $\mathcal{Q}$ without the root, starting and ending with an edge. Therefore, the replacement leads to the substitution $y \mapsto y\frac{1}{1-Q(x,y,z,w)}$ (see Figure 4.3).

This results in a 1-to-1 correspondence between the two generating functions $\overline{W}_g^{\mathrm{fw}\geq 2}(x, y, z, w)$ and $B_g^{\mathrm{fw}\geq 2}(x, \overline{y}, \overline{z})$ for non-planar multigraphs. The replacement for double edges results from replacing a set of two edges each as above, except when the two edges are left intact, then they should still be treated as a double edge instead of two simple edges. We thus have the correction term $-\frac{y^2}{2} + z$.

As in Lemma 4.4.4, if C is planar, the above argument does not necessarily result in a bijection. We thus have to subtract all corresponding planar multigraphs again in order to avoid overcounting on the left-hand side. Therefore, we get the claimed result analogously to Lemma 4.4.4. $\square$

Combining Lemmas 4.4.4 and 4.4.5, we have the following upper and lower bounds for the generating function $C_g(x, y, z, w)$.

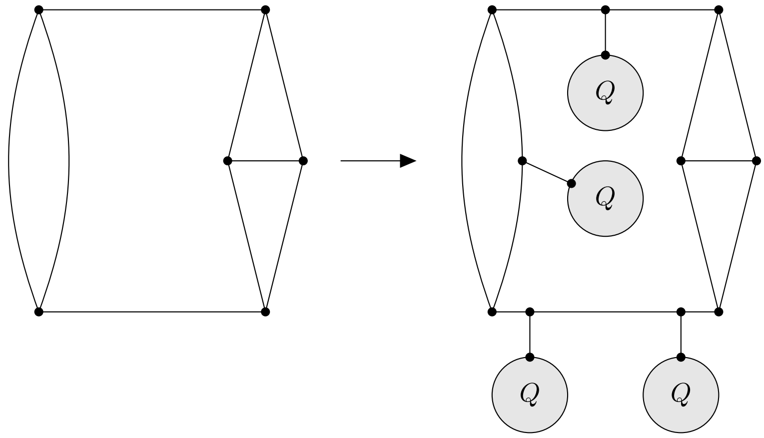

FIGURE 4.3. Obtaining connected graphs from 2-connected graphs by substituting edges with sequences of Q-graphs.

Corollary 4.4.6. *For $g \geq 1$, the generating function $C_g(x, y, z, w)$ satisfies*

$$\begin{aligned}
&D_g^{\mathrm{fw}\geq 3}\left(x, \overline{y}\left(1 + N(x, \overline{y}, \overline{z})\right)\right) - D_0\left(x, \overline{y}\left(1 + N(x, \overline{y}, \overline{z})\right)\right) \\
&\quad + B_g^{\mathrm{fw}=2}(x, \overline{y}, \overline{z}) - B_0(x, \overline{y}, \overline{z}) + \overline{W}_g^{\mathrm{fw}=1}(x, y, z, w) \\
&\preceq \overline{W}_g(x, y, z, w) \\
&\preceq D_g^{\mathrm{fw}\geq 3}\left(x, \overline{y}\left(1 + N(x, \overline{y}, \overline{z})\right)\right) + B_g^{\mathrm{fw}=2}(x, \overline{y}, \overline{z}) + \overline{W}_g^{\mathrm{fw}=1}(x, y, z, w),
\end{aligned} \tag{4.9}$$

where $\overline{y} = \frac{y}{1-Q(x,y,z,w)}$ and $\overline{z} = \frac{1}{2}\left(\frac{y}{1-Q(x,y,z,w)}\right)^2 - \frac{y^2}{2} + z$ as in Lemma 4.4.5.

We note that the upper and lower bounds of $\overline{W}_g(x, y, z, w)$ differ only by terms involving generating functions of planar graphs. In Section 4.4.3 we will show that those generating functions are subdominant. Therefore, the two bounds match in asymptotics. We will also provide the asymptotic expressions for the other terms in Section 4.4.3. In order to do that, we first establish bounds on the generating functions for multigraph classes with fixed facewidth.

Lemma 4.4.7. *For $g \geq 1$ the following relations hold.*

$$\begin{aligned}
B_g^{\mathrm{fw}=2}(x,y,z) \preceq 2\left(y + \frac{z}{y}\right)^2 \left(\frac{1}{y} + \frac{y}{z}\right)^2 \Bigg(&(\delta_y + \delta_z)^2 \left(B_{g-1}^{\mathrm{fw}\geq 2}(x, y, z)\right) \\
&+ \sum_{g'=1}^{g-1} (\delta_y + \delta_z)\left(B_{g'}^{\mathrm{fw}\geq 2}(x, y, z)\right)(\delta_y + \delta_z)\left(B_{g-g'}^{\mathrm{fw}\geq 2}(x, y, z)\right)\Bigg),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\overline{W}_g^{\mathrm{fw}=1}(x,y,z,w) \preceq (xyw)^2 \left(\frac{1}{y} + \frac{y}{z}\right)\Bigg(&\delta_w^2\left(\overline{W}_{g-1}(x, y, z, w)\right) \\
&+ \sum_{g'=1}^{g-1} \delta_w\left(\overline{W}_{g'}(x, y, z, w)\right)\delta_w\left(\overline{W}_{g-g'}(x, y, z, w)\right)\Bigg).
\end{aligned} \tag{4.11}$$

PROOF. In order to show (4.10), let B be a multigraph in $\mathcal{B}_g^{\mathrm{fw}=2}$. Consider a fixed 2-cell embedding M of B on $\mathbb{S}_g$ with facewidth two and let $\{e_1 = \{v_1, w_1\}, e_2 = \{v_2, w_2\}\}$ be two edges such that there exists an essential circle C on $\mathbb{S}_g$ meeting M only in e_1 and e_2. Note that e_1, e_2 do not share vertices, because otherwise the facewidth would have been one. Then we delete e_1 and e_2, cut the surface along

C and close both holes with a disk[2] (see Figure 4.4). By this surgery we either disconnect the surface or we reduce its genus by one.

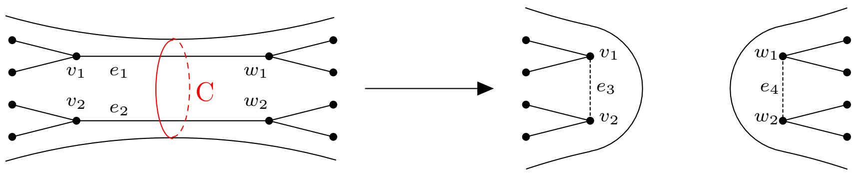

FIGURE 4.4. Surgery along an essential circle.

Case 1: Cutting along C disconnects the surface. As C was an essential loop, both components have genus at least one. Therefore, we obtain two multigraphs B_1^* and B_2^*, strongly embeddable on $\mathbb{S}_{g_1}$ and $\mathbb{S}_{g_2}$ respectively, with $g_1, g_2 \geq 1$ and $g_1 + g_2 = g$. Without loss of generality, we can assume that $v_1, v_2 \in B_1^*$ and $w_1, w_2 \in B_2^*$. Furthermore, $\{e_1, e_2\}$ was a 2-edge-separator in B. Thus, B_1^* and B_2^* are connected as B is 2-edge-connected. Let B_1 be obtained from B_1^* by adding an edge $e_3 = \{v_1, v_2\}$ and marking e_3. Note that B_1 is also strongly embeddable on $\mathbb{S}_{g_1}$. We claim that B_1 is 2-connected. Indeed, any path in B between vertices in B_1 gives rise to a path in B_1 between the same vertices by replacing any sub-path in $B \setminus B_1$ by the edge e_3. Thus, B_1 is 2-connected as B is. Analogously, we add the edge $e_4 = \{w_1, w_2\}$ to B_2^* to obtain a 2-connected multigraph B_2 strongly embeddable on $\mathbb{S}_{g_2}$. We also mark e_4. Furthermore, we claim that both B_1 and B_2 cannot have facewidth 1. Indeed, suppose that B_1 has facewidth 1 for a certain embedding M_1, then we can perform the reverse direction of the surgery to get an embedding of B. Since B is of facewidth at least 2, the only possibility is that the face containing the essential circle of length 1 in M_1 is one of those created in the surgery from B to B_1 and B_2, which is not possible by construction. Therefore, B_1 has facewidth at least 2, and analogously, B_2 has facewidth at least two as well. Therefore, in this case, we can conclude that a multigraph B can be constructed from a 2-connected multigraph embeddable on $\mathbb{S}_{g'}$ with one marked edge and a 2-connected multigraph embeddable on $\mathbb{S}_{g-g'}$ with one marked edge, with both multigraphs of facewidth at least 2, resulting in the term

$$\sum_{g'=1}^{g-1} (\delta_y + \delta_z) \left(B_{g'}^{\mathrm{fw}\geq 2}(x,y,z) \right) (\delta_y + \delta_z) \left(B_{g-g'}^{\mathrm{fw}\geq 2}(x,y,z) \right).$$

Note that e_3 and e_4 might be single edges or part of double edges. Therefore, differentiating with respect to both possibilities results in an upper bound. The factor $\left(\frac{1}{y} + \frac{y}{z}\right)^2$ accounts for the deletion of e_1 and e_2, each of which might have been a single edge or part of a double edge (hence deleting it turns a double edge into a single edge). The factor $\left(y + \frac{z}{y}\right)^2$ represents the insertion of e_3 and e_4, each of which either adds a single edge or turns a single edge into a double edge. Both factors contribute to the the upper bound as they overcount. Furthermore, we obtain a factor of two for the ways to obtain the original multigraph from B_1 and B_2.

Case 2: Cutting along C does not disconnect the surface. As the embedding after cutting is still a 2-cell embedding, $B \setminus \{e_1, e_2\}$ is connected. We can connect v_1,

[2]For a formal definition of this operation see Section 4.5.1

v_2, w_1, w_2 in $B \setminus \{e_1, e_2\}$ by two edges (without loss of generality $e_3 = \{v_1, v_2\}$ and $e_4 = \{w_1, w_2\}$) so as to obtain a multigraph B^*. The graph $\overline{B} = B \cup \{e_3, e_4\}$ has a 2-cell embedding $\overline{M}$ on $\mathbb{S}_g$ such that $e_1 \cup e_2 \cup e_3 \cup e_4$ bounds a face. Indeed, starting from M, e_3 and e_4 can be embedded so that they run close to e_1, e_2, and C. Let M^* be the embedding of B^* induced by $\overline{M}$. Suppose that B^* is not 2-connected, that is, it has a bridge e. Note that e cannot be e_3 or e_4 as $B^* \setminus \{e_3, e_4\} = B \setminus \{e_1, e_2\}$ is connected.

There is a (not necessarily essential) circle C' on $\mathbb{S}_g$ hitting M^* only in e. As e has not been a bridge in B, C' has to meet e_1 and e_2 as well. If it met neither e_1 nor e_2, it would either contradict B having facewidth two (if C' is essential) or the 2-connectivity of B (if C' is not essential). If it met only one of them, it would have to meet one of e_3, e_4, because e_1, e_2, e_3 and e_4 bound a disk in $\overline{M}$. This contradicts the fact that C' meets M^* only in e.

We now construct the circle C'' as follows. First, we follow C' from e to e_1 without traversing it. Then, we follow e_1 until reaching C and switch to C to reach e_2 without crossing e_1 and e_2. Finally, we return to C' along e_2 and then return to e (see Figure 4.5). C'' meets M only in e. Either C'' is an essential circle, contradicting the fact that B has facewidth two, or it is planar, contradicting the 2-connectivity of B. Similarly, we can also prove that B^* has facewidth at least 2. Thus we conclude that every multigraph B, where the surgery does not result in disconnecting the surface, can be constructed from a 2-connected multigraph embeddable on $\mathbb{S}_{g-1}$ with two marked edges and facewidth at least two, which results in the term

$$(\delta_y + \delta_z)^2 B_{g-1}(x, y, z).$$

The factor $2\left(y + \frac{z}{y}\right)^2 \left(\frac{1}{y} + \frac{y}{z}\right)^2$ follows as in Case 1. We thus conclude (4.10).

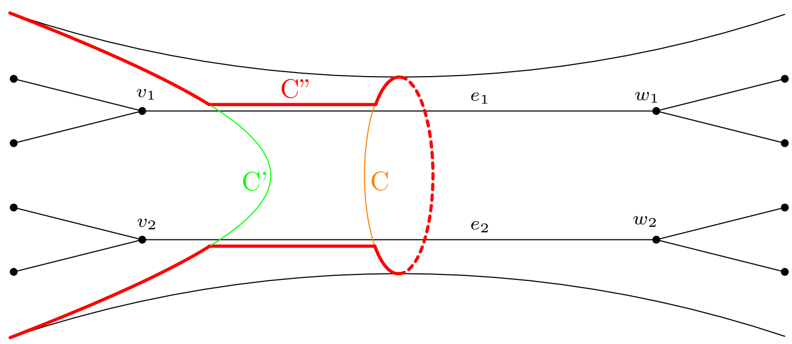

FIGURE 4.5. Finding an essential circle witnessing small facewidth

To prove (4.11), let G be a multigraph in $\overline{\mathcal{W}}_g^{\mathrm{fw}=1}$. We fix a 2-cell embedding M of G on $\mathbb{S}_g$ of facewidth one and let $e_1 = \{v_1, v_2\}$ be an edge such that there exists an essential circle C on $\mathbb{S}_g$ meeting G only in e_1. Then we perform the following surgery: we delete e_1, cut the surface along C, close both holes with a disk, and attach an edge, an additional vertex and a loop to both v_1 and v_2. Remark that the edge deleted may be a single edge or part of a double edge. Thus, we have a factor of $(xyw)^2\left(\frac{1}{y} + \frac{y}{z}\right)$, overcounting all possibilities. The deleted edge cannot be a loop, since in cubic maps on orientable surfaces loops are always on the boundary of two different faces, and as such cannot be the only intersection of an embedding of a multigraph with an essential circle. This can easily be seen, as there is only one other edge at the base of the loop. Thus, the boundary of the face of the loop without this additional edge consists only of traversing the loop once. By this surgery, we either disconnect the surface or we reduce its genus by one.

If we separate the surface, we obtain two connected multigraphs each with one marked loop. These multigraphs are counted by $\delta_w(\overline{W}_{g'})$ and by $\delta_w(\overline{W}_{g-g'})$, as the genera of the two parts sum up to g and the embeddings resulting from the surgery are still 2-cell embeddings.

If the surface is not separated, the resulting embedding is a 2-cell embedding and hence the multigraph remains connected. Therefore, we obtain a multigraph counted by $\delta_w^2(\overline{W}_{g-1})$. The factor in front of the generating function is once again obtained by marking two loops. This proves (4.11). $\square$

In subsequent calculations, it will be more convenient to change the differential operators to operators with respect to x instead of y, z, and w.

Corollary 4.4.8. *For $g \geq 1$, we have*

$$\begin{aligned} B_g^{\mathrm{fw}=2}(x,y,z) \preceq \frac{9}{2}\left(y+\frac{z}{y}\right)^2\left(\frac{1}{y}+\frac{y}{z}\right)^2 \Bigg(\delta_x^2\left(B_{g-1}^{\mathrm{fw}\geq 2}(x,y,z)\right) \\ + \sum_{g'=1}^{g-1}\delta_x\left(B_{g'}^{\mathrm{fw}\geq 2}(x,y,z)\right)\delta_x\left(B_{g-g'}^{\mathrm{fw}\geq 2}(x,y,z)\right)\Bigg), \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \overline{W}_g^{\mathrm{fw}=1}(x,y,z,w) \preceq (xyw)^2\left(\frac{1}{y}+\frac{y}{z}\right)\Bigg(\delta_x^2\left(\overline{W}_{g-1}(x,y,z,w)\right) \\ + \sum_{g'=1}^{g-1}\delta_x\left(\overline{W}_{g'}(x,y,z,w)\right)\delta_x\left(\overline{W}_{g-g'}(x,y,z,w)\right)\Bigg). \end{aligned} \tag{4.13}$$

Proof. Since the generating function $B_g^{\mathrm{fw}\geq 2}$ counts cubic graphs, it is the sum of monomials of the form $x^{2k}y^{3k-2\ell}z^{\ell}$ for some nonnegative integers k, ℓ with $2l \leq 3k$. We thus have

$$\delta_x x^{2k}y^{3k-2\ell}z^{\ell} = 2kx^{2k}y^{3k-2\ell}z^{\ell} \geq \frac{2}{3}(3k-\ell)x^{2k}y^{3k-2\ell}z^{\ell} = \frac{2}{3}\left(\delta_y+\delta_z\right)x^{2k}y^{3k-2\ell}z^{\ell}.$$

Therefore, we have $\frac{3}{2}\delta_x B_g^{\mathrm{fw}\geq 2} \geq \left(\delta_y+\delta_z\right)B_g^{\mathrm{fw}\geq 2}$. Combining this with (4.10) proves (4.12).

To show (4.13), we note that a cubic graph has at most as many loops as vertices, and thus replacing δ_w by δ_x increases each coefficient. Thus, we still have an upper bound when replacing δ_w by δ_x in (4.11). $\square$

Corollary 4.4.8 will be used to show that the number of multigraphs with small facewidth in Corollary 4.4.6 is negligible. Additionally, we need equations for the generating functions of the auxiliary classes $\mathcal{N}$ and $\mathcal{Q}$. Recall that $\mathcal{N}$ is the class of edge-rooted 2-connected vertex-labelled cubic planar multigraphs, and $\mathcal{Q}$ is the class of all edge-rooted connected vertex-labelled cubic planar multigraphs whose root edge is a loop.

Proposition 4.4.9. *The generating function $N(x,y,z)$ of $\mathcal{N}$ satisfies the system of equations*

$$\begin{aligned} N(x,y,z) &= \frac{u(1-2u)-x^2y(1+N(x,y,z))(y^2-2z)}{2}, \\ x^2y^3(1+N(x,y,z))^3 &= u(1-u)^3, \end{aligned} \tag{4.14}$$

and the generating function $Q(x, y, z, w)$ *of* $\mathcal{Q}$ *satisfies*

$$\begin{aligned} Q &= \frac{Q^2}{2} + \frac{x^2y^3}{2}\big(A + \overline{Q}\big) + x^2y^2w, \\ A &= Q + S + P + H, \\ S &= \frac{A^2}{A+1}, \\ P &= \frac{x^2y^3}{2}A^2 + x^2y^3A + x^2yz, \\ 2H(1 + A) &= u(1 - 2u) - u(1 - u)^3, \\ x^2y^3A^3 &= u(1 - u)^3, \end{aligned} \tag{4.15}$$

where

$$\overline{Q} = \begin{cases} -Q & \text{for simple graphs,} \\ 0 & \text{for weighted multigraphs,} \\ Q & \text{for multigraphs.} \end{cases}$$

Proof. We obtain (4.15) by following the lines of Section 3 in [**66**] or Section 3 in [**87**]. The only difference is that we account for loops and double edges in the initial conditions. In order to derive the system for Q, one starts with an edge-rooted connected cubic planar graph and recursively decomposes it depending on the placement of the root. One of the following mutually exclusive cases occurs:

(i) the root is a loop;
(ii) the root is a bridge;
(iii) the root is part of a minimal separating edge set of size two;
(iv) the end vertices of the root separate the graph; or
(v) the root is part of a 3-connected component.

In Case (i) we obtain an equation for Q, while Case (ii) results in an equation that can immediately be eliminated from the system, Case (iii) in the equation for S, Case (iv) in the equation for P and Case (v) in the parametric equations for H in terms of u. It is shown in [**66**] that these cases are indeed exhaustive. For each of these cases, there is a decomposition of the graph resulting in the corresponding equation in the system. The difference for the three values of $\overline{Q}$ is due to the difference of how to deal with loops and double edges in the three different weightings. The only difference in the systems of all three weightings comes from Case (i), when the third edge at the root vertex is incident to a double edge (see Figure 4.6). While this case cannot happen for simple graphs (and thus it is not possible to obtain a loop as a root in this case), the difference regarding weighted and unweighted multigraphs is due to the weighting of $\frac{1}{2}$ of the double edge.

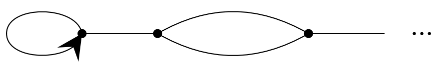

Figure 4.6. The exceptional case which has to be dealt with differently for simple graphs, weighted and unweighted multigraphs.

To obtain the equations for $N(x, y, z)$, we start with (4.15). Because $N(x, y, z)$ enumerates edge-rooted 2-connected planar cubic multigraphs, setting $w = 0$ and $Q(x, y, z, w) = 0$ results in a system of equations for $N(x, y, z) = 1 + A$. The given equations follow by eliminating S, P, and H from the new system. □

4.4.3. **Asymptotics.** The goal of this section is to obtain asymptotics for $\overline{W}_g(v)$ via Corollary 4.4.6. The analysis for $\overline{W}_g^u(v)$ and $\overline{W}_g^s(v)$ are analogous; we will point out the differences when they occur.

To use Corollary 4.4.6, we will prove asymptotic formulas for each of the occurring terms. In order to simplify notations, we define

$$\widetilde{D}_g(x,y,z,w) = D_g\left(x, \overline{y}\left(1+N(x,\overline{y},\overline{z})\right)\right),$$

where

$$\overline{y} = \frac{y}{1-Q(x,y,z,w)}, \quad \text{and} \quad \overline{z} = \frac{1}{2}\left(\frac{y}{1-Q(x,y,z,w)}\right)^2 - \frac{y^2}{2} + z.$$

This change of variables comes from Lemma 4.4.5. Facewidth conditions can be added in the usual way. Additionally, we define

$$\begin{aligned}\widetilde{N}(x,y,z,w) &= N\left(x,\overline{y},\overline{z}\right)\\ \widetilde{B}_g(x,y,z,w) &= B_g\left(x,\overline{y},\overline{z}\right).\end{aligned}$$

Furthermore, when writing v as the sole variable, we are always using the corresponding substitution from Definition 4.4.1.

In order to determine the dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$, one of the summands in Corollary 4.4.6, we first analyse the dominant singularity of $Q(v)$. The numerical values of the dominant singularities and other constants are different for $Q(v)$, $Q^u(v)$, and $Q^s(v)$, but the analysis works in exactly the same way.

Lemma 4.4.10. *The dominant singularity of $Q(v)$ is $\rho_Q = \frac{54}{79^{3/2}}$. Furthermore, $Q(v)$ is Δ-analytic and*

$$Q(v) \cong q_0(1-\rho_Q^{-1}v)^{3/2} + O\Big((1-\rho_Q^{-1}v)^2\Big),$$

where q_0 is a constant and we have $Q(\rho_Q) = 1 - \frac{17}{2\sqrt{79}}$.

Proof. Substituting v as in Definition 4.4.1 in (4.15) and eliminating S, P, H, u, and A in this order from (4.15), we get the following implicit equation for Q.

$$\begin{aligned}0 =&256Q^4 - 512Q^5 + 384Q^6 - 128Q^7 + 16Q^8\\ &+v(-320Q^3 - 224Q^4 + 2352Q^5 - 3304Q^6 + 2008Q^7 - 576Q^8 + 64Q^9)\\ &+v^2(144Q^2 + 136Q^3 - 384Q^4 + 210Q^5 - 35Q^6)\\ &+v^3(-28Q + 42Q^2 - 14Q^3) + 2v^4.\end{aligned} \tag{4.16}$$

Using standard methods for implicitly defined functions (see for example [**47**, VII.7.1]), we determine the dominant singularity to be at $\rho_Q = \frac{54}{79^{3/2}}$ and obtain the stated expression for $Q(v)$ and the value of $Q(v)$ at ρ_Q. □

This lemma is already strong enough to deal with the planar case. By unrooting the classes in Lemma 4.4.10, we obtain the asymptotic expansion of $\overline{W}_0(v)$ as a corollary.

Corollary 4.4.11. *The dominant singularity of the generating function $\overline{W}_0(v)$ of planar connected cubic vertex-labelled weighted multigraphs is $\rho_Q = \frac{54}{79^{3/2}}$. Furthermore, the generating function $\overline{W}_0(v)$ is Δ-analytic and*

$$\begin{aligned}\overline{W}_0(v) =&a_0 + a_1\left(1-\rho_{\overline{W}}^{-1}v\right) + a_2\left(1-\rho_{\overline{W}}^{-1}v\right)^2\\ &+ c_0\left(1-\rho_{\overline{W}}^{-1}v\right)^{5/2} + O\bigg(\left(1-\rho_{\overline{W}}^{-1}v\right)^3\bigg),\end{aligned}$$

where c_0, a_0, a_1, a_2 are constants.

Proof. The class of planar connected cubic multigraphs is given by unrooting the sum of the classes used in Lemma 4.4.10. It is easy to see that all those classes have the same dominant singularity as $Q(v)$, see [**20, 66, 88**] for more details. $\square$

Similar results also hold for unweighted multigraphs and simple graphs with the same dominant singularities as $Q^u(v)$ and $Q^s(v)$, respectively.

Next we determine the asymptotic behaviour of networks in $\widetilde{N}(v)$. The only difference to the other two cases are the numerical values of n_0 and $\widetilde{N}(\rho_N)$.

Lemma 4.4.12. *The dominant singularity of $\widetilde{N}(v)$ occurs at $\rho_N = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore, $\widetilde{N}(v)$ is Δ-analytic, and*

$$\widetilde{N}(v) \cong n_0\Big((1-\rho_N^{-1}v)^{3/2}\Big) + O\big((1-\rho_N^{-1}v)^2\big),$$

where n_0 is a constant and $\widetilde{N}(\rho_N) = 1/16$.

Proof. Starting from (4.14), we obtain a system of equations that is satisfied by $\widetilde{N}(v)$ by performing the appropriate substitutions on y and z in (4.14): first $y = \overline{y}$ and $z = \overline{z}$, then the substitutions from Definition 4.4.1. We thus obtain a system of two equations involving $\widetilde{N}(v)$, u, v and $Q(v)$. We then add (4.16), relating $Q(v)$ and v, and obtain a determined system. Eliminating $Q(v)$ and u from this system results in an implicit equation in $\widetilde{N}(v)$ and v.

Using standard methods (see for example [**47**, VII.7.1]) to deal with implicitly defined functions we determine the dominant singularity to be at $\rho_N = \rho_Q$ and derive the claimed properties of $\widetilde{N}(v)$. $\square$

With the help of these two lemmas we obtain the singularity and singular expansion of the main term $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$ in Corollary 4.4.6.

Lemma 4.4.13. *The generating function $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$ has its dominant singularity at $\rho_D = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore, we have*

$$\begin{aligned}
\widetilde{D}_0(v) &\cong c_0(1-\rho_Q)^{5/2} + O\big((1-\rho_Q)^3\big),\\
\widetilde{D}_1^{\mathrm{fw}\geq 3}(v) &\cong c_1\log(1-\rho_Q^{-1}v) + O\Big((1-\rho_Q^{-1}v)^{1/4}\Big),\\
\widetilde{D}_g^{\mathrm{fw}\geq 3}(v) &\cong c_g(1-\rho_Q^{-1}v)^{-5g/2+5/2} + O\Big((1-\rho_Q^{-1}v)^{-5g/2+11/4}\Big) \quad \textit{for } g \geq 2.
\end{aligned}$$

Analogous results to Lemma 4.4.13 also hold for unweighted multigraphs and simple graphs. The only difference are the numerical values of the constants and the dominant singularities, where the latter coincide with the dominant singularities of $Q^u(v)$ and $Q^s(v)$, respectively.

Proof. The dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$ is given either by the singularity ρ_Q of $Q(v)$ and $\widetilde{N}(v)$, or by a solution of $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3} = \rho_S^3$, where $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3}$ is obtained by substituting v in $x^2(\overline{y}(1+N(x,\overline{y},\overline{z})))^3$, and ρ_S^3 is the dominant singularity of $D_g(v)$. By Proposition 4.3.2 and Lemma 4.4.12 we verify that $\frac{\rho_Q(1+\widetilde{N}(\rho_Q))^3}{(1-Q(\rho_Q))^3} = \rho_S^3$. This is the only solution of this equation, as $\frac{v(1+\widetilde{N}(v))^3}{(1-Q(v))^3}$ is a power series with positive coefficients, and thus monotone on the interval $[0,\rho_Q)$. Therefore, the dominant singularity of $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$ is ρ_Q, and the composition is critical (in the sense of [**47**, pp. 411ff]). We thus conclude the proof by Proposition 4.4.2, and noting that $D_g^{\mathrm{fw}\geq 3}(v)$ has same asymptotic behaviour as $D_g(v)$. $\square$

The next lemma shows the asymptotic behaviour of $\widetilde{B}_g^{\mathrm{fw}=2}(v)$, which is the next term occurring in the bounds of Corollary 4.4.6.

Lemma 4.4.14. *For $g \geq 1$, we have*

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \cong O\Big(\big(1-\rho_N^{-1}v\big)^{-5g/2+11/4}\Big). \tag{4.17}$$

Proof. First we observe that by Lemmas 4.4.4 and 4.4.13, the generating function $\widetilde{B}_g^{\mathrm{fw}\geq 3}(v)$ has its dominant singularity at $\rho_B = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore,

$$\begin{aligned}
\widetilde{B}_1^{\mathrm{fw}\geq 3}(v) &\cong c_1 \log(1-\rho_Q^{-1}v) + O\Big((1-\rho_Q^{-1}v)^{1/4}\Big),\\
\widetilde{B}_g^{\mathrm{fw}\geq 3}(v) &\cong c_g (1-\rho_Q^{-1}v)^{-5g/2+5/2} + O\Big((1-\rho_Q^{-1}v)^{-5g/2+11/4}\Big) \quad \text{for } g \geq 2.
\end{aligned}$$

We prove the claim by induction on g. Suppose that our claim is correct for all $g' < g$. By Corollary 4.4.8 and the fact that both $\overline{y}$ and $\overline{z}$ are formal power series with positive coefficients, we have

$$\begin{aligned}
B_g^{\mathrm{fw}=2}(x,\overline{y},\overline{z}) \preceq \frac{9}{2}\Big(\overline{y}+\frac{\overline{z}}{\overline{y}}\Big)^2\Big(\frac{1}{\overline{y}}+\frac{\overline{y}}{\overline{z}}\Big)^2 \Big(\delta_x^2 B_{g-1}^{\mathrm{fw}\geq 2}(x,\overline{y},\overline{z})\Big)\\
+\sum_{g'=1}^{g-1} \delta_x B_{g'}^{\mathrm{fw}\geq 2}(x,\overline{y},\overline{z})\delta_x B_{g-g'}^{\mathrm{fw}\geq 2}(x,\overline{y},\overline{z}).
\end{aligned} \tag{4.18}$$

We now perform the substitutions as in Definition 4.4.1. Note that x is substituted by $v^{1/4}$, while $\overline{y}$ and $\overline{z}$ are formal power series in x, y, z, w, all substituted by positive powers of v. Therefore, we can replace δ_x by $4\delta_v$ while keeping an upper bound. We thus have

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \preceq 648\left(\delta_v^2(\widetilde{B}_{g-1}^{\mathrm{fw}\geq 2}(v)) + \sum_{g'=1}^{g-1}\delta_v(\widetilde{B}_{g'}^{\mathrm{fw}\geq 2}(v))\delta_v(\widetilde{B}_{g-g'}^{\mathrm{fw}\geq 2}(v))\right). \tag{4.19}$$

The precise coefficient may change for unweighted multigraphs or simple graphs. By Lemma 4.2.4 and the fact that all generating functions on the right-hand side of (4.19) are for genus smaller than g, we deduce by induction that

$$\begin{aligned}
\delta_v^2(\widetilde{B}_{g-1}^{\mathrm{fw}\geq 2}(v)) &\cong O\Big(\big(1-\rho_B^{-1}v\big)^{-5g/2+3}\Big),\\
\delta_v(\widetilde{B}_{g'}^{\mathrm{fw}\geq 2}(v)) &\cong O\Big(\big(1-\rho_B^{-1}v\big)^{-5g'/2+3/2}\Big),\\
\delta_v(\widetilde{B}_{g-g'}^{\mathrm{fw}\geq 2}(v)) &\cong O\Big(\big(1-\rho_B^{-1}v\big)^{-5(g-g')/2+3/2}\Big).
\end{aligned}$$

Substituting these congruences into (4.19) results in

$$\widetilde{B}_g^{\mathrm{fw}=2}(v) \preceq O\Big(\big(1-\rho_B^{-1}v\big)^{-5g/2+3}\Big),$$

which immediately implies (4.17).

For the base case $g = 1$, the computation is the same, except that we have a term $\delta_v^2\widetilde{B}_0(v)$, which is $\delta_v^2\overline{W}_0(v)$, since $B_0(x,\overline{y},\overline{z}) = \overline{W}_0(x,y,z,w)$. By Corollary 4.4.11, we have

$$\delta_v^2\widetilde{B}_0(v) \cong a_2 + O\Big((1-\rho_B v)^{1/2}\Big),$$

completing the proof. □

We use the asymptotic results in Corollary 4.4.11 and Lemmas 4.4.13 and 4.4.14 to examine the bounds in Corollary 4.4.6 and determine the dominant term of connected cubic multigraphs embeddable on $\mathbb{S}_g$.

THEOREM 4.4.15. *For $g \geq 1$, the dominant singularity of the generating function $\overline{W}_g(v)$ of connected cubic vertex-labelled weighted multigraphs that are strongly embeddable on $\mathbb{S}_g$ is $\rho_{\overline{W}} = \rho_Q = \frac{54}{79^{3/2}}$. Furthermore, we have*

$$\overline{W}_1(v) \cong c_1 \log\left(1 - \rho_{\overline{W}}^{-1} v\right) + O\left(\left(1 - \rho_{\overline{W}}^{-1} v\right)^{1/4}\right),$$

$$\overline{W}_g(v) \cong c_g \left(1 - \rho_{\overline{W}}^{-1} v\right)^{-5g/2+5/2} + O\left(\left(1 - \rho_{\overline{W}}^{-1} v\right)^{-5g/2+11/4}\right) \quad \textit{for } g \geq 2,$$

where c_g is a constant depending only on g.

PROOF. We first do the substitution of Definition 4.4.1 in (4.9), which leads to

$$\begin{aligned}\widetilde{D}_g^{\mathrm{fw}\geq 3}(v) - \widetilde{D}_0(v) + \widetilde{B}_g^{\mathrm{fw}=2}(v) - \widetilde{B}_0(v) + \overline{W}_g^{\mathrm{fw}=1}(v)& \\ \preceq \overline{W}_g(v) \preceq \widetilde{D}_g^{\mathrm{fw}\geq 3}(v) + \widetilde{B}_g^{\mathrm{fw}=2}(v) + \overline{W}_g^{\mathrm{fw}=1}(v).&\end{aligned}$$

Comparing the terms other than $C_g^{\mathrm{fw}=1}(v)$ in these bounds, we obtain by Corollary 4.4.11, Lemma 4.4.13, and Lemma 4.4.14 that the dominant term is $\widetilde{D}_g^{\mathrm{fw}\geq 3}(v)$, which has the claimed singularity and decomposition.

To conclude the proof, it remains to show that

$$\overline{W}_g^{\mathrm{fw}=1} \cong O\left(\left(1 - \rho_{\overline{W}}^{-1} v\right)^{-5g/2+11/4}\right). \tag{4.20}$$

By Corollary 4.4.8 and the fact that by replacing δ_x by $4\delta_v$, the coefficients do not decrease, we have the relation

$$\overline{W}_g^{\mathrm{fw}=1}(v) \preceq \frac{27}{4} \delta_v^2(\overline{W}_{g-1}(v)) + \frac{27}{4} \sum_{g'=1}^{g-1} \delta_v(\overline{W}_{g'}(v)) \delta_v(\overline{W}_{g-g'}(v)). \tag{4.21}$$

By the fact that all generating functions on the right-hand side of (4.21) are for genus smaller than g, we can use induction on g as in the proof of Lemma 4.4.14 to deduce (4.20), concluding the proof. □

From Theorem 4.4.15 and Lemma 4.2.6, we can immediately evaluate the coefficients of $\overline{W}_g(v)$.

Corollary 4.4.16. *The asymptotic number of connected cubic vertex-labelled multigraphs that are weighted by their compensation factor and are strongly embeddable on $\mathbb{S}_g$ is given by*

$$[v^n]\overline{W}_g(v) = \left(1 + O\left(n^{-1/4}\right)\right) c_g n^{5g/2-7/2} \rho_{\overline{W}}^{-n}.$$

Here $\rho_{\overline{W}} = \rho_Q = \frac{54}{79\sqrt{79}}$ and c_g is a constant only depending on g.

Again, both Theorem 4.4.15 and Corollary 4.4.16 work analogously for unweighted multigraphs and simple graphs with different constants and dominant singularities of $Q^u(v)$ and $Q^s(v)$, respectively.

4.4.4. **Proof of Theorem 4.1.2.** Using the results from Section 4.4.3 we can now prove Theorem 4.1.2. Recall that a cubic multigraph embeddable on $\mathbb{S}_g$ is given by a set of connected cubic multigraphs embeddable on $\mathbb{S}_{g_i}$ such that $\sum g_i \leq g$ (see Proposition 4.2.2). Therefore, we have the relation

$$G_g(v) \preceq \sum_{k=1}^{\infty} \sum_{\sum g_i \leq g} \frac{1}{k!} \prod_{i=1}^{k} \left(\overline{W}_{g_i}(v) + \frac{v}{6}\right). \tag{4.22}$$

The summand $\frac{v}{6}$ accounts for the fact that a component might also be a triple edge, which is not taken into account in $\overline{W}_{g_i}(v)$ (this additional summand will differ when proving Theorems 4.1.1 and 4.1.3). We only get an upper bound, because we overcount if a multigraph is strongly embeddable on surfaces of multiple genera. Later we will also obtain a lower bound with the same asymptotics to complete the proof.

If $g = 0$, the relation (4.22) simplifies to $G_0 = \exp(\overline{W}_0 + \frac{v}{6})$, as there is no overcounting in this case. This coincides with Theorem 1 of [**66**] and therefore we can conclude our statement in this case. (Although it is not directly shown there, the same arguments can be used for unweighted planar cubic multigraphs. For simple graphs, see [**20**].)

Now suppose $g \geq 1$. The first step to obtain asymptotics from (4.22) is to rearrange the sum in such a way that all planar components are singled out. This results in

$$G_g(v) \preceq \sum_{k=0}^{g} \sum_{\substack{\sum g_i \leq g \\ g_i \geq 1}} \frac{1}{k!} \prod_{i=1}^{k} \Big(\overline{W}_{g_i}(v) + \frac{v}{6}\Big) \sum_{j=0}^{\infty} \frac{k!}{(k+j)!} \Big(\overline{W}_0(v) + \frac{v}{6}\Big)^j. \tag{4.23}$$

By the dominant term and the value of $\overline{W}_0(v)$ at the singularity $\rho_{\overline{W}}$ from Corollary 4.4.11, we observe that the last sum contributes only a constant factor. Thus, it remains to derive the dominant term of $\frac{1}{k!}\prod\big(\overline{W}_{g_i}(v) + \frac{v}{6}\big)$. As the first sum consists only of a constant number of summands, the dominant term of the right-hand side of (4.23) will be the (sum of the) dominant terms from $\frac{1}{k!}\prod\big(\overline{W}_{g_i}(v) + \frac{v}{6}\big)$ up to the constant obtained from the planar components. That is, we shall compute the dominant term of

$$A(v) := \frac{1}{k!} \prod_{i=1}^{k} \Big(\overline{W}_{g_i}(v) + \frac{v}{6}\Big),$$

where the g_i are positive and sum up to $g' \leq g$.

For $g = 1$, either $k = g' = 0$ or $k = g' = 1$. By Theorem 4.4.15, we have

$$A(v) = C_1(v) + \frac{v}{6} \cong c_1 \log(1 - \rho_{\overline{W}}^{-1} v) + \frac{v}{6} + O\bigg(\Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{1/4}\bigg)$$

and thus

$$A(v) \preceq P_1(v) + c_1 \log(1 - \rho_{\overline{W}}^{-1} v) + O\bigg(\Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{1/4}\bigg)$$

with $P_1(v)$ a polynomial and c_1 a constant.

Suppose now $g \geq 2$. Without loss of generality let $g_1, \ldots, g_l = 1$ and let $g_{l+1}, \ldots, g_k > 1$. Then

$$\begin{aligned} A(v) &\cong \bigg(1 + O\bigg(\Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{1/4}\bigg)\bigg) \frac{c_1^l}{k!} \Big(\log\Big(1 - \rho_{\overline{W}}^{-1} v\Big)\Big)^l \prod_{i=l+1}^{k} c_{g_i} \Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{\frac{5(1-g_i)}{2}} \\ &\cong \bigg(c + O\bigg(\Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{1/4}\bigg)\bigg) \Big(\log\Big(1 - \rho_{\overline{W}}^{-1} v\Big)\Big)^l \Big(1 - \rho_{\overline{W}}^{-1} v\Big)^{-5g'/2 + 5k/2}. \end{aligned} \tag{4.24}$$

For $k = 1$ and $g' = g$ (and hence $l = 0$) we thus have

$$\frac{1}{1!} \prod_{g'=1}^{1} \Big(\overline{W}_{g_i} + \frac{v}{6}\Big) \cong c_g (1 - \rho_{\overline{W}}^{-1} v)^{-5g/2 + 5/2} + O\Big((1 - \rho_{\overline{W}}^{-1} v)^{-5g/2 + 11/4}\Big). \tag{4.25}$$

For $k \geq 2$ or $g' < g$, (4.24) yields

$$\frac{1}{k!}\prod_{i=1}^{k}\overline{W}_{g_i}(v) \cong O\Big((1-\rho_{\overline{W}}^{-1}v)^{-5g/2+5/2+2}\Big) \tag{4.26}$$

and thus

$$G_g(v) \preceq c_g(1-\rho_{\overline{W}}^{-1}v)^{-5g/2+5/2} + O\Big((1-\rho_{\overline{W}}^{-1}v)^{-5g/2+11/4}\Big).$$

We derive a lower bound for $g \geq 1$ as follows. Let $\tilde{\mathcal{G}}_g$ be the class of graphs in $\mathcal{G}_g$ with one component of genus g and all other components planar. Then

$$\sum_{j=0}^{\infty}\frac{C_g(v)C_0^j(v)}{(j+1)!} \succeq \tilde{G}_g(v) \succeq \sum_{j=0}^{\infty}\frac{C_g(v)C_0^j(v)}{(j+1)!} - \sum_{j=0}^{\infty}\frac{(j+1)C_0^{j+1}(v)}{(j+1)!}.$$

Indeed, if the component of genus g is also planar, then the graph might be counted up to $j+1$ times (once for each component) on the left-hand side. Substituting the corresponding summands thus yields a lower bound of $G_g(v)$.

$$G_g(v) \succeq \tilde{G}_g(v) \succeq \sum_{j=0}^{\infty}\frac{1}{(j+1)!}\overline{W}_g(v)\overline{W}_0^j(v) - \sum_{j=1}^{\infty}\frac{j}{j!}\overline{W}_0^j(v). \tag{4.27}$$

Applying Theorem 4.2.3 to the upper and lower bounds and setting $\gamma_2 = \rho_{\overline{W}}^{-1}$ completes the proof. $\square$

4.4.5. **Proofs of Theorems 4.1.1, 4.1.3 and 4.1.4.** Theorems 4.1.1 and 4.1.3 can be proven in a similar way as Theorem 4.1.2. We obtain $\rho_1 = \gamma_1^{-1}$ as the square root of the smallest positive solution of

$$0 = 46656 - 279936u - 7293760u^2 - 513216u^3 + 148716u^4 - 17469u^5 + 729u^6$$

and $\rho_3 = \gamma_3^{-1}$ as the square root of the smallest positive solution of

$$0 = 46656 + 279936u - 7293760u^2 + 513216u^3 + 148716u^4 + 17469u^5 + 729u^6.$$

Theorem 1.3.3 follows immediately from (4.25), (4.26), and Theorem 4.2.3. Indeed, (4.25) and Theorem 4.2.3 imply that the number of weighted multigraphs in $\mathcal{G}_g(n)$ that have a unique non-planar component that is not embeddable on $\mathbb{S}_{g-1}$ is

$$\Big(1+O\Big(n^{-1/4}\Big)\Big)e_g n^{5g/2-5/2}\gamma_2^{2n}(2n)!\,.$$

On the other hand, (4.26) and Theorem 4.2.3 imply that the number of weighted multigraphs in $\mathcal{G}_g(n)$ that do not have such a component is

$$O\Big(n^{5g/2-5/2-2}\gamma_2^{2n}(2n)!\Big).$$

Thus, Theorem 4.1.4 follows. Observe that the probability $1-O(n^{-2})$ is not sharp. Indeed, the exponent in (4.26) could be improved to $-5g/2+5-\varepsilon$ for every $\varepsilon > 0$, which would yield a probability $1-O(n^{-5/2+\varepsilon})$. The statements of Theorem 4.1.4 for unweighted multigraphs and simple graphs are proved analogously. $\square$

Remark 4.4.17. *Observe that the polynomials $p_1(u), p_3(u)$ whose smallest positive zeroes u_1 and u_3 give rise to the exponential growth constants γ_1 for cubic multigraphs and γ_3 for simple cubic graphs, respectively, satisfy the relation $p_1(u) = p_3(-u)$. It would be interesting to know whether this fact has a combinatorial meaning.*

4.5. Triangulations

In this appendix, we compute the asymptotic numbers of triangulations in $\mathcal{T}_g$ and $\hat{\mathcal{T}}_g$, as stated in Propositions 4.3.2 and 4.3.3, respectively. Our proof follows the approach of [**7**].

4.5.1. **Surgeries.** When dealing with maps on $\mathbb{S}_g$ we will perform operations on the surfaces that are commonly known as *cutting* and *gluing*. In the course of these operations we will encounter *surfaces with holes*. A *surface with k holes* is a surface $\mathbb{S}_g$ from which the disjoint interiors $D_1, \dots, D_k$ of k closed disks have been deleted. Each D_i is called a *hole*. Let S be the disjoint union of finitely many orientable surfaces, at least one of them with holes, and suppose that X and Y are homeomorphic subsets of the boundary of S. By *gluing S along X and Y* we mean the operation of identifying every point $x \in X$ with $f(x)$ for any fixed homeomorphism $f\colon X \to Y$. The identification of X and Y induces a surjection σ from S onto the resulting space $\tilde{S}$. We write $\tilde{X}$ for the subset $\sigma(X) = \sigma(Y)$ of $\tilde{S}$.

We will glue along subsets in two particular situations: when X and Y are

(i) disjoint boundaries of holes of S, or
(ii) sub-arcs of the boundary of the same hole that meet precisely in their endpoints.

For (ii), we shall additionally assume that the homeomorphism $f\colon X \to Y$ induces the identity on $X \cap Y$. In either case, the space $\tilde{S}$ resulting from gluing along X and Y is again the disjoint union of finitely many orientable surfaces with holes, with the number of components being either the same or one less than that for S. The subset $\tilde{X}$ of $\tilde{S}$ is a circle in Case (i) and homeomorphic to the closed unit interval in Case (ii). If S has k holes, then $\tilde{S}$ has $k-2$ holes in Case (i) and $k-1$ holes in Case (ii). A special case of (i) is when one of the components of S is a disk (i.e. a sphere with one hole) and Y is its boundary. In this case, we say that we *close the hole bounded by X by inserting a disk*.

If in addition we are given a map M on S, then we will glue along X and Y only if either both of them are contained in a face (not necessarily the same face for X and Y) or both are unions of the same number of vertices and the same number of edges of M. We also assume the homeomorphism $f\colon X \to Y$ to map vertices to vertices and edges to edges. Under these conditions, we obtain a map $\tilde{M}$ on $\tilde{S}$. The subset $\tilde{X}$ of $\tilde{S}$ is then either a subgraph of $\tilde{M}$ or a subset of a face of $\tilde{M}$. Observe that the surjection $\sigma\colon S \to \tilde{S}$ induces a bijection between the sets of corners of M and of corners of $\tilde{M}$. We will refer to this bijection by saying that every corner of M *corresponds* to a corner of $\tilde{M}$.

If $\tilde{S}$ is obtained from S by gluing along X and Y, we also say that vice versa, S is obtained from $\tilde{S}$ by *cutting along $\tilde{X}$*. The operation of cutting along a circle or interval is well defined in the sense that if $\tilde{S}$ and $\tilde{X}$ are given, then S, X, and Y are unique up to homeomorphism. If S has more components than $\tilde{S}$, we call $\tilde{X}$ *separating*. Cutting along a separating circle on $\mathbb{S}_g$ and closing the resulting holes by inserting disks will yield two surfaces $\mathbb{S}_{g_1}, \mathbb{S}_{g_2}$ with $g_1 + g_2 = g$. Cutting along a non-separating circle and closing the holes by inserting disks reduces the genus by one.

A combination of cutting and gluing surfaces along some subsets of their boundaries is called a *surgery*. Again, if a map $\tilde{M}$ results from performing surgeries on a map M, then every corner of $\tilde{M}$ corresponds to a corner of M.

4.5.2. **Quasi-triangulations.** We begin with some notations. The *valency* of a face f in a map is the number of corners of f. We call a rooted map M a *quasi-triangulation* if all faces except the root face f_r are bounded by triangles. Let $\mathcal{P}_g$ be the class of *simple* quasi-triangulations and $P_g(y,u)$ its generating function, where y marks the number of edges, and u marks the valency of f_r. Given an index set I and an injective function $h\colon I \to F(M)\setminus\{f_r\}$, we call M an *I-quasi-triangulation with respect to h* if all faces in $F(M)\setminus(h(I)\cup\{f_r\})$ are bounded by triangles. If in addition f_r is also bounded by a triangle, we say that M is an *I-triangulation* (with respect to h). Let $\mathcal{T}_{g,I}$ and $\mathcal{P}_{g,I}$ be the classes of simple I-triangulations and simple I-quasi-triangulations, respectively, with their generating functions denoted by $T_g(y,z_I)$ and $P_g(y,u,z_I)$, respectively. Here u again marks the valency of the root face f_r and $z_I=(z_i)_{i\in I}$ is a set of variables indexed by I, where z_i marks the valency of $h(i)$. Additionally, let $\hat{\mathcal{P}}_g$, $\hat{\mathcal{T}}_{g,I}$, and $\hat{\mathcal{P}}_{g,I}$ be the analogous classes for triangulations without separating loops or separating double edges and $\hat{P}_g(y,u)$, $\hat{T}_g(y,z_I)$, and $\hat{P}_g(y,u,z_I)$ their generating functions, respectively.

Note that $\mathcal{T}_g=\mathcal{T}_{g,\emptyset}$ and $\mathcal{P}_g=\mathcal{P}_{g,\emptyset}$. In the case $I=\emptyset$, we will therefore always use the generating functions $T_g(y)$ and $P_g(y,u)$ without mentioning variables z_i. To simplify notations, the one-vertex map is put into $\mathcal{P}_0$, although it is not a quasi-triangulation, since it does not have any corners and thus cannot be rooted. We say that a face f is *marked* if $f\in h(I)$ and that we are *marking a face f* if we add a new index i to the set I with $h(i)=f$. We use the same convention also for the corresponding classes and generating functions for triangulations without loops and double edges.

To prove Propositions 4.3.2 and 4.3.3, we first derive a recursive formula relating $\mathcal{P}_{g,I}$ (and $\hat{\mathcal{P}}_{g,I}$) for different genera and different sizes of the set I. We then prove Propositions 4.3.2 and 4.3.3 by applying this formula inductively. In order to derive the recursive formula, we delete the root edge of a given quasi-triangulation and then perform surgeries that either separate the given surface or decrease its genus. One part of the reverse operation then consists of adding a new edge to a map. Let S be a map and $c_1=(v_1,e_1^-,e_1^+)$ and $c_2=(v_2,e_2^-,e_2^+)$ be two (not necessarily distinct) corners of the same face f of S. For T is a map with $V(T)=V(S)$ and $E(T)=E(S)\cup\{e_{\text{new}}\}$, we say that e_{new} is an *edge from c_1 to c_2* if

- e_{new} is contained in f and its end vertices are v_1 and v_2;
- in the cyclic order of edges of T at v_1, e_{new} is the predecessor of e_1^+; and
- in the cyclic order of edges of T at v_2, e_{new} is the successor of e_2^-.

If $c_1=c_2=:c$, we also say that e_{new} is a *loop at c*.

4.5.3. **The planar case.** Before we derive the recursive formula, we study the base case of *planar* quasi-triangulations.

Proposition 4.5.1. *The generating functions of planar quasi-triangulations satisfy $\hat{P}_0(y,u)=P_0(y,u)$ and*

$$P_0(y,u)=1+yu^2P_0^2(y,u)+\frac{y(P_0(y,u)-1)}{u}-y^2uP_0(y,u)-T_0(y)(P_0(y,u)-1). \tag{4.28}$$

Proof. As planar quasi-triangulations cannot have non-separating loops or double edges, $\hat{P}_0(y,u)=P_0(y,u)$ follows immediately.

The first summand in (4.28) corresponds to the one-vertex map. Let $S\in\mathcal{P}_0$ be a planar quasi-triangulation with at least one edge. We distinguish two cases.

First, suppose that the root edge e_r is a bridge; then the only face incident with e_r is the root face f_r. The union $f_r\cup e_r$ is not a disk and thus contains a non-contractible circle C. We delete e_r, cut along C, and close the two resulting holes by inserting disks. By this surgery, S is separated into two quasi-triangulations S_1, S_2. Let v_1 and v_2 be the end vertices of e_r in S_1 and S_2 respectively. One of these two vertices is the root vertex of S; by renaming S_1, S_2 we may assume that v_1 is the root vertex of S. In the cyclic order of the edges of S at v_1, let e_1^- and e_1^+ be the predecessor and successor of e_r respectively. Define e_2^- and e_2^+ analogously at v_2. We let (v_1, e_1^-, e_1^+) and (v_2, e_2^-, e_2^+) be the roots of S_1 and S_2 respectively. We thus have $S_1, S_2 \in \mathcal{P}_0$. Furthermore, S_1 and S_2 together have one edge less than S and the sum of the valencies of their root faces is two less than the valency of f_r. Thus, we obtain $yu^2P_0^2(x,u)$, the second term of the right-hand side of (4.28).

Now suppose that e_r is not a bridge. Then it lies on the boundary of the root face and of another face, which is bounded by a triangle. In the cyclic order of edges at the root vertex v_r, let e_r^- and e_r^+ be the predecessor and successor of e_r, respectively. We delete e_r and obtain a quasi triangulation S' that we root at $c_r' := (v_r, e_r^-, e_r^+)$. The valency of the root face of S' is larger by one than the valency of f_r. This is reflected by $\frac{y}{u}(P_0(y,u) - 1)$, the third term of the right-hand side of (4.28), because S' cannot be the quasi-triangulation consisting only of a single vertex. However, with this summand we have overcounted. Indeed, the reverse construction is as follows. Let f_r' be the root face of S'. Then the corners of f_r' can be ordered by walking along the boundary of f_r' in counterclockwise direction. In this order, starting from c_r', let $c' = (v, e, e')$ be the corner after the next; then S is obtained from S' by inserting an edge from c_r' to c'. If $v_r = v$ this results in a loop; if v_r and v are adjacent, we obtain a double edge (see Figure 4.7).

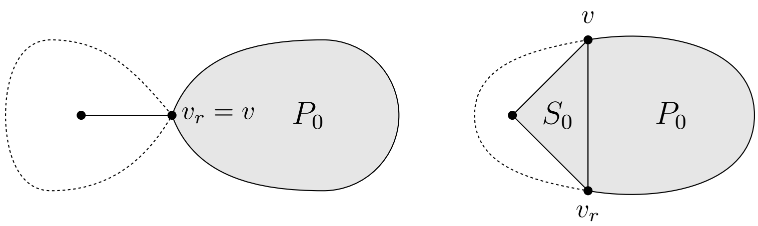

FIGURE 4.7. Obtaining a loop or a double edge by inserting an edge.

These cases have to be subtracted again in order to obtain a valid formula. First suppose that $v_r = v$. As we do not have double edges in S', this is only possible if the corner between c_r' and c' is at a vertex of degree one. We have to subtract $y^2uP_0(y,u)$, i.e. the fourth term of the right-hand side of (4.28), for this case (we add one vertex and two edges to a quasi-triangulation and increase the root face valency by one). Now suppose that v_r and v are adjacent, i.e. inserting an edge between them creates a double edge. In this case zipping the double edge separates the quasi-triangulation into two quasi-triangulations S_1, S_2. For one of them, without loss of generality for S_1, the root face valency is the same as for S, while the root face of S_2 has valency three. Thus, S_1 is in $\mathcal{P}_0$ but not the one-vertex map, while $S_2 \in \mathcal{T}_0$. Summing up we have to subtract $T_0(y)(P_0(y,u) - 1)$, the fifth term of the right-hand side of (4.28). $\square$

We can use the quadratic method (see e.g. [**60**]) to obtain the main result for planar triangulations from Proposition 4.5.1. Those were already obtained by Tutte [**103**] with slightly different parameters.

Lemma 4.5.2. *It holds that $\hat{T}_0(y) = T_0(y)$. The dominant singularity of $T_0(y)$ is $\rho_T = \frac{3}{2^{8/3}}$, $T_0(y)$ is Δ-analytic and satisfies*

$$T_0(y) = \frac{1}{8} - \frac{9}{16}\left(1 - \rho_T^{-1}y\right) + \frac{3}{2^{5/2}}\left(1 - \rho_T^{-1}y\right)^{3/2} + O\Big(\left(1 - \rho_T^{-1}y\right)^2\Big). \tag{4.29}$$

Furthermore, for $u = f(y)$ with

$$f(y) = \frac{t^{1/3}}{1+t} \qquad \textit{and} \qquad y = t^{1/3}(1-t),$$

the equations

$$\begin{aligned} P_0(y, f(y)) &= \frac{5}{4} - \frac{3}{2^{5/2}}\big(1-\rho_T^{-1}y\big)^{1/2} + O\big(1-\rho_T^{-1}y\big), \\ \left(\frac{\partial}{\partial u}P_0(y,u)\right)\bigg|_{u=f(y)} &= \frac{75}{2^{13/3}} - \frac{125\cdot 3^{3/4}}{2^{23/3}}\big(1-\rho_T^{-1}y\big)^{1/4} + O\Big(\big(1-\rho_T^{-1}y\big)^{1/2}\Big) \end{aligned}$$

hold and $P_0(y, f(y))$ is Δ-analytic. Let $n \geq 2$ be an integer. Then

$$\left(\frac{\partial^n}{\partial u^n}P_0(y,u)\right)\bigg|_{u=f(y)} = c(n)\big(1-\rho_T^{-1}y\big)^{-n/2+3/4} + O\Big(\big(1-\rho_T^{-1}y\big)^{-n/2+1}\Big),$$

where $c(n)$ is a positive constant depending only on n.

Proof. As that planar quasi-triangulations cannot have non-separating loops or double edges, $\hat{T}_0(y) = T_0(y)$ follows immediately.

Multiplying (4.28) by $4yu^4$ and rearranging the terms yields

$$\big(2yu^3P_0(y,u) + q(y,u)\big)^2 = q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4T_0(y), \tag{4.30}$$

where $q(x,u) = y - y^2u^2 - u - uT_0(y)$. Let

$$\begin{aligned} Q(y,u) &= 2yu^3P_0(y,u) + q(y,u) \qquad\qquad \text{and} \\ R(x,u) &= q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4T_0(y). \end{aligned}$$

Then (4.30) reduces to $Q^2(y,u) = R(y,u)$. To obtain the claimed asymptotic behaviour one chooses $u = f(y)$ in such a way that $Q(y, f(y)) = 0$. This u is a double zero of $Q^2(y,u)$ and therefore both $R(y,u)$ and $\frac{\partial}{\partial u}R(y,u)$ are 0 at $u = f(y)$, giving

$$\begin{aligned} 0 &= q(y,u)^2 + 4y^2u^3 - 4yu^4 - 4yu^4T_0(y), \\ 0 &= 2q(y,u)(1 + T_0(y) + 2y^2u) + 16yu^3 + 16yu^3T_0(y) - 12y^2u^2. \end{aligned}$$

By eliminating $f(y)$ from this system we obtain the implicit equation

$$T_0(y)^4 + 3T_0(y)^3 + T_0(y)^2(3+8y^3) + T_0(y)(1-20y^3) = (1-16y^3)y^3.$$

By standard methods for implicitly given functions (e.g. [**47**, VII.7.1]) we obtain the dominant singularity and the singular expansion of $T_0(y)$ as stated in (4.29).

Conversely, by eliminating $T_0(y)$ and substituting $y = t^{1/3}(1-t)$ we obtain $f(y) = \frac{t^{1/3}}{1+t} = \frac{y}{1-t^2}$ and $T_0(y) = t(1-2t)$. Since $\frac{1}{1-t^2}$ has only nonnegative coefficients in t and $t = t(y)$ has only nonnegative coefficients by Lagrange Inversion, $f(y)$ has only nonnegative coefficients as well. From the implicit equation for $f(y)$ we deduce that

$$f(y) = \frac{2^{4/3}}{5} - \frac{2^{11/6}}{25}\big(1-\rho_T^{-1}y\big)^{1/2} + O\big(1-\rho_T^{-1}y\big). \tag{4.31}$$

From $2yf(y)^3P_0(y, f(y)) + q(y, f(y)) = Q(y, f(y)) = 0$, (4.29), (4.31), and $y = \rho_T - \rho_T(1-\rho_T^{-1}y)$ we derive the claimed expression

$$P_0(y, f(y)) = \frac{5}{4} - \frac{3}{2^{5/2}}\big(1-\rho_T^{-1}y\big)^{1/2} + O\big(1-\rho_T^{-1}y\big).$$

Given $n \in \mathbb{N}_0$, let us write $R^{(n)}(y,u) = \frac{\partial^n}{\partial u^n}R(y,u)$. By the choice of $f(y)$ we know that $R^{(0)}(y, f(y)) = R^{(1)}(y, f(y)) = 0$. As $R(y,u)$ is a polynomial of degree four in u, we have $R^{(n)}(y,u) = 0$ for all $n \geq 5$. For $n \in \{2,3,4\}$, we obtain the dominant term of $R^{(n)}(y, f(y))$ by first differentiating $R(y, u)$ with respect to u and then substituting $u = f(y)$, (4.29), (4.31), and $y = \rho_T - \rho_T(1 - \rho_T^{-1}y)$. This yields

$$\begin{aligned}
R^{(2)}(y, f(y)) &= \frac{27}{2^{7/2}}\big(1-\rho_T^{-1}y\big)^{1/2} + O\big(1-\rho_T^{-1}y\big),\\
R^{(3)}(y, f(y)) &= -\frac{675}{2^{16/3}} + O\Big(\big(1-\rho_T^{-1}y\big)^{1/2}\Big),\\
R^{(4)}(y, f(y)) &= -\frac{10125}{2^{23/3}} + O\Big(\big(1-\rho_T^{-1}y\big)^{1/2}\Big).
\end{aligned}$$

We define $Q^{(n)}(y, u)$ and $P_0^{(n)}(y, u)$ analogously to $R^{(n)}(y, u)$. From the facts that $Q(y, f(y)) = 0$ and $\frac{\partial^n}{\partial u^n}\big(Q^2(y, u)\big) = R^{(n)}(y, u)$ we deduce that

$$\begin{aligned}
2nQ^{(1)}(y, f(y))Q^{(n-1)}(y, f(y)) &= R^{(n)}(y, f(y))\\
&\quad - \sum_{k=2}^{n-2}\binom{n}{k}Q^{(k)}(y, f(y))Q^{(n-k)}(y, f(y))
\end{aligned}\tag{4.32}$$

for every $n \in \mathbb{N}$. For $n = 2$, this implies that

$$Q^{(1)}(y, f(y)) = \overline{c}(1-\rho_T^{-1}y)^{1/4} + O\Big((1-\rho_T^{-1}y)^{3/4}\Big),$$

where $\overline{c} = \pm\frac{3^{3/2}}{2^{11/4}}$. By differentiating $Q(y, u) = 2yu^3P_0(y, u) + q(y, u)$ with respect to u, we deduce that

$$P_0^{(1)}(y, f(y)) = \frac{75}{2^{13/3}} + \overline{c}\frac{125}{2^{59/12}3^{3/4}}\big(1-\rho_T^{-1}y\big)^{1/4} + O\Big(\big(1-\rho_T^{-1}y\big)^{1/2}\Big).$$

Since $P_0^{(1)}(y, u)$ is a generating function of a combinatorial class, its coefficients $[y^ku^l]P_0^{(1)}(y, u)$ are nonnegative. As $f(y)$ has only nonnegative coefficients as well, all coefficients of $P_0^{(1)}(y, u)|_{u=f(y)}$ are nonnegative, implying that $\overline{c} = -\frac{3^{3/2}}{2^{11/4}}$.

For $n = 3$, we deduce from (4.32) that

$$Q^{(2)}(y, f(y)) = -\frac{675}{6\overline{c}2^{16/3}}(1-\rho_T^{-1}y)^{-1/4} + O\Big((1-\rho_T^{-1}y)^{1/4}\Big).$$

For $n \geq 4$, the term $R^{(n)}(y, f(y))$ is constant, while the sum on the right-hand side is nonempty. Since the sum only involves terms $Q^{(j)}(y, f(y))$ with $2 \leq j \leq n-2$, we deduce by induction that

$$Q^{(n)}(y, f(y)) = \overline{c}(n)\big(1-\rho_T^{-1}y\big)^{-n/2+3/4} + O\Big(\big(1-\rho_T^{-1}y\big)^{-n/2+5/4}\Big), \tag{4.33}$$

where $\overline{c}(n)$ is a constant depending only on n and $\overline{c}(n) > 0$ for $n \geq 2$.

The claimed expressions of $P_0^{(n)}(y, f(y))$ are now obtained by differentiating

$$Q(y, u) = 2y^2u^3P_0(y, u) + q(y, u)$$

n times and by (4.29), (4.31), (4.33), and induction.

As all generating functions in this proof are given by a system of algebraic equations, they are Δ-analytic. □

4.5.4. **Recurrence for higher genus.** Our next aim is to derive a recursion formula for $P_g(y, u, z_I)$ and $\hat{P}_g(y, u, z_I)$. Using the planar case in Lemma 4.5.2 as the base case, inductively applying the recursion formula allows us to derive similar statements as Lemma 4.5.2 for all g and I. In order to derive the recursion formula, we will perform different surgeries on the surface depending on the placement of the root. We distinguish four cases.

(A) The root edge e_r is only incident with the root face f_r and is a bridge;
(B) e_r is only incident with f_r and is not a bridge;
(C) e_r is incident with f_r and one marked face; and

(D) e_r is incident with f_r and one unmarked face.

The recursion formula is then of the form

$$P_g(y,u,z_I) = A_g(y,u,z_I) + B_g(y,u,z_I) + C_g(y,u,z_I) + D_g(y,u,z_I), \tag{4.34}$$

where $A_g(y,u,z_I)$, $B_g(y,u,z_I)$, $C_g(y,u,z_I)$, and $D_g(y,u,z_I)$ are the generating functions of the sub-classes $\mathcal{A}_{g,I}$, $\mathcal{B}_{g,I}$, $\mathcal{C}_{g,I}$, and $\mathcal{D}_{g,I}$ of $\mathcal{P}_{g,I}$ corresponding to the four cases (A), (B), (C), and (D) respectively. Each of the generating functions can be further decomposed as

$$\begin{aligned}
A_g(y,u,z_I) &= a(y,u)P_g(y,u,z_I) + M_A(g;y,u,z_I) - E_A(g;y,u,z_I),\\
B_g(y,u,z_I) &= b(y,u)P_g(y,u,z_I) + M_B(g;y,u,z_I) - E_B(g;y,u,z_I),\\
C_g(y,u,z_I) &= c(y,u)P_g(y,u,z_I) + M_C(g;y,u,z_I) - E_C(g;y,u,z_I),\\
D_g(y,u,z_I) &= d(y,u)P_g(y,u,z_I) + M_D(g;y,u,z_I) - E_D(g;y,u,z_I),
\end{aligned} \tag{4.35}$$

where $a(y,u)$, $b(y,u)$, $c(y,u)$, and $d(y,u)$ are functions only involving the generating functions P_0 and S_0 of the planar case, while the other functions involve terms of the type $P_{g'}(y,u,z_{I'})$ for $g' < g$ or $I' \subsetneq I$. This will enable us to use (4.34) to recursively determine the dominant terms of $P_g(y,u,z_I)$. In this recursion, the functions M_A, M_B, M_C, and M_D will contribute to the dominant term; the functions E_A, E_B, E_C, and E_D turn out to be of smaller order.

The classes $\hat{\mathcal{A}}_{g,I}$, $\hat{\mathcal{B}}_{g,I}$, $\hat{\mathcal{C}}_{g,I}$, and $\hat{\mathcal{D}}_{g,I}$ together with the functions $\hat{A}_g(y,u,z_I)$, $\hat{a}(y,u)$, $\hat{M}_A(g;y,u,z_I)$, and $\hat{E}_A(g;y,u,z_I)$ (and similarly for B, C, and D) are defined analogously.

We start by determining the functions for Case (A). In this case, after deleting the root edge we can split the map into two maps whose genera add up to g.

Lemma 4.5.3. *The three functions $a(y,u,z_I)$, $M_A(g;y,u,z_I)$, and $E_A(g;y,u,z_I)$ in* (4.35) *are given by*

$$\begin{aligned}
a(y,u,z_I) &= 2yu^2P_0(y,u),\\
M_A(g;y,u,z_I) &= yu^2\sum_{t,J} P_t(y,u,z_J)P_{g-t}\big(y,u,z_{I\setminus J}\big),\\
E_A(g;y,u,z_I) &= 0.
\end{aligned}$$

The sum is over $t = 0,\dots,g$ and $J \subseteq I$ such that $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g,I)$.

Proof. Let S be an I-quasi-triangulation in $\mathcal{A}_{g,I}$, with respect to $h\colon I \to F(S)$, say. By (A), the union $f_r \cup e_r$ is not a disk and thus contains a non-contractible circle C. We delete e_r, cut along C, and close the two resulting holes by inserting disks. Since e_r was a bridge, this surgery results in two components S_1 and S_2. We define the roots of S_1 and S_2 like in Proposition 4.5.1: let v_1 and v_2 be the end vertices of e_r in S_1 and S_2 respectively. Without loss of generality we may assume that v_1 is the root vertex of S. In the cyclic order of the edges of S at v_1, let e_1^- and e_1^+ be the predecessor and successor of e_r, respectively. Define e_2^- and e_2^+ analogously at v_2. We let (v_1,e_1^-,e_1^+) and (v_2,e_2^-,e_2^+) be the root of S_1 and S_2 respectively. Denote the root faces by f_1 and f_2 respectively. These are the faces of S_1 and S_2 into which the disks were inserted.

Since every face in $F(S)\setminus\{f_r\}$ corresponds to a face in $F(S_1)\setminus\{f_1\}$ or in $F(S_2)\setminus\{f_2\}$, h induces a function $\tilde{h}\colon I \to (F(S_1)\cup F(S_2))\setminus\{f_1,f_2\}$. If we write $J = \tilde{h}^{-1}(F(S_1))$, then S_1 is a J-quasi-triangulation on a surface of genus $t \le g$; consequently, S_2 is an $(I\setminus J)$-quasi-triangulation on a surface of genus $g-t$. By deleting e_r, we decreased the number of corners of f_r by two; the surgery then distributed the remaining corners of f_r to f_1 and f_2. Therefore, the sum of valencies of f_1 and f_2 is smaller by two than the valency of f_r. On the other hand, we clearly have $|E(S_1)| + |E(S_2)| = |E(S)| - 1$. The reverse operation of the surgery is to delete an open disk from each of f_1, f_2, glue the surfaces along the boundaries of these disks, and add an edge from the root corner of S_1 to the root corner of S_2. As this operation is uniquely defined, we deduce that

$$A_g(y, u, z_I) = yu^2 \sum_{t=0}^{g} \sum_{J \subseteq I} P_t(y, u, z_J) P_{g-t}\big(y, u, z_{i \setminus J}\big).$$

Extracting the terms for $(t, J) = (0, \emptyset)$ and $(t, J) = (g, I)$ finishes the proof. $\square$

Remark 4.5.4. *Analogously to Lemma 4.5.3, we have*

$$\begin{aligned} \hat{a}(y, u, z_I) &= 2yu^2 \hat{P}_0(y, u), \\ \hat{M}_A(g; y, u, z_I) &= yu^2 \sum_{t,J} \hat{P}_t(y, u, z_J) \hat{P}_{g-t}\big(y, u, z_{I \setminus J}\big), \\ \hat{E}_A(g; y, u, z_I) &= 0, \end{aligned}$$

where the sum is over $t = 0, \ldots, g$ and $J \subseteq I$ such that $(t, J) \neq (0, \emptyset)$ and $(t, J) \neq (g, I)$.

This follows by the same proof as for Lemma 4.5.3, because no loops or double edges occur in that construction.

For Case (B), we will cut the surface along a circle contained in $f_r \cup e_r$ and close the holes by inserting disks. However, because the surface will not be separated by this surgery, one needs to keep track of where to cut and glue to reverse the surgery. To this end we have to mark faces. Therefore, the index set I will increase.

Lemma 4.5.5. *The three functions $b(y, u, z_I)$, $M_B(g; y, u, z_I)$, and $E_B(g; y, u, z_I)$ in* (4.35) *are given by*

$$\begin{aligned} b(y, u, z_I) &= 0, \\ M_B(g; y, u, z_I) &= yu^2 \delta_{z_{i_0}} \big(P_{g-1}(y, u, z_{I \cup \{i_0\}})\big)\big|_{z_{i_0} = u}, \\ 0 \preceq E_B(g; y, u, z_I) &\preceq \big(1 + yu^2\big) \delta_u (P_{g-1}(y, u, z_I)). \end{aligned}$$

Proof. Let S be an I-quasi-triangulation in $\mathcal{B}_{g,I}$, with respect to $h\colon I \to F(S)$. We use the analogous surgery as in Lemma 4.5.3, with the difference that S is not separated by cutting along the circle C. Therefore we only obtain one map T. One of the end vertices of e_r is the root vertex v_r of S. Let e_r^- and e_r^+ be the predecessor and successor of e_r in the cyclic order of edges of S at v_r, respectively. Then we define the root of T to be (v_r, e_r^-, e_r^+). Denote the root face of T by f_r'; this is one of the two faces into which we inserted disks to close the holes during our surgery. Denote the other such face by f_2. We mark f_2 by adding a new index i_0 to the index set I and extend the function h to $I \cup \{i_0\}$ by setting $h(i_0) := f_2$. Then T is an $(I \cup \{i_0\})$-quasi-triangulation on $\mathbb{S}_{g-1}$.

To reverse the surgery, we delete an open disk from each of f_r', f_2, glue the surface along the boundaries of these disks, add a new edge e_{new} from the root corner of T to a corner c_2 of f_2, and let $(v_r, e_{\mathrm{new}}, e_r^+)$ be the new root corner. We thus have to mark a corner of f_2, which corresponds to applying the operator $\delta_{z_{i_0}}$ to the generating function. After gluing, the corners of f_2 become corners of the new root face; we thus have to remove i_0 from the index set and replace z_{i_0} by u in the generating function. Like in the previous cases, adding e_{new} increases the total number of edges by one and the valency of the root face by two, as e_{new} lies only on the boundary of the new root face. This results in the term $yu^2 \delta_{z_{i_0}} \big(P_{g-1}(y, u, z_{I \cup \{i_0\}})\big) \,|_{z_{i_0} = u}$.

However, by this construction we have overcounted. If the vertex v of the corner c_2 is adjacent to v_r, then e_{new} will be part of a double edge; if $v = v_r$, e_{new} will be a loop. We want to subtract all resulting maps $\tilde{S}$ for which e_{new} is a loop or part of a double edge. Suppose first that e_{new} is part of a double edge. Since e_{new} lies only on the boundary of the root face of $\tilde{S}$, the double edge is not separating. Thus, zipping it results in an I-quasi-triangulation $\tilde{T}$ on $\mathbb{S}_{g-1}$. One of the two zipped edges is the root edge e'_r of $\tilde{T}$, denote the other zipped edge by e'. Both e'_r and e' lie on the boundary of the root face (see Figure 4.8). Let v'_r be the root vertex of $\tilde{T}$; then v'_r is one of the two copies of v_r. If we denote the other copy by v', then v' is an end vertex of e' and thus there is a corner $c' = (v', e'', e')$ of the root face of $\tilde{T}$. We can reconstruct $\tilde{S}$ from $\tilde{T}$ in the following way: cut along e'_r and e' and glue the surface along the boundaries of the resulting holes in the unique way that identifies v'_r and v'. Identifying the corner c' is bounded by marking an *arbitrary* corner of the root face of $\tilde{T}$. This corresponds to applying the operator δ_u to the generating function $P_{g-1}(y, u, z_I)$. As zipping a double edge does neither change the number of edges nor the valencies of faces, $\delta_u(P_{g-1}(y, u, z_I))$ is an upper bound in this case.

Suppose now that e_{new} is a loop and recall that $(v_r, e_{\text{new}}, e_r^+)$ is the root corner of $\tilde{S}$. Since e_{new} lies only on the boundary of the root face of $\tilde{S}$, there is a unique edge $e_2 \neq e_r^+$ such that $(v_r, e_{\text{new}}, e_2)$ is a corner of the root face. We cut along e_{new}, close the two resulting holes by inserting disks, and delete the two copies of e_{new}. Again, cutting does not separate the surface. Thus, we obtain a map $\tilde{T}$ on $\mathbb{S}_{g-1}$ that does not have loops or double edges. Let v'_r be the copy of v_r in $\tilde{T}$ that is incident with e_r^+ and let v'_2 be the other copy. Then the root of $\tilde{T}$ is (v'_r, e', e_r^+) for some edge e'. Furthermore, the root face of $\tilde{T}$ has a corner (v'_2, e'_2, e_2). Now $\tilde{S}$ can be reconstructed from $\tilde{T}$ in the following way (see Figure 4.8).

(i) Add a loop at each of (v'_r, e', e_r^+) and (v'_2, e'_2, e_2);
(ii) delete the resulting two faces of valency one;
(iii) identify the two loops.

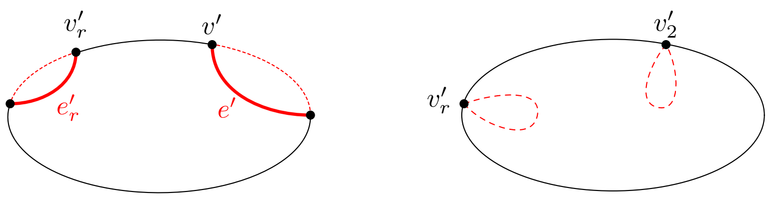

FIGURE 4.8. Deriving an upper bound for E_B.

In order to identify the corner (v'_2, e'_2, e_2), we mark an arbitrary corner of the root face, which is again overcounting. Since we have to add one edge to $\tilde{T}$ and increase the valency of the root face by two to reconstruct $\tilde{S}$, we have an additional factor of yu^2, resulting in the claimed upper bound for E_B. $\square$

Similar arguments also show the corresponding result for $\hat{B}_g$.

Lemma 4.5.6. *The functions $\hat{b}(y, u, z_I)$, $\hat{M}_B(g; y, u, z_I)$, and $\hat{E}_B(g; y, u, z_I)$ are given by*

$$\begin{aligned}
\hat{b}(y, u, z_I) &= 0, \\
\hat{M}_B(g; y, u, z_I) &= yu^2 \delta_{z_{i_0}} \left(\hat{P}_{g-1}(y, u, z_{I \cup \{i_0\}}) \right) \big|_{z_{i_0} = u}, \\
\hat{E}_B(g; y, u, z_I) &= 0.
\end{aligned}$$

The only difference to the proof of Lemma 4.5.5 is that we do not need to compensate for overcounting in $\hat{E}_B$, as all loops and double edges occurring in the proof are non-separating and thus allowed.

In Case (C), the root edge is not a bridge. Therefore, we will not be able to find a circle C like in the previous two cases. On the other hand, deleting the root edge does not produce any faces that are not disks. Our construction in this case will thus start without cutting the surface.

Lemma 4.5.7. *The three functions $c(y,u,z_I)$, $M_C(g;y,u,z_I)$, and $E_C(g;y,u,z_I)$ in* (4.35) *are given by*

$$c(y,u,z_I) = 0,$$

$$M_C(g;y,u,z_I) = y\sum_{i\in I}\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k},$$

$$0 \preceq E_C(g;y,u,z_I) \preceq \sum_{i\in I}\Bigg((1+yuz_i)\delta_{z_i}(P_{g-1}(y,u,z_I)) + (1+yuz_i)\sum_{t=0}^{g}\sum_{J\subseteq I\setminus\{i\}} P_t(y,u,z_J)P_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})})\Bigg),$$

where $\beta(T)$ and $\beta_j(T)$ denote the valencies of the root face of T and of the face with index j in T, respectively.

Note that the sum in M_C is over all $i\in I$ and all $I\setminus\{i\}$-quasi-triangulations. As such, M_C can be written as

$$M_C = y\sum_{i\in I}\frac{u^2 z_i P_g(y,u,z_{I\setminus\{i\}}) - uz_i^2 P_g(y,z_i,z_{I\setminus\{i\}})}{u-z_i}.$$

However, similarly to Lemma 4.5.2, we shall replace u and z_i by $f(y)$ in order to derive the desired asymptotic formulas, which would result in a division by 0. For that reason we will use M_C as stated in Lemma 4.5.7. We will derive a more convenient formulation in Proposition 4.5.11.

Proof of Lemma 4.5.7. Let S be an I-quasi-triangulation in $\mathcal{C}_{g,I}$ with respect to $h\colon I\to F(S)$. We delete the root edge e_r, thus obtaining a map T on $\mathbb{S}_g$. The root of T is defined as follows. Let e_r^- and e_r^+ be the predecessor and successor of e_r at v_r, respectively; then (v_r,e_r^-,e_r^+) is the root of T. By (C), e_r was incident with a marked face $h(i)$. The root face of T is $f_r' := f_r\cup e_r\cup h(i)$ and T is an $(I\setminus\{i\})$-quasi-triangulation with respect to $h|_{I\setminus\{i\}}$.

Let c be a corner of f_r' and let $\tilde{S}$ be obtained from T by adding an edge e_{new} from (v_r,e_r^-,e_r^+) to c and let the root of $\tilde{S}$ be

- $(v_r,e_{\mathrm{new}},e_r^+)$ if $c\neq(v_r,e_r^-,e_r^+)$ and
- either $(v_r,e_{\mathrm{new}},e_r^+)$ or $(v_r,e_{\mathrm{new}},e_{\mathrm{new}})$ otherwise.

These cases are illustrated in Figure 4.9. Adding e_{new} divides f_r' into two faces. One of these faces is the root face of $\tilde{S}$; we mark the other face with the index i and denote the corresponding function $I\to F(\tilde{S})$ by $\tilde{h}$. Clearly, there is a unique choice of $c\neq(v_r,e_r^-,e_r^+)$ such that $\tilde{S}=S$. If c is a corner at v_r (in particular if $c=(v_r,e_r^-,e_r^+)$), then e_{new} will be a loop. If c is a corner at a vertex adjacent to v_r, then e_{new} will be part of a double edge. In either case, $\tilde{S}$ will not be simple and thus not an I-quasi triangulation. Although the case $c=(v_r,e_r^-,e_r^+)$ is clearly one of the cases when $\tilde{S}$ is not simple, it is slightly easier to derive the formulas including this case.

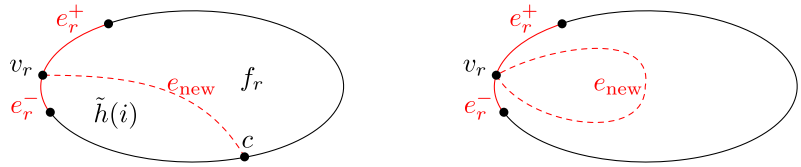

FIGURE 4.9. Adding the edge e_{new} from (v_r, e_r^-, e_r^+) to c to obtain $\tilde{S}$. If $c = (v_r, e_r^-, e_r^+)$, then each of the two faces can either be the root face or $\tilde{h}(i)$.

As f_r' has valency $\beta(T)$, there are $\beta(T)+1$ choices for $\tilde{S}$. The valency of the root face of $\tilde{S}$ is one if $c = (v_r, e_r^-, e_r^+)$ and the root face is $(v_r, e_{\text{new}}, e_{\text{new}})$. If $c = (v_r, e_r^-, e_r^+)$ and the root face is $(v_r, e_{\text{new}}, e_r^+)$, the valency is $\beta(T)+1$. Depending on which corner is chosen as c, the valency of the root face can take any value k between 1 and $\beta(T)+1$; the face $\tilde{h}(i)$ then has valency $\beta(T)+2-k$. The generating function of maps that can occur as $\tilde{S}$ from this particular $(I \setminus \{i\})$-quasi-triangulation T is thus given by

$$y^{|E(T)+1|}\left(\prod_{j \in I\setminus\{i\}} z_j^{\beta_j(T)}\right)\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}.$$

This holds as the number of edges is increased by one and the valencies of all other marked faces do not change. After summing over all possible marked faces and all possible T, we obtain M_C.

As already mentioned, we overcount whenever the chosen corner c is at v_r or at a vertex adjacent to v_r, making e_{new} a loop or part of a double edge, respectively. Suppose first that e_{new} is part of a double edge. We zip the double edge. If it does not separate the surface, we have an upper bound $\delta_{z_i}(P_{g-1}(y, u, z_I))$ analogous to Lemma 4.5.5. Indeed, the only difference to the corresponding case in Lemma 4.5.5 is that we mark a corner of (the face corresponding to) $\tilde{h}(i)$ instead of a corner of the root face, because e_{new} was incident with both the root face and $\tilde{h}(i)$. If the double edge separates the surface, we obtain two maps T_1 on $\mathbb{S}_t$ for $0 \le t \le g$ and T_2 on $\mathbb{S}_{g-t}$. One of the two maps, without loss of generality T_2, contains (the face corresponding to) $\tilde{h}(i)$. As T_2 is rooted at a corner of that face and the root face is never marked, the number of marks decreases by one. Thus, T_1 is a J-quasi-triangulation on $\mathbb{S}_t$ and T_2 is a $(I \setminus (J \cup \{i\}))$-quasi-triangulation on $\mathbb{S}_{g-t}$, where $J \subseteq I \setminus \{i\}$. Going back, all corners of the root face of T_2 become corners of the face with index i, meaning that we have to replace u by z_i in $P_{g-t}(x, z_i, z_{I\setminus(J\cup\{i\})})$. This gives us an upper bound of

$$\sum_{t=0}^{g}\sum_{J\subseteq I\setminus\{i\}} P_t(x, u, J) P_{g-t}(x, z_i, z_{I\setminus(J\cup\{i\})}).$$

If e_{new} is a loop, then we proceed the same way as in Lemma 4.5.5: we cut along e_{new}, close the two resulting holes by inserting disks, and delete the two copies of e_{new}. Like in Lemma 4.5.5, the reverse construction yields the same bounds as in the case of e_{new} being part of a double edge; the additional factor yuz_i is due to the fact that we add one edge and increase the valencies of the root face and of $\tilde{h}(i)$ by one. $\square$

Similar to Lemmas 4.5.5 and 4.5.6, the only difference when using triangulations without separating loops or separating double edges instead of simple triangulations is in the calculation of $\hat{E}_C$. The corresponding results are obtained by only keeping the terms in which separating loops or separating double edges are involved.

Lemma 4.5.8. *The functions $\hat{c}(y,u,z_I)$, $\hat{M}_C(g;y,u,z_I)$, and $\hat{E}_C(g;y,u,z_I)$ are given by*

$$\hat{c}(y,u,z_I) = 0,$$

$$\hat{M}_C(g;y,u,z_I) = y\sum_{i\in I}\sum_{T\in\hat{\mathcal{P}}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k},$$

$$0\preceq \hat{E}_C(g;y,u,z_I)\preceq \sum_{i\in I}2(1+yuz_i)\sum_{t=0}^{g}\sum_{J\subseteq I\setminus\{i\}}\hat{P}_t(y,u,z_J)\hat{P}_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})}),$$

where $\beta(T)$ and $\beta_j(T)$ denote the valencies of the root face of T and of the face with index j in T, respectively.

The difference between Lemma 4.5.7 and Lemma 4.5.8 is that we do not need to compensate for the case where non-separating loops and double edges appear, since they are allowed in $\hat{\mathcal{P}}_{g,I}$.

The construction in Case (D) is similar to Case (C). The fact that the second face incident to e_r is not marked makes the analysis easier.

Lemma 4.5.9. *The functions $d(y,u,z_I)$, $M_D(g;y,u,z_I)$, and $E_D(g;y,u,z_I)$ in* (4.35) *are given by*

$$d(y,u,z_I) = yu^{-1} - y^2u - T_0(y),$$

$$M_D(g;y,u,z_I) = -T_g(y,z_I)P_0(y,u),$$

$$0\preceq E_D(g;y,u,z_I)\preceq 3P_{g-1}(y,u,z_{I\cup\{i_0\}})|_{z_{i_0}=u} + \sum_{t,J}T_t(y,J)P_{g-t}(y,u,z_{I\setminus J}),$$

where the sum is taken over all $0\leq t\leq g$ and $J\subseteq I$ with $(t,J)\neq(0,\emptyset)$ and $(t,J)\neq(g,I)$.

Proof. Let S be an I-quasi-triangulation in $\mathcal{D}_{g,I}$. We delete e_r and choose the root of the resulting map T to be $c_r' := (v_r, e_r^-, e_r^+)$ like in Lemma 4.5.7. As the second face f incident with e_r is not marked and S is an I-quasi-triangulation, f is bounded by a triangle. Thus, T is also an I-quasi-triangulation and the valency of its root face f_r' is larger by one than the valency of f_r. For the reverse construction, consider the ordering of the corners of f_r' in clockwise direction along its boundary and let c be the corner after the next, starting from c_r'. We add an edge e_{new} from c_r' to c and let $(v_r, e_{\text{new}}, e_r^+)$ be the root of the resulting I-quasi-triangulation $\tilde{S}$. If T was obtained from S by deleting e_r, then $\tilde{S}=S$. However, if T is an *arbitrary* I-quasi-triangulation on $\mathbb{S}_g$, then e_{new} might be a loop or part of a double edge. Thus,

$$yu^{-1}P_g(y,u,z_I) \tag{4.36}$$

is only an upper bound for $D_g(y,u,z_I)$. Again, we have to subtract the cases when $\tilde{S}$ is not simple.

The case when e_{new} is a loop yields a term of

$$-y^2uP_g(x,u,z_I) \tag{4.37}$$

analogously to Proposition 4.5.1. When e_{new} is part of a double edge, we need to distinguish whether this double edge separates the surface. If it does separate, we obtain

$$-\sum_{t=0}^{g}\sum_{J\subseteq I}T_t(y,J)P_{g-t}(y,u,z_{I\setminus J}) \tag{4.38}$$

by zipping the double edge, similar to Lemma 4.5.3. The only differences are that the number of edges and the valencies of the faces do not change and that one of the two components is a J-triangulation, since its root face is f and thus has valency three. Finally, if the double edge does not separate, then after zipping it we have to mark f with a new index i_0 like in Lemma 4.5.5. However, since the valency of f is three, we only have three possible ways to reverse the construction. As the number of edges and all valencies remain unchanged, we have a summand

$$-3P_{g-1}(y, u, z_{I\cup\{i_0\}})|_{z_{i_0}=u}\,. \tag{4.39}$$

Note that (4.39) is overcounting as the reverse construction can lead to additional loops or double edges.

Combining (4.36), (4.37), and the term from (4.38) with $(t, J) = (0, \emptyset)$, we deduce the claimed expression for $d(y, u, z_I)$. The term from (4.38) with $(t, J) = (g, I)$ yields $M_D(g; y, u, z_I)$; the remaining terms form the upper bound for $E_D(g; y, u, z_I)$. □

Throughout the proof of Lemma 4.5.9, we do not encounter loops or multiple edges. Thus, the corresponding result for $\hat{D}$ follows immediately.

Lemma 4.5.10. *The functions $\hat{d}(y, u, z_I)$, $\hat{M}_D(g; y, u, z_I)$, and $\hat{E}_D(g; y, u, z_I)$ are given by*

$$\begin{aligned}
\hat{d}(y, u, z_I) &= yu^{-1} - y^2u - T_0(y),\\
\hat{M}_D(g; y, u, z_I) &= -\hat{T}_g(y, z_I)\hat{P}_0(y, u),\\
0 \preceq \hat{E}_D(g; y, u, z_I) &\preceq 3\hat{P}_{g-1}(y, u, z_{I\cup\{i_0\}})|_{z_{i_0}=u} + \sum_{t,J} \hat{T}_t(y, J)\hat{P}_{g-t}(y, u, z_{I\setminus J}),
\end{aligned}$$

where the sum is taken over all $0 \le t \le g$ and $J \subseteq I$ with $(t, J) \neq (0, \emptyset)$ and $(t, J) \neq (g, I)$.

4.5.5. **Asymptotics.** We now compute the asymptotics of all generating functions involved. Among the generating functions of all cases, the only one with a different structure than the others is M_C which cannot be easily expressed in terms of $P_{g'}(y, u, z_{I'})$ and $T_{g'}(y, I')$ with some genus g' and set I'. From M_B and E_B we observe that we need to calculate derivatives with respect to u and z_{i_0} and that we want to set $z_{i_0} = u$ in the end. We will be interested in the dominant term of $P_g(y, u, z_I)$ when we set $u = f(y)$ and $z_i = f(y)$ for all $i \in I$; we will abbreviate this by $u = z_I = f(y)$. Observe that setting $u = z_I = f(y)$ does not have any influence on the functions $T_g(y)$, as they only depend on the variable y.

The following proposition enables us to express arbitrary derivatives of M_C (and $\hat{M}_C$) at $u = z_I = f(y)$ in terms of derivatives of P_g (or $\hat{P}_g$).

Proposition 4.5.11. *Let $|y| < \rho_T$, $n \in \mathbb{N}_0$, and $\alpha_i \in \mathbb{N}_0$ for all $i \in I$. Write $|\alpha_I|$ for $\sum \alpha_i$. Then*

$$\begin{aligned}
&\left.\frac{\partial^{n+|\alpha_I|}}{\partial u^n \prod_{i\in I}\partial z_i^{\alpha_i}} M_C(g; y, u, z_I)\right|_{u=z_I=f(y)} \qquad (4.40)\\
&= y\left.\left(\sum_{i\in I} \frac{n!\alpha_i!}{(n+\alpha_i+1)!}\frac{\partial^{n+1+|\alpha_I|}}{\partial u^{n+1+\alpha_i}\prod_{j\in I\setminus\{i\}}\partial z_j^{\alpha_j}}\left(u^3P_g(y, u, z_{I\setminus\{i\}})\right)\right)\right|_{u=z_I=f(y)}.
\end{aligned}$$

Proof. The generating function $yu^3P_g(y, u, z_{I\setminus\{i\}})$ is given by

$$yu^3P_g(y, u, z_{I\setminus\{i\}}) = y\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}u^{\beta(T)+3}\prod_{j\in I\setminus\{i\}} z_j^{\beta_j(T)}.$$

By comparing this term with the summand in

$$M_C(g;y,u,z_I) = y\sum_{i\in I}\sum_{T\in\mathcal{P}_g(I\setminus\{i\})} y^{|E(T)|}\prod_{j\neq i} z_j^{\beta_j(T)}\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}$$

for a fixed index $i\in I$, one sees that the difference between them is that the factor $u^{\beta(T)+3}$ is replaced by $\sum_{k=1}^{\beta(T)+1} u^k z_i^{\beta(T)+2-k}$. Taking the derivatives with respect to u and z_i the given number of times and comparing the coefficients yields factors

$$\frac{(\beta(T)+3)!}{(\beta(T)+2-n-\alpha_i)!}u^{\beta(T)+2-n-\alpha_i} \qquad \text{and}$$

$$\sum_{k=n}^{\beta(T)+2-\alpha_i}\frac{k!(\beta(T)+2-k)!}{(k-n)!(\beta(T)+2-k-\alpha_i)!}u^{\beta(T)+2-n-\alpha_i},$$

respectively, when $n+\alpha_i+1\le\beta(T)+3$ and factors 0 otherwise. The quotient of these two coefficients equals $\frac{n!\alpha_i!}{(n+\alpha_i+1)!}$ by a binomial identity. Summing over $i\in I$ finishes the proof. $\square$

Remark 4.5.12. *With the same proof, the analogous result for $\hat{M}_C$ and $\hat{P}_g$ holds as well.*

The only other term where differentiating is not straight forward is M_B. By using the chain rule n times we obtain

$$\begin{aligned}
&\frac{\partial^{n+|\alpha_I|}}{\partial u^n\prod_{i\in I}\partial z_i^{\alpha_i}}M_B(g;y,u,z_I) \qquad (4.41)\\
&\quad=\frac{\partial^{n+|\alpha_I|}}{\partial u^n\prod_{i\in I}\partial z_i^{\alpha_i}}\left(yu^2\Big(z_{i_0}\frac{\partial}{\partial z_{i_0}}P_{g-1}\big(y,u,z_{I\cup\{i_0\}}\big)\Big)\Big|_{z_{i_0}=u}\right)\\
&\quad=y\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\partial^{n-k+|\alpha_I|}}{\partial u^{n-k}\prod_{i\in I}\partial z_i^{\alpha_i}}\frac{\partial^{k+1}}{\partial z_{i_0}^{k+1}}\big(u^3P_{g-1}\big(y,u,z_{I\cup\{i_0\}}\big)\big)\right)\Big|_{z_{i_0}=u}.
\end{aligned}$$

Using (4.40) and (4.41) we can now determine the dominant terms of the derivatives of T_g and P_g (and analogously the derivatives of $\hat{T}_g$ and $\hat{P}_g$).

Theorem 4.5.13. *Let $\alpha_i\in\mathbb{N}_0$, $i\in I$, and $|\alpha_I|:=\sum\alpha_i$. If $(g,I)\neq(0,\emptyset)$, then*

$$\frac{\partial^{|\alpha_I|}}{\prod\partial z_i^{\alpha_i}}T_g(y,z_I)\Big|_{z_I=f(y)} \cong a_0+c_g\big(1-\rho_T^{-1}y\big)^{e_1}+O\Big(\big(1-\rho_T^{-1}y\big)^{e_1+1/4}\Big), \qquad (4.42)$$

where a_0 and $c_g=c_g(\alpha_i,i\in I)$ are positive constants and

$$e_1=-\frac{5g}{2}-\frac{5|I|}{4}-\frac{|\alpha_I|}{2}+\frac{3}{2}.$$

$\frac{\partial^n}{\partial u^n}P_0(y,u)\big|_{u=f(y)}$ is given as in Lemma 4.5.2. If $(g,I,n)\neq(0,\emptyset,0)$, then

$$\frac{\partial^{n+|\alpha_I|}}{\partial u^n\prod\partial z_i^{\alpha_i}}P_g(y,u,z_I)\Big|_{u=z_I=f(y)} \cong c\big(1-\rho_T^{-1}y\big)^{e_2}+O\Big(\big(1-\rho_T^{-1}y\big)^{e_2+1/4}\Big), \qquad (4.43)$$

where $c=c(g,|I|,n,|\alpha_I|)$ is a positive constant and

$$e_2=e_1-\frac{n}{2}-\frac{3}{4}.$$

Proof. We show Theorem 4.5.13 by induction on $(g,|I|,n)$ in lexicographic order. Lemma 4.5.2 shows that (4.43) is true for $(g,I)=(0,\emptyset)$ and $n>0$. Note that $|\alpha_I|=0$ for $I=\emptyset$.

Suppose now that (4.43) is true for all $(g,|I|,n)<(g_0,|I_0|,0)$ and (4.42) is true for all $(g,|I|)<(g_0,|I_0|)$ with $(g,I)\neq(0,\emptyset)$. We first prove that (4.42) holds for $(g_0,|I_0|)$. By multiplying (4.34) by u and applying Lemmas 4.5.3, 4.5.5, 4.5.7 and 4.5.9 we obtain

$$u(1-a-d)P_{g_0}(y,u,z_{I_0}) = u(M_A+M_B+M_C) - uT_{g_0}(y,z_{I_0})P_0(y,u) - uE,$$

where $E = E_B + E_C + E_D$. The term

$$u(1-a-d) = u - 2yu^3P_0(y,u) - uT_0(y) - y + y^2u^2$$

is equal to $-Q(y,u)$ in (4.30) and thus

$$-Q(y,u)P_{g_0}(y,u,z_{I_0}) = u(M_A+M_B+M_C) - uT_{g_0}(y,z_{I_0})P_0(y,u) - uE. \quad (4.44)$$

Therefore, the left-hand side is zero when replacing u by $f(y)$. As this factor is independent of z_{I_0}, this does also hold when differentiating the equation α_i times with respect to z_i. Thus we obtain

$$uP_0(y,u)\frac{\partial^{|\alpha_{I_0}|}T_{g_0}(y,z_{I_0})}{\prod \partial z_i^{\alpha_i}}\bigg|_{u=z_{I_0}=f(y)} = u\frac{\partial^{|\alpha_{I_0}|}(M_A+M_B+M_C-E)}{\prod \partial z_i^{\alpha_i}}\bigg|_{u=z_{I_0}=f(y)}.$$

By inspecting the formulas for M_A to E_D in Lemmas 4.5.3, 4.5.5, 4.5.7 and 4.5.9, one sees that all occurring terms are lexicographically smaller than $(g_0,|I_0|,0)$,and the induction hypothesis can thus be used. Inspection of the exponents of $(1-\rho_T^{-1}y)$ in all those terms shows the following.

M_A: The summands have the form

$$\frac{\partial^{|\alpha_{I_0}|}}{\prod \partial z_i^{\alpha_i}}\big(P_t(y,u,z_J)P_{g_0-t}(y,u,z_{I_0\setminus J})\big)$$

for $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g_0,I_0)$. Thus, for $g_0+|I_0| \leq 1$, we have $M_A = 0$. For all other values of $(g_0,|I_0|)$, each summand is of the form $c_A(1-\rho_T^{-1}y)^{m_A} + O\big((1-\rho_T^{-1}y)^{m_A+1/4}\big)$ with

$$m_A = -\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{4} - \frac{5(g_0-t)}{2} - \frac{5|I_0\setminus J|}{4} - \frac{|\alpha_{I_0\setminus J}|}{2} + \frac{3}{4} = e_1$$

by induction. Furthermore, all coefficients are positive by induction.

M_B: We have $M_B = yu^2\delta_{z_{i_0}}\big(P_{g_0-1}(y,u,z_{I\cup\{i_0\}})\big)\ |_{z_{i_0}=u}$. Thus, for $g_0 = 0$ we have $M_B = 0$ and $M_B = c_B(1-\rho_T^{-1}y)^{m_B} + O\big((1-\rho_T^{-1}y)^{m_B+1/4}\big)$ otherwise with

$$m_B = -\frac{5(g_0-1)}{2} - \frac{5|I_0\cup\{i_0\}|}{4} - \frac{|\alpha_{I_0}|+1}{2} + \frac{3}{4} = e_1$$

by induction. Again, the coefficient is positive by induction.

M_C: We determine the expression for M_C by Proposition 4.5.11. For $g_0 = 0$ and $|I_0| = 1$ we have $M_C \cong c_{C,1}(1-\rho_T^{-1}y)^{1/4} + O\big((1-\rho_T^{-1}y)^{1/2}\big)$, which is of the desired order, since $e_1 = 1/4$ in this case. For all other (g_0,I_0), induction yields that $M_C = c_{C,2}(1-\rho_T^{-1}y)^{m_C} + O\big((1-\rho_T^{-1}y)^{m_C+1/4}\big)$ with

$$m_C = -\frac{5g_0}{2} - \frac{5(|I_0|-1)}{4} - \frac{|\alpha_{I_0}|}{2} - \frac{1}{2} + \frac{3}{4} = e_1\,.$$

Like in the previous cases, the coefficient is positive by induction.

E_B: The function E_B is bounded from above by $\big(yu^2+1\big)\delta_u(P_{g_0-1}(y,u,z_{I_0}))$. For $g_0 = 0$, we thus have $E_B = 0$ and otherwise $E_B = O\big((1-\rho_T^{-1}y)^{e_B}\big)$ with

$$e_B = -\frac{5(g_0-1)}{2} - \frac{5|I_0|}{4} - \frac{|\alpha_{I_0}|}{2} - \frac{1}{2} + \frac{3}{4} \geq e_1 + \frac{1}{4}\,.$$

E_C: The first summand in the expression of E_C from Lemma 4.5.7 is 0 if $g_0 = 0$ and otherwise $O((1-\rho_T^{-1}y)^{e_{C,1}})$ with

$$e_{C,1} = -\frac{5(g_0-1)}{2} - \frac{5|I_0|}{4} - \frac{|\alpha_{I_0}|+1}{2} + \frac{3}{4} \geq e_1 + \frac{1}{4}.$$

The second summand is $a_E + O\big((1-\rho_T^{-1}y)^{1/2}\big)$ if $g_0 = 0$ and $|I_0| = 1$. Suppose $(g_0, |I_0|) \neq (0,1)$. Then every term

$$(1+yuz_i)P_t(y,u,z_J)P_{g-t}(y,z_i,z_{I\setminus(J\cup\{i\})})$$

with $(t,J) \neq (0,\emptyset)$ and $(t,J) \neq (g_0, I_0\setminus\{i\})$ is $O((1-\rho_T^{-1}y)^{e_{C,2}})$ with

$$\begin{aligned} e_{C,2} = &-\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{4} - \frac{5(g_0-t)}{2} - \frac{5(|I_0\setminus J|-1)}{4} - \frac{|\alpha_{I_0\setminus J}| - \alpha_i}{2} + \frac{3}{4} \\ &\geq e_1 + \frac{1}{4} \end{aligned}$$

by induction. The corresponding terms for $(t,J) = (0,\emptyset)$ and $(t,J) = (g_0, I_0\setminus\{i\})$ are $O\big((1-\rho_T^{-1}y)^{e_{C,3}}\big)$ with

$$e_{C,3} = -\frac{5g_0}{2} - \frac{5(|I_0|-1)}{4} - \frac{|\alpha_{I_0}| - \alpha_i}{2} + \frac{3}{4} \geq e_1 + \frac{1}{4}.$$

In total, we have $E_C = O\big((1-\rho_T^{-1}y)^{e_1+1/4}\big)$.

E_D: The first summand in the expression of E_D from Lemma 4.5.9 is 0 if $g_0 + |I_0| \leq 1$ and otherwise each of its summands is $O\big((1-\rho_T^{-1}y)^{e_{D,1}}\big)$ with

$$\begin{aligned} e_{D,1} = &-\frac{5t}{2} - \frac{5|J|}{4} - \frac{|\alpha_J|}{2} + \frac{3}{2} - \frac{5(g_0-t)}{2} - \frac{5|I_0\setminus J|}{4} - \frac{|\alpha_{I_0\setminus J}|}{2} + \frac{3}{4} \\ &\geq e_1 + \frac{1}{4} \end{aligned}$$

by induction. The second term is 0 for $g_0 = 0$ and $O\big((1-\rho_T^{-1}y)^{e_{D,2}}\big)$ otherwise with

$$e_{D,2} = -\frac{5(g_0-1)}{2} - \frac{5(|I_0|+1)}{4} - \frac{|\alpha_{I_0}|}{2} + \frac{3}{4} \geq e_1 + \frac{1}{4}.$$

Note that for $\hat{P}_g$ the only difference is in the formulas of $\hat{E}_B$ and $\hat{E}_C$, both of which satisfy the same inequalities as E_B and E_C above. Thus the following conclusions also hold for $\hat{P}_g$.

Combining these results, we have proved (4.42), where $a_0 = a_M - a_E$ for $g_0 = 0$ and $|I_0| = 1$. As this constant is the value of the generating function $T_0(y, z_{I_0})|_{z_{I_0}=f(y)}$ at its singularity ρ_T, a_0 is positive. For $|\alpha_{I_0}| > 0$ or $(g_0, |I_0|) \neq (0,1)$, the exponent e_1 is negative and (4.42) is thus true with the same value for a_0. Finally, c_g is positive, since it is the sum of positive numbers.

To prove (4.43), recall that we assume that (4.43) is true for $(g, |I|, n) < (g_0, |I_0|, 0)$ and we have already shown that (4.42) is true for $(g, |I|) \leq (g_0, |I_0|)$. Let $n_0 \in \mathbb{N}_0$ and assume that (4.43) is also true for $(g_0, |I_0|, n)$ with $n < n_0$. Consider the derivative $\frac{\partial^{n+1}}{\partial u^{n+1}}$ of (4.44) and set $u = f(y)$; as $Q(y, f(y)) = 0$, this yields

$$-\sum_{k=0}^{n} \binom{n+1}{k} \frac{\partial^k}{\partial u^k} P_{g_0}(y,u,z_{I_0}) \frac{\partial^{n+1-k}}{\partial u^{n+1-k}} Q(y,u)\bigg|_{u=z_{I_0}=f(y)}$$

for the left-hand side of (4.44). For the derivatives of M_A, M_B, and M_C, we obtain

$$\begin{aligned}
M_A &= c_A(1-\rho_T^{-1}y)^{m_A} + O\Big((1-\rho_T^{-1}y)^{m_A+1/4}\Big),\\
M_B &= c_B(1-\rho_T^{-1}y)^{m_B} + O\Big((1-\rho_T^{-1}y)^{m_B+1/4}\Big),\\
M_C &= c_C(1-\rho_T^{-1}y)^{m_C} + O\Big((1-\rho_T^{-1}y)^{m_C+1/4}\Big)
\end{aligned}$$

with positive constants c_A, c_B, c_C and $m_A = m_B = m_C = e_1 - \frac{n+1}{2} = e_2 + \frac{1}{4}$. For the derivatives of E_B, E_C, and E_D, the exponents $e_B, e_{C,1}, \ldots$ in the considerations above reduces by $\frac{n+1}{2}$ as well. By (4.33) and the induction hypothesis, each term

$$\frac{\partial^k}{\partial u^k} P_{g_0}(y,u,z_{I_0}) \frac{\partial^{n+1-k}}{\partial u^{n+1-k}} Q(y,u)\Big|_{u=z_{I_0}=f(y)}$$

for $k < n$ is of the form $\overline{c}(k)(1-\rho_T^{-1}y)^{e_1-(n+1)/2} + O\big((1-\rho_T^{-1}y)^{e_1-(n+1)/2+1/4}\big)$ with $\overline{c}(k) > 0$. Since $\frac{\partial}{\partial u}Q(y,u)|_{u=f(y)} = \overline{c}(1-\rho_T^{-1}y)^{1/4} + O\big((1-\rho_T^{-1}y)^{1/2}\big)$ with $\overline{c} < 0$, (4.43) follows. □

An analogous proof yields the corresponding result for $\hat{\mathcal{T}}_g$ and $\hat{\mathcal{P}}_g$, with identical constants a_0, c_g, c, e_1, e_2. By Lemma 4.5.2 and setting $I = \emptyset$ in (4.42) we deduce Proposition 4.3.2 and, from the corresponding result for $\hat{T}_g$, also Proposition 4.3.3.

CHAPTER 5

Cubic graphs with non-constant genus

In this section we prove Theorem 1.3.5. Therefore, we derive upper and lower bounds for $|\mathcal{S}_g(2n)|$ for general $g = g(n)$ via some double counting arguments. Throughout this section, let $g = g(n)$ be a function. Theorem 1.3.5 distinguishes three cases for the range of g:

(i) $g > \frac{n-1}{2}$;
(ii) $g \leq \frac{n-1}{2}$;
(iii) $g = o\big(n \log(n)^{-2}\big)$.

We will prove the statement in these three regimes in different ways. Throughout these proofs, we will use Euler's formula at various points

$$|V| - |E| + |F| = 2 - 2g, \tag{5.1}$$

where $|V|, |E|, |F|$ are the vertices, edges, and faces of a map on $\mathbb{S}_g$, respectively.

5.1. Fast-growing genus

We start with the fast growing regime, that is, with $g > \frac{n-1}{2}$. In this case we want to show that all cubic graphs are embeddable on $\mathbb{S}_g$. We even prove a slightly stronger result. Let $\mathcal{M}_g(2n)$ be the class of all cubic *maps* on $\mathbb{S}_g$. We show that if the genus is too large, then no such maps exist. Furthermore, we state that in this case indeed all graphs are embeddable on $\mathbb{S}_g$. Recall that in a map, all faces are homeomorphic to discs.

Proposition 5.1.1. *Suppose that $g > \frac{n-1}{2}$. Then $\mathcal{M}_{g+1}(2n) = \emptyset$ and all cubic graphs are embeddable on $\mathbb{S}_g$, i.e. $\mathcal{S}_g(2n) = \mathcal{S}(2n)$.*

Proof. The statement $\mathcal{M}_{g+1}(2n) = \emptyset$ is a simple consequence of Euler's formula (5.1), which states that for a cubic map on $\mathbb{S}_{g+1}$ with $2n$ vertices and $3n$ edges, we have $|F| = n + 2 - 2(g+1) < 1$ faces, which is clearly a contradiction.

For the second statement, let $G \in \mathcal{S}(2n)$. We have to show that G is embeddable on $\mathbb{S}_g$. Let M be any embedding of G on some surface $\mathbb{S}_{g'}$ (that is, $M \in \mathcal{M}_{g'}(2n)$). As $\mathcal{M}_{g+k}(2n) = \emptyset$ for $k \geq 1$ by the first part of the proposition, we have $g' \leq g$. Therefore $G \in \mathcal{S}_g(2n)$, proving Proposition 5.1.1. $\square$

5.2. Intermediate regime

Next we prove the intermediate regime $g \leq \frac{n-1}{2}$. In this case, we use double counting. Via various intermediate steps, we will deduce upper and lower bounds in terms of unicellular maps. More precisely, we use four different classes of graphs and maps and prove bounds between them by multiple double counting arguments.

- We show upper and lower bounds for cubic graphs in $\mathcal{S}_g(2n)$ in terms of *connected* cubic graphs embeddable on the same surface $\mathbb{S}_g$, denoted by $\overline{\mathcal{S}}_g(2n)$.
- We bound the number of connected cubic graphs in $\overline{\mathcal{S}}_g(2n)$ by cubic *maps* on $\mathbb{S}_g$, denoted by $\mathcal{M}_g(2n)$.

- We bound the number of cubic maps in $\mathcal{M}_g(2n)$ by maps on $\mathbb{S}_g$ with exactly one face and maximum degree three. We denote this class by $\mathcal{E}_g(2n)$.

The maps in $\mathcal{E}_g(2n)$ are also called *unicellular* maps. For convenience, we will show the bounds starting from one-faced maps in the reverse order of this list.

As we only bound the maximum degree of maps in $\mathcal{E}_g(2n)$, we further subdivide this class. For $a \geq 0$, we denote by $\mathcal{E}_g(2n, a)$ the subclass of $\mathcal{E}_g(2n)$ with exactly a vertices of degree three. Note that all maps in $\mathcal{E}_g(2n, a)$ have the same number of vertices of degree two and one.

Lemma 5.2.1. *Every graph $G \in \mathcal{E}_g(2n, a)$ has exactly $a - 4g + 2$ vertices of degree one and $2n - 2a + 4g - 2$ vertices of degree two.*

Proof. Let b be the number of vertices of degree one and c be the number of vertices of degree two. Then

$$a + b + c = 2n \quad \text{and} \quad \frac{3a + b + 2c}{2} = |E(G)|.$$

Additionally we have by Euler's formula that

$$|E(G)| = 2n + 2g - 1.$$

Substituting this and solving the linear system of equations for b and c proves the lemma. $\square$

This lemma gives some bounds on possible values of a, as all of the values a, $a - 4g + 2$, and $2n - 2a + 4g - 2$ have to be non negative. We denote by

$$A := \{a \in \mathbb{Z} : \max\{0, 4g - 2\} \leq a \leq n + 2g - 1\}$$

the set of all possible values of a. Note that for $g \leq \frac{n-1}{2}$, A is not empty. In order to simplify notation we assume for the remainder of this section that $g > 0$.

As a starting point, the number of unicellular maps with given degree sequence was determined by Walsh and Lehman [**105**]. In our special case with a vertices of degree three we have the following formula.

Lemma 5.2.2 ([**105**])**.** *Let $a \in A$. Then the number of unicellular maps in $\mathcal{E}_g(2n, a)$ is given by*

$$|\mathcal{E}_g(2n, a)| = 2^{2n-2a+2g-2} 3^{a-g} \binom{a}{g} (2n + 2g - 2)!. \tag{5.2}$$

By summing over all possible values for a, where the numbers of vertices of degree one and two are not negative, we derive bounds for the number of unicellular maps in $\mathcal{E}_g(2n)$.

Lemma 5.2.3. *There exist constants b_E, c_E such that for all $a \in A$, the number of unicellular maps in $\mathcal{E}_g(2n)$ satisfies*

$$b_E^{n+g}(2n + 2g)! \leq |\mathcal{E}_g(2n, a)| \leq |\mathcal{E}_g(2n)| \leq c_E^{n+g}(2n + 2g)!. \tag{5.3}$$

Proof. As $\mathcal{E}_g(2n)$ is the disjoint union of $\mathcal{E}_g(2n, a)$ over all $a \in A$, we have

$$|\mathcal{E}_g(2n)| = \sum_{a \in A} |\mathcal{E}_g(2n, a)|.$$

Substituting (5.2) and sorting the terms results in

$$|\mathcal{E}_g(2n)| = \left(\frac{4}{3}\right)^g 4^{n-1} (2n + 2g - 2)! \sum_{a \in A} \binom{a}{g} \left(\frac{3}{4}\right)^a.$$

Next we calculate bounds for the sum. We use the bounds (3.6) for the binomial coefficient and derive

$$\sum_{a\in A}\binom{a}{g}\left(\frac{3}{4}\right)^a = \Theta(1)\sum_{a\in A} f_1(a)f_2(a),$$

where

$$f_1(a) = \sqrt{\frac{a}{g(a-g)}} \quad \text{and} \quad f_2(a) = \left(\frac{3}{4}\right)^a \left(\frac{a-g}{g}\right)^g \left(\frac{a}{a-g}\right)^a.$$

In order to prove the upper bound, first note that $f_1(a)$ is strictly decreasing for $a \in A$. To derive the maximum of $f_2(a)$ we calculate the derivative and observe that it is zero at $a = 4g$. In addition, as the second derivative at $a = 4g$ is negative, this is indeed a maximum. Therefore, we deduce an upper bound for $|\mathcal{E}_g(2n)|$ of

$$|\mathcal{E}_g(2n)| = O\left(\left(\frac{4}{3}\right)^g 4^n (2n+2g)!|A| f_1(4g-2) f_2(4g)\right).$$

The claimed upper bound follows by noting that $f_2(4g) = 3^g$ and choosing suitable exponential bounds for the polynomial terms.

To show the lower bound, note that for $a \geq 4g-2$, we have $\frac{a-g}{g} \geq 2$ and $\frac{a}{a-g} \geq 1$. Therefore, $f_2(a) \geq \left(\frac{3}{4}\right)^a 2^g$ and

$$|\mathcal{E}_g(2n, a)| = \Omega\left(\frac{f_1(a)}{(2n+2g)(2n+2g-1)}\left(\frac{8}{3}\right)^g \left(\frac{3}{4}\right)^a 4^n (2n+2g)!\right). \tag{5.4}$$

Choosing suitable exponential bounds for the fraction and choosing the constants as in $a = n + 2g - 2$ for the exponential terms in (5.4) proves the lemma. $\square$

For the next step, we prove bounds for the class $\mathcal{M}_g(2n)$ of all vertex-labelled cubic maps on $\mathbb{S}_g$.

Lemma 5.2.4. *Let $g \leq \frac{n-1}{2}$. Then*

$$2^{-3n}|\mathcal{E}_g(2n)| \leq |\mathcal{M}_g(2n)| \leq d_0 \gamma_p^{4n-2+4g} |\mathcal{E}_g(2n)|,$$

where $\gamma_p \approx 5.828$ is the growth constant of dissections, i.e. 2-connected outerplanar graphs, and d_0 is a constant.

Proof. We will prove both inequalities by a double counting argument.

For the upper bound, let $M \in \mathcal{M}_g(2n)$. As the map is vertex-labelled, there is a canonical order on the edges. We go through the edges in order and if the edge is in the boundary of two different faces we delete it. By this construction we obtain a unicellular map U. As we did not delete any vertices, U has exactly $2n - 2g + 1$ edges by Euler's formula. For the reverse direction, we need an upper bound for the number of maps where the construction results in U. First note that both M and U are labelled, have the same number of vertices and during the construction no labels change. Therefore the only possible ambiguity in the construction is in how to delete edges of M in the unique face of U. This has to be done in such a way that no intersections occur. Therefore we need an upper bound on the number of ways to insert $2n - 2g + 1$ non-intersecting diagonals into a face with $2n + 4g - 2$ vertices (some vertices are on the face multiple times). This is bounded from above by the number of dissections of such a polygon and thus bounded from above by $d_0 \gamma_p^{2n+4g-2}$.

To show the lower bound we will construct a map in $\mathcal{M}_g(2n)$ from a map $N \in \mathcal{E}_g(2n, 4g-2)$. Note that N has no vertices of degree one and, as $g \leq \frac{n-1}{2}$, at least four vertices of degree two. We have to construct a cubic map in such a way that we do not generate any double edges. Therefore we cannot connect two adjacent vertices of degree two. In order to ensure this, for each vertex v of degree two we denote by $\beta(v)$ the number of vertices of degree two (itself included) that can be reached from v on a path where all internal vertices have degree two as well. That is, on a maximal path of vertices of degree two, all vertices have the same value $\beta(v)$ and this value is the number of vertices of degree two on this path.

Suppose first there are no vertices v with $\beta(v) = 2$. Then we use the following construction for vertices with $\beta(v) \geq 3$. If $\beta(v)$ is odd, we iteratively add edges between the outermost degree two vertices until only one degree two vertex is left. All these edges are on the same side of the path. Which side is chosen will be determined later by the edge to the central vertex. If $\beta(v)$ is even, we again iteratively add edges between the outermost vertices of degree two until exactly four vertices are left. All those edges are on the left side of the path. For the final four vertices we connect the first and third vertex and the second and fourth vertex as in Figure 5.1 . Finally, we connect all vertices that are left (vertices with $\beta(v) = 1$ or central vertices of paths with $\beta(v)$ odd), to the next one along the boundary of the one face of N, starting from the root corner.

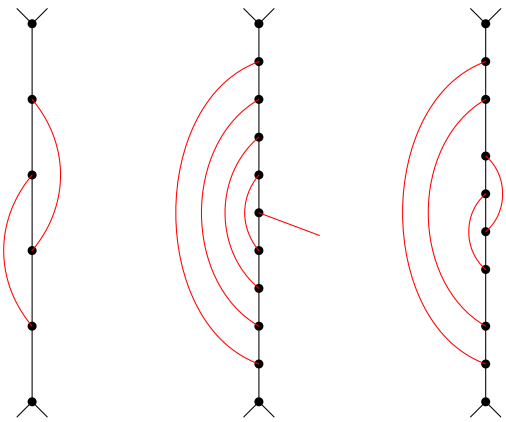

FIGURE 5.1. How to connect multiple vertices along a path: $\beta = 4$ on the left hand side; $\beta \geq 3$ odd in the centre; $\beta \geq 6$ even on the right hand side.

Suppose now that we have vertices with $\beta(v) = 2$. Then we iteratively reduce the size of the problem. Let v_1, v_2 be the lexicographically smallest such pair of connected vertices of degree two. As the number of vertices is at least four, there exists another vertex v_3, where v_3 is adjacent (i.e. the next vertex of degree two, disregarding possible vertices of degree three in between, see Figure 5.2) to one of the two copies of v_1 along the boundary of the face (if there are two such vertices, choose the one with smaller label). We connect v_3 with v_1 along the boundary of the face where there are on other vertices of degree two. Note that if there are still pairs of vertices with $\beta(v) = 2$ in the newly constructed graph, there are still at least four vertices of degree two left, as $\beta(v_2) = 1$ in the new graph and the number of vertices of degree two has to be even. We now iterate this construction.

As no edge connects neighbouring vertices, the resulting map is simple. We deduce an upper bound for the number of unicellular maps resulting in the same map by stating that the underlying graph of the unicellular map has to be a subgraph of the underlying graph of the resulting map. As every graph has at most $2^{|E|}$ many subgraphs, we prove the claimed bound. $\square$

By substituting the bounds of Lemma 5.2.3 in the bounds of Lemma 5.2.4, we derive bounds on the number of all vertex labelled cubic maps with $2n$ vertices on $\mathbb{S}_g$ for any $g = g(n)$.

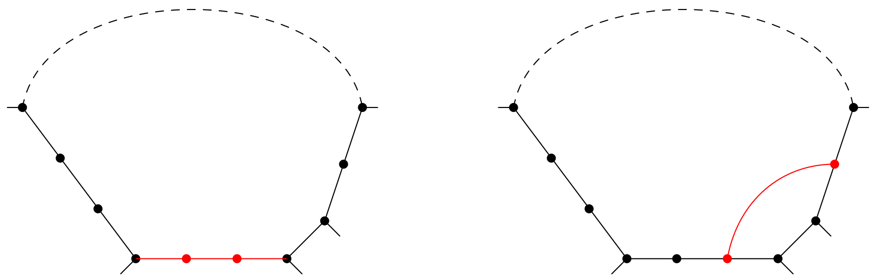

FIGURE 5.2. The construction for $\beta = 2$.

Corollary 5.2.5. *Let $g \le \frac{n-1}{2}$. Then there exist constants b_M, c_M such that*

$$b_M^{n+g}(2n+2g)! \le |\mathcal{M}_g(2n)| \le c_M^{n+g}(2n+2g)!.$$

□

Next we compare the classes $\mathcal{M}_g(2n)$ of vertex-labelled cubic maps and $\overline{\mathcal{S}}_g(2n)$ of connected cubic graphs. As we do not have any analogues to Whitney's theorem as in the constant genus case, we only have very general bounds.

Lemma 5.2.6. *Let $g \le \frac{n-1}{2}$. Then*

$$2^{-2n}|\mathcal{M}_g(2n)| \le |\overline{\mathcal{S}}_g(2n, 3n)| \le \frac{|\mathcal{M}_g(2n+4g)|}{(4g)!}.$$

PROOF. To prove the lower bound observe first that the underlying graph of each map in $\mathcal{M}_g(2n)$ is in $\overline{\mathcal{S}}_g(2n)$. For a given graph G, all possible 2-cell embeddings of G can be described using so-called *rotation systems* (see e.g. [**86**]). This means that the embedding is uniquely defined as soon as, for each vertex v, the cyclic order is fixed in which the three edges incident with v are arranged around v. For cubic G we have two possibilities for each vertex and thus at most (not all such choices will result in maps on $\mathbb{S}_g$, the genus may vary) 2^{2n} 2-cell embeddings on $\mathbb{S}_g$.

To show the upper bound we give an injection from $\overline{\mathcal{S}}_g(2n)$ to the class $\mathcal{N}$ of cubic maps on $\mathbb{S}_g$ with $2n + 4g$ vertices, n of which are labelled. Therefore $|\mathcal{N}| = \frac{|\mathcal{M}_g(2n+4g)|}{(4g)!}$ and the upper bound will follow.

Let $G \in \overline{\mathcal{S}}_g(2n)$. We will construct a map in $\mathcal{N}$ as follows. Let M be a 2-cell embedding of G on a surface of genus $g_0 \le g$. As G is embeddable on $\mathbb{S}_g$ such an embedding exists. Use a canonical way to choose a root vertex, a root face and root edge (together defining a root corner) on the vertex-labelled map M. Let Ψ be one (canonically chosen) map from the class $\mathcal{E}_{g-g_0}(4(g-g_0), 4(g-g_0)-1)$, that is, a one-faced map of genus $g - g_0$ with one vertex of degree one and $4(g-g_0)-1$ vertices of degree three. Replace the root face with the unique face of Ψ, and insert the one vertex of degree one in the root edge. That is, we delete the root edge and instead connect its two end vertices with the vertex of degree one. In this way we construct a cubic map on the surface $\mathbb{S}_g$ with $2n + 4g - 4g_0$ vertices. In order to construct a map with $2n + 4g$ vertices, we subdivide the two non-root edges attached to the root vertex $2g_0$ times and connect them in pairs (see Figure 5.3).

This results in a cubic 2-cell embedding on $\mathbb{S}_g$ with $2n+4g$ vertices and thus in a map in $\mathcal{N}$. This is an injection as G can uniquely be determined from the result of this construction. Indeed, there are exactly three *labelled* vertices connected to the root vertex via a path where all intermediate vertices are unlabelled. These were the original neighbours of the root vertex. □

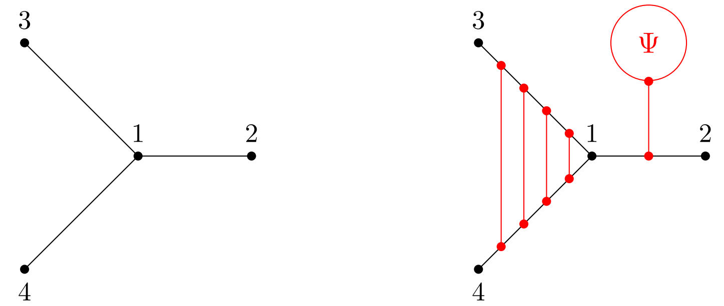

FIGURE 5.3. Constructing a map with the correct number of vertices and genus.

Again, we derive bounds for $|\overline{\mathcal{S}}_g(2n)|$ from Lemma 5.2.6 and Corollary 5.2.5.

Corollary 5.2.7. *Let $g \le \frac{n-1}{2}$. Then there exist constants $b_{\overline{S}}, c_{\overline{S}}$ such that*

$$b_{\overline{S}}^{n+g}(2n+2g)! \le |\overline{\mathcal{S}}_g(2n,3n)| \le c_{\overline{S}}^{n+g}\frac{(2n+6g)!}{(4g)!}.$$

$\square$

Finally, we prove bounds for general cubic graphs embeddable on $\mathbb{S}_g$ via connected cubic graphs on $\mathbb{S}_g$.

Lemma 5.2.8. *Let $g \le \frac{n-1}{2}$. Then*

$$\left|\overline{\mathcal{S}}_g(2n,3n)\right| \le |\mathcal{S}_g(2n,3n)| \le \frac{\left|\overline{\mathcal{S}}_g(4n-2,6n-3)\right|}{(2n-2)!}.$$

PROOF. Let $K \in \mathcal{S}_g(2n,3n)$ be a graph and let c be the number of its components. To prove Lemma 5.2.8 we construct $(2n-2)!$ graphs in $\overline{\mathcal{S}}_g(4n-2,6n-3)$ from K such that for different graphs $K, \overline{K}$, all resulting graphs are pairwise different.

We sort the components by the vertex with the smallest label occurring in each component and call them $C_1,\dots,C_c$. For a component C_i let $e_{i,1}$ and $e_{i,2}$ be the lexicographically smallest and largest edge adjacent to the vertex with smallest label in C_i, respectively. For $1 \le i \le c-2$ we subdivide $e_{i,2}$ and $e_{i+1,1}$ and connect the new vertices. Then we subdivide $e_{c-1,2}$ and $e_{c,1}$ both $n-c$ times and connect these vertices in order (see Figure 5.3).

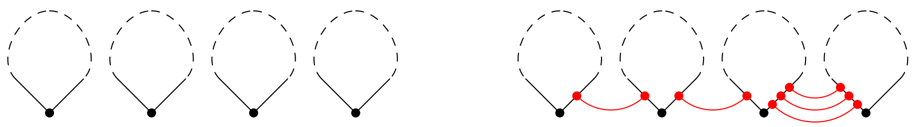

FIGURE 5.4. Constructing a connected graph from multiple connected components.

Labelling the new vertices from $2n+1$ to $4n-2$ results in $(2n-2)!$ different graphs.

Conversely, there is a unique way to reconstruct the original graph from a graph constructed in this way, completing the proof. $\square$

5.3. Slowly growing genus

In this section we prove Theorem 1.3.5(i). We do this via a double counting argument comparing $\mathcal{S}_g(2n)$ to cubic planar graphs via *planarising* edge sets.

Djidjev and Venkatesan [**33**] proved an upper bound on the size of the smallest planarising edge set.

Lemma 5.3.1. [**33**] *For any m-edge graph with maximum degree d embeddable on $\mathbb{S}_g$, there exists a planarising set of at most $4\sqrt{dgm}$ edges.*

Deleting such a planarising set will be one direction of the double counting argument.

Lemma 5.3.2. *For $g = o((\log n)^{-1}n)$, there exists a constant c such that*

$$|\mathcal{S}_g(2n)| \leq |\mathcal{S}_0(2n)|c^{\log n\sqrt{ng}}.$$

Proof. Let $K \in \mathcal{S}_g(2n)$ and let E be a minimal planarising edge set of K with $|E| := k \leq 12\sqrt{gn}$, which exists by Lemma 5.3.1. We construct a cubic planar graph from K as follows. First we delete all edges in E to obtain a graph K'. Note that K' does not have isolated vertices as that would contradict the minimality of E. Let ψ be the graph in Figure 5.5.

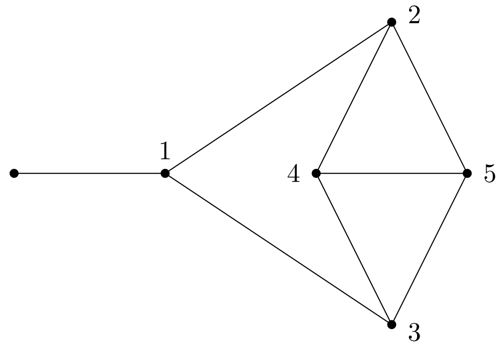

Figure 5.5. The graph ψ used as a building block in the construction.

We now attach one copy of ψ to each vertex of degree two in K' and two copies of ψ to each vertex of degree one. In this way, we attach two copies of ψ for each deleted edge and therefore have $10k$ new vertices. Furthermore, we label these new vertices with labels $2n+1$ to $2n+10k$ (with five consecutive labels in the correct order for each of the copies of ψ) starting from the neighbouring vertex with smallest label. In this way, we construct a unique graph in $\mathcal{S}_0(2(n+5k))$.

Conversely, in order to construct the original graph, we have to delete the pending copies of ψ with labels $2n+1$ to $2n+5k$. Then we have to add k new edges such that all vertices have degree three. An upper bound for this construction is to add the correct number of half edges at each vertex and choose a perfect matching among these half edges. There are at most $(2k)(2k-2)(2k-4)\cdots 2 = \frac{(2k)!}{2^k k!}$ possible matchings.

Summing up over all possible values of k results in

$$\begin{aligned}|\mathcal{S}_g(2n)| &\leq \sum_{k=0}^{12\sqrt{gn}} |\mathcal{S}_0(2(n+5k))|\frac{(2k)!}{2^k k!}\\ &\leq 12\sqrt{gn}|\mathcal{S}_0(2n+120\sqrt{gn})|\frac{(24\sqrt{gn})!}{2^{12\sqrt{gn}}(12\sqrt{gn})!},\end{aligned}$$

as we have

$$\frac{\left|\mathcal{S}_0(2n+120\sqrt{gn})\right|}{|\mathcal{S}_0(2n)|} \leq \Theta(1)\left(1+60\sqrt{\frac{g}{n}}\right)^{-7/2} \gamma_0^{120\sqrt{gn}} \frac{(2n+120\sqrt{gn})!}{(2n)!}$$
$$\overset{(3.4),(3.2)}{\leq} \exp(\alpha_1 \log n\sqrt{gn})$$

and also

$$\frac{(24\sqrt{gn})!}{2^{12\sqrt{gn}}(12\sqrt{gn})!} \overset{(3.4),(3.2)}{\leq} \exp(\alpha_2 \log n\sqrt{gn}).$$

For suitable constants α_1, α_2, the lemma follows. □

5.4. Proof of the final main result

By Lemma 5.3.2, we know that

$$\left(\frac{|\mathcal{S}_g(2n)|}{(2n)!}\right)^{\frac{1}{2n}} \leq \left(\frac{|\mathcal{S}_0(2n)|}{(2n)!}\right)^{\frac{1}{2n}} c^{\frac{1}{2}\log n\sqrt{\frac{g}{n}}} = (1+o(1))\gamma_0,$$

which, together with the trivial fact that $\mathcal{S}_0(2n) \subseteq \mathcal{S}_g(2n)$ proves Theorem 1.3.5(i).

Secondly, Theorem 1.3.5(ii) follows by substituting the bounds of Lemma 5.2.3 in the bounds of Lemma 5.2.4, deducing

$$b_A^{n+g}(2n+2g)! \leq |\mathcal{S}_g(2n,3n)| \leq c_A^{4n-2+g}\frac{(4n+6g-2)!}{(4g)!(2n-2)!}.$$

Using the bounds in (3.4) for the factorials results in the claimed bounds.

Finally, Theorem 1.3.5(iii) follows immediately from Proposition 5.1.1.

CHAPTER 6

Discussion

6.1. Comparisons

Let us first compare $G_g(n,m)$ to the Erdős-Rényi random graph $G(n,m)$. In the first phase transition, the main differences between Theorems 1.1.1 and 1.2.1 are observed in the *supercritical regime*, which is when $\lambda \to \infty$. Firstly, the order of the giant component is only about half as large in $G_g(n,m)$ as it is in $G(n,m)$. Secondly, the i-th-largest component H_i for fixed $i \geq 2$ is much larger in $G_g(n,m)$ than in $G(n,m)$. This second difference is only the outward visible difference stemming from a much deeper and more interesting distinction. If one deletes the giant component from $G(n,m)$ in the supercritical regime, the remaining graph behaves as if it were a graph drawn uniformly at random from the *subcritical regime* and thus only small components occur. In contrast, in $G_g(n,m)$, deleting the giant component results in a graph in the *critical* regime, thus resulting in larger orders for H_i with $i \geq 2$. Finally, while each such H_i is a tree whp for the Erdős–Rényi random graph, this is not necessarily the case for $G_g(n,m)$.

The other major difference between $G_g(n,m)$ and $G(n,m)$ is the appearance of the second phase transition. While for $G(n,m)$, the number of vertices outside the giant component remains linear as long as m is linear, this is not the case for $G_g(n,m)$ for any fixed g. To understand this phenomenon better, even larger classes of graphs have to be studied, for example classes with non-constant genus.

Comparing the results of this thesis with the results on $G_0(n,m)$ by Kang and Łuczak[**66**], the main improvement is in the second phase transition. Whereas Kang and Łuczak only proved their results up to $m = n + O(n^{2/3})$, we improved that bound to $m = n + O\big(n(\log n)^{-2/3}\big)$. Additionally, in this thesis we improved on the order and structure of the i-th-largest component ($i \geq 2$) and showed that the giant component is also the unique non-planar component and cannot be embedded on a surface of smaller genus.

6.2. Two phase transitions

Although we obtained a multitude of results regarding the component structure of sparse random graphs embeddable on orientable surfaces in Theorems 1.2.1, 1.2.3 and 1.2.5, there are still many unsolved problems.

Even in the first critical regime, that is, for $m = (1 + \lambda\, n^{-1/3})\frac{n}{2}$ with constant $\lambda \in \mathbb{R}$, there are still open questions. Noy, Ravelomanana, and Rué [**87**] answered a challenging open question of Erdős and Rényi [**41**] about the limiting probability of $G(n,m)$ being planar at the critical phase 1Crit, the limit $p(\lambda)$ of the probability that $G\big(n, \big(1 + \lambda\, n^{-1/3}\big)\frac{n}{2}\big)$ is planar. For graphs embeddable on a surface of positive genus, they gave a general strategy for how to determine the corresponding probability up to a given error. However, determining the *exact* limiting probability for $g \geq 1$ is still an open problem.

Other open questions appear in the second phase transition. Comparing the range for m that we cover in Theorem 1.2.3 with the 'dense' regime $m = \lfloor \mu n \rfloor$ for $1 < \mu < 3$ considered in [**30, 59**], a gap of order $(\log n)^{2/3}$ becomes apparent—a significant improvement over the planar case in [**66**], where the gap had order $n^{1/3}$. The order term $\zeta^{-3/2}\, n^{3/5}$ in Theorem 1.2.3 becomes constant when $\zeta = \Theta(n^{2/5})$, which matches the results from [**30, 59**] that the giant component covers all but finitely many vertices in the dense regime. Therefore, we expect Theorem 1.2.3 to hold for *all* $m = (1 + o(1))n$.

The gap of order $(\log n)^{2/3}$ originates from the fact that we can only determine the number of cores up to an exponential error term in the second phase transition. The cause of this gap is the fact that we determine the number of cores of embeddable graphs only up to an exponential error (see Lemma 3.4.5 and Lemma 3.4.6). Obtaining better bounds for the number of cores should therefore close this gap and the results of Theorem 1.2.3 should hold for all $m = (1 + o(1))n$. Moreover, such better enumeration results could show a similar strong relation between the complex part and the kernel of the graph, as in the first phase transition and the intermediate regime. It might therefore open the possibility of proving an analogous version of Theorem 3.5.4 in the second phase transition. This would further improve the understanding of the second phase transition.

6.3. Changing the model

Throughout this thesis, we have always considered graphs embeddable on orientable surfaces, as a way to understand the differences between planar graphs and the Erdős-Rényi random graphs. This is by far not the only feasible graph class. An obvious generalisation is the class of graphs embeddable on non-orientable surfaces. The calculations done in Chapter 3 would still work even for that case, provided we obtain an analogous result to Theorem 1.3.3 for weighted multigraphs embeddable on non-orientable surfaces. The main problem is that there is no easy analogue to Whitney's theorem or Lemma 4.2.8 for non-orientable surfaces. Thus, connecting embeddable graphs and maps is much harder.

Another possible expansion of the model is to let the genus also be dependent on n. As seen when comparing Theorem 1.1.1 and Theorems 1.2.1 and 1.2.3, there are various differences in the behaviour of graphs in $\mathcal{G}(n, m)$ and $\mathcal{G}_g(n, m)$. Investigating graphs embeddable on surfaces where the genus may depend on n is one idea to deduce the reason. Those classes are intermediate steps between $\mathcal{G}(n, m)$ and $\mathcal{G}_g(n, m)$. Intuitively, the faster the genus is allowed to grow, the more the graph will behave like a graph from $\mathcal{G}(n, m)$. Indeed, as soon as the genus is bigger than the expected excess in $\mathcal{G}(n, m)$, such graphs are embeddable with high probability. This is due to the fact that any graph has the same genus as its kernel. If the genus is larger than the number of edges in the kernel, such a graph is therefore embeddable.

On the other hand, heuristically, if the genus is so small that the kernel of $\mathcal{G}(n, m)$ is not embeddable on $\mathbb{S}_g$, there should not be much difference between growing and non-growing genus. The methods provided in this thesis should also work for the case of non-constant genus. The main starting point for this is the number of cubic graphs embeddable on a surface of growing genus. We proved first results in this direction in Chapter 5. The results in Theorem 1.3.5 about cubic graphs are relatively weak compared to the results of the constant genus case in Theorem 1.3.1. Nonetheless, we believe that they are strong enough to prove phase transition results for graphs embeddable on surfaces with non-constant genus similar to the results in this thesis. On the other hand, more exact enumeration results for cubic graphs would be of interest in themselves, as, until now, very few classes of graphs or maps on surfaces of non-constant genus have been studied.

Another interesting problem emerges for m beyond 1Crit. There we know that whp $G(n, m)$ is not embeddable on any surface of fixed genus. This immediately raises the question of what genus $g = g(n)$ is needed in order to embed $G(n, m)$ on $\mathbb{S}_g$. That is, given $m = m(n)$, for what functions $g = g(n)$ is $G(n, m)$ embeddable on $\mathbb{S}_g$ with high probability? Or given $m = m(n)$, what is the expected genus of $G(n, m)$?

Another interesting direction, which might provide insight into the behaviour of $G(n, m)$, is to reverse these questions. Suppose we are given $g = g(n)$ that tends to infinity with n. Does $G_g(n, m)$ admit a second phase transition? Is its behaviour more closely related to the constant genus case or to $G(n, m)$?

Heuristically, if g grows 'fast enough' (e.g. as $\binom{n}{2}$), then $G_g(n, m)$ will coincide with $G(n, m)$ and will therefore not exhibit the second phase transition described in Theorem 1.2.3. For 'slowly' growing g, on the other hand, it is to be expected that the second phase transition does take place. The interesting question therefore is, where this change takes place and whether there is a 'phase transition phenomenon' occurring for the appearance of the second phase transition.

CHAPTER 7

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