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ON STOCHASTIC INTEREST RATE MODELS AND THEIR EXTENSIONS TO INCORPORATE DEFAULT RISK

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Abstract

The goal of this thesis is to give a comprehensive overview of stochastic interest rate models, which are of great importance in financial and actuarial mathematics, not only since new regulations require market-consistent valuation of assets and liabilities. After establishing the basics of interest rate theory and looking at the evolution of those models over time, a special focus will be laid on the G2++ model, a two-factor Gaussian model with a deterministic shift, which is used in Germany to simulate market scenarios for retirement provisions.
Additionally, an intensity-based approach to incorporate default risk is going to be presented. It turns out that, with suitable assumptions, including default risk simply leads to a stochastic spread added to the risk-free interest rate. Therefore, the results and methods derived for risk-free interest rate models can often be readily transferred to the case where default risk is present. Methods for calibrating the models to the market will be presented and implemented.
This thesis only brushes the surface of this exciting topic, which is still adjusting to the new market environment as an aftermath of the global financial crisis.

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And last but not least, I would like to say thank you to my friends, who make my life outside of university so very enjoyable. Without you, life would be bleak and unexciting.

Abbreviations and notation

- ATM = At the money
- CDF = Cumulative distribution function
- CDS = Credit default swap
- CIR = Cox-Ingersoll-Ross model
- FRA = Forward rate agreement
- G2++ = Two-factor Gaussian-model with deterministic shift
- IRS = Interest Rate Swap
- ITM = In the money
- MC = Monte Carlo
- OTM = Out of the money
- RFS = Receiver IRS
- SDE = Stochastic differential equation
- B_t = Bank account at time t
- $r(t)_{t\geq 0}$ = Instantaneous spot interest rate (”short rate”) process
- $\lambda(t)_{t\geq 0}$ = Intensity process driving survival probabilities
- $D(t,T)$ = Stochastic discount factor for the interval $[t,T]$
- $P(t,T)$ = Zero-coupon bond price for maturity T at time t
- $P^M(0,T)$ = Zero-coupon bond price for maturity T currently observed in the market
- $\bar{P}(t,T)$ = Defaultable zero-coupon bond price for maturity T at time t
- $R(t,T)$ = Continuously-compounded spot rate for the interval $[t,T]$
- $L(t,T)$ = Simply-compounded spot rate for the interval $[t,T]$
- $Y(t,T)$ = Annually-compounded spot rate for the interval $[t,T]$
- $Y^k(t,T)$ = k-times annually-compounded spot rate for the interval $[t,T]$
- $F(t,T,S)$ = Simply-compounded forward rate at time t for the interval $[T,S]$
- $f(t,T,S)$ = Continuously-compounded forward rate at time t for the interval

$[T, S]$

- $S_{i,j}(t)$ = Forward swap rate at time t for initial reset time T_i and payment times $T_{i+1}, \cdots, T_j$
- $R_{i,j}(t)$ = CDS forward rate at time t for payment times $T_i, \cdots, T_j$
- L = Loss given default of a CDS
- R = premium payment rate of a CDS
- $\mathbb{P}$ = physical (or real-world) measure
- $\mathbb{Q}$ = risk-neutral (or equivalent martingale) measure
- $\mathbb{Q}^T$ = T-forward measure (corresponding to numeraire $P(\cdot, T)$)
- $(W(t))_{t\geq 0}$ = Brownian motion with respect to the risk-neutral measure
- $(W^T(t))_{t\geq 0}$ = Brownian motion with respect to the T-forward measure
- $\mathbb{E}$ = Expectation w.r.t. to a previously specified measure
- $\mathbb{E}^{\mathbb{Q}}$ = Expectation w.r.t. to the risk-neutral measure
- $\mathbb{E}^T$ = Expectation w.r.t. to the T-forward measure
- $\mathcal{T}$ = Set of relevant dates $\{T_\alpha, T_{\alpha+1} \cdots, T_\beta\}$ of an interest rate derivative. T_α is the first reset date, T_β the last settlement date
- τ = Set of year fraction $\{\tau_{\alpha+1}, \cdots, \tau_\beta\}$ corresponding to $\mathcal{T}$, i.e. $\tau_i = T_i - T_{i-1}$.
- τ = Time of default of a company
- $\mathbb{1}_A$ = Indicator function for set A; equals 1 for $\omega \in A$, else equals zero
- Φ = Cumulative distribution function of the standard normal distribution
- $CDS_{a,b}(t, R, L)$ = Price at time t of CDS with payment dates $T_a, \cdots, T_b$, premium rate R and loss given default L
- $\mathbf{Cap}(t, T_{i-1}, T_i, N, K)$ = Price of a cap at time t with relevant dates $\mathcal{T}$, strike rate K and notional amount N
- $\mathbf{Cpl}(t, \mathcal{T}, \tau, N, K)$ = Price of a caplet at time t with reset date T_{i-1}, settlement date T_i, strike rate K and notional amount N
- $\mathbf{Flr}(t, T_{i-1}, T_i, N, K)$ = Same as $\mathbf{Cap}(t, T_{i-1}, T_i, N, K)$, just as a floor
- $\mathbf{Fll}(t, \mathcal{T}, \tau, N, K)$ = Same as $\mathbf{Cpl}(t, T_{i-1}, T_i, N, K)$, just as a floorlet
- $\mathbf{PS}(0, \mathcal{T}, \tau, N, K)$ = Price of a payer swaption at time t with relevant dates $\mathcal{T}$, strike rate K and notional amount N
- $\mathbf{RS}(0, \mathcal{T}, \tau, N, K)$ = Same as $\mathbf{PS}(0, \mathcal{T}, \tau, N, K)$, just as a receiver swaption
- $\mathbf{ZBC}(t, T, S, N, K)$ = Price of a European call at time t option with maturity T and strike K with an S-bond with notional N as an underlying
- $\mathbf{ZBP}(t, T, S, N, K)$ = Same as $\mathbf{ZBC}(t, T, S, N, K)$, just as a put option.

Contents

Chapter 1

Introduction

Interest rate models are a very important tool in today's world of financial and actuarial mathematics. To give an example, consider the Solvency II directive of the European Union, which came into effect at the beginning of 2016. Among other things, it requires insurance companies to perform market consistent valuation of their assets and liabilities to be able to verify that the solvency capital requirements are met. For liquid financial contracts, one can just take the price quoted on the market. However, if a financial contract is illiquid, mathematical models are necessary to valuate them. In the case of bonds or some kinds of interest rate derivatives, interest rate models are a way of performing this market-consistent valuation.

The first part of this thesis will deal with various types of such interest rate models. In this first part, the counterparties concluding the financial contracts are assumed to be default-free. However, in reality this is often not case. For example, it is possible that a corporation, who issued a corporate bond, goes bankrupt and is not able to meet its debts. For this reason, the second part of this thesis presents an approach to incorporate the possibility of default into interest rate models.

To give the reader an idea of what to expect, the contents of each chapter will now briefly be presented: Chapter 2 will give an overview of the basics of interest rate theory, including the definition of different types of interest rates and their derivatives, as well as pricing formulas for those derivatives.

Chapter 3 will briefly introduce the notion of affine term-structure models and then present the most famous models in that category. They are exclusively one-factor models, i.e., driven by a single source of randomness. Their dynamics and distributional properties will be discussed and analytical expressions for pricing bonds within those models will be derived. Most of those models feature considerable weaknesses, yet they are important to gain a first insight into the topic and are very interesting in a historical sense. However, they are not the main focus of this thesis and will therefore not be dealt with in great detail.
Chapter 4 will encompass an in-depth treatment of the G2++ model, a two-factor Gaussian model with a deterministic shift, which was proposed by Brigo & Mercurio in [5]. Next to stating the dynamics and the distributional behaviour, prices of zero-coupon bonds, caps and floors will be derived. The equivalence to the two-factor Hull-White Model will briefly be mentioned and methods of calibration to current market data will be described and performed. The motivation behind mainly focusing on the G2++ model is that it is used in Germany for simulation of market scenarios in order to divide tariffs of retirement provision into risk-reward categories[1].
Chapter 5 gives an overview of how to incorporate default risk. The focus will lie on the so-called intensity-based approach, where the evolution of the default probabilities is dependent on a stochastic intensity process. In the corresponding deterministic setting, this intensity matches the mortality rate in life-insurance mathematics. Under certain assumptions, it will be shown that the general pricing formulas are equal or similar to the ones in the default-free setting, with the difference that discounting occurs w.r.t. to the default-free interest rate plus an interest rate spread. The second section of this chapter will give an introduction to credit default swaps, since their rates on the market are an indicator for the creditworthiness of a company. Therefore, they can be used to calibrate the stochastic intensity to the market. One method of calibration will be presented in the last section of the

[1] The full description of the applied model can be found here (German only): `http://www.produktinformationsstelle.de/assets/PIA-Kapitalmarktmodell-Basisprozesse-2017.pdf`

chapter, along with a practical example.
Finally, the appendix will give a brief introduction to the existence and uniqueness of solutions of a special type of stochastic differential equations, namely time-homogeneous diffusion processes.

Chapter 2

Interest rate basics

This chapter deals with the basics of interest rate theory. Firstly, the notion of a bank-account, bonds and different types of interest rates will be presented within a stochastic framework. Secondly, the most important interest rate derivatives will be introduced along with Black's pricing formulas. Finally, some model independent pricing formulas will be presented and Black's formula for caps will be derived . Notation and structure of this chapter follow mainly [5], Chapter 1. Furthermore it is based on [12], Chapter 2 and [25], Chapter 16.

2.1 Some definitions

Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ be a filtered probability space, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration fulfilling the usual conditions[1] and the sigma-field $\mathcal{F}_t$ contains all information available up to time t.

[1] A filtration is said to fulfil the usual conditions if

- it is right-continuous, i.e., $\mathcal{F}_{t+} = \mathcal{F}_t$ hold for all t, where $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$
- and $\mathbb{P}$-complete, i.e., $\mathcal{F}_t$ contains all $\mathbb{P}$ null sets.

2.1.1 Bank account and zero-coupon bond

Definition 2.1.1 (Bank account process). *The process* $(B(t))_{t\geq 0}$ *given by*

$$B(t) := \exp\left(\int_0^t r_s ds\right), \quad B(0) = 1$$

is called bank account (process). The real-valued, $\mathcal{F}_t$*-adapted stochastic process* $(r(t))_{t\geq 0}$ *is called instantaneous spot rate, or more commonly* ***short rate****.* r_t *can be interpreted as the interest rate prevailing at time* t *for the infinitesimal interval* $[t, t + dt]$*.*

Remark: In former times, it was a desirable property for the interest rate to be positive, which posed a problem for models which relaxed this property in exchange for analytical tractability. However, small negative interest rates became a common sight on the market in certain instances, turning the disadvantage of those models into an advantage.[2]

Definition 2.1.2 (Stochastic discount factor). *The process*

$$D(t, T) := \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s ds\right)$$

is called stochastic discount factor. It can be interpreted as the money amount at time t*, which enables to pay a unit of money at time* T*.*

It is important to note that $D(t, T)$ is $\mathcal{F}_T$-measurable, but not $\mathcal{F}_t$-measurable, since r_t is stochastic. For the motivation of the discount factor one returns to the deterministic setup: If one wants to have a unit amount available at the bank account at time T, the initial investment (i.e. at $t = 0$) needs to be $B(T)^{-1}$, since $B(T)^{-1}B(T) = 1$. At time $t \in [0, T]$ the initial investment is

[2] For example, on July 21^{th}, 2017 the yields of German governmental bonds up to a time to maturity of 6.5 years were negative (See `https://www.bundesbank.de/Redaktion/DE/Downloads/Service/Bundeswertpapiere/Rendite/kurse_renditen_bundeswertpapiere_2017_07.pdf?__blob=publicationFile`) Another example can be found here `https://www.bundesbank.de/Redaktion/EN/Downloads/Statistics/Money_Capital_Markets/Interest_Rates_Yields/stat_geldmarkts.pdf?__blob=publicationFile`, where one can see, that the overnight-rate EONIA and all EURIBOR rates up to a year were also negative on July 21^{th}, 2017.

worth $B(T)^{-1}B(t) = D(t,T)$, i.e. the discount factor describes the value at time t of a unit of money available at time T.
On the market, where the time value of money is traded, the primary financial instruments are bonds. The most basic form of a bond, and also the most important one for developing an interest rate theory, is the zero-coupon bond.

Definition 2.1.3 (Zero-coupon bond)**.** *A zero-coupon bond is a contract concluded at the current time t, which guarantees its holder a unit amount of money at a fixed future date T, the so-called maturity of the bond, without any coupon-payments in $[t,T)$. The value of such a zero-coupon bond, also called T-bond, is denoted by $P(t,T)$.*

An immediate consequence of this definition is that $P(T,T) = 1\,\forall T$ holds. Since the price of a T-bond at time t expresses the value of one unit of money at maturity T, it can be seen as a discount factor. Indeed, if the short rate r is deterministic, then so is $D(t,T)$, thus implying $P(t,T) = D(t,T)$, whereas in the stochastic setting $P(t,T)$ is $\mathcal{F}_t$-measurable, but $D(t,T)$ is not.

2.1.2 Spot interest rates

By means of the bonds prices $P(t,T)$ one can define various types of spot interest rates. Spot interest rates are fixed rates which prevail at time $t < T$ till maturity T and depend on the way of compounding.

Definition 2.1.4 (Simply-compounded spot interest rate)**.** *The constant interest rate, which yields a unit of money at maturity T when investing an amount of $P(t,T)$ at time t and accruing proportionally to the time to maturity $T-t$, is called simply-compounded spot interest rate and denoted by $L(t,T)$. Thus, mathematically speaking*

$$L(t,T) := \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right). \tag{2.1.1}$$

This definition makes sense, since rearranging it leads to

$$P(t,T)(1 + (T-t)L(t,T)) = 1, \tag{2.1.2}$$

which fits exactly the situation described in the definition.

Definition 2.1.5 (Annually-compounded spot interest rate)**.** *The constant interest rate, which yields a unit of money at maturity T when investing an amount of $P(t,T)$ at time t and compounding annually, is called annually-compounded spot interest rate and denoted by $Y(t,T)$. Thus,*

$$Y(t,T) := \frac{1}{P(t,T)^{\frac{1}{T-t}}} - 1.$$

Once again, because of

$$P(t,T)(1+Y(t,T))^{(T-t)} = 1, \tag{2.1.3}$$

the definition is useful.
Compounding an arbitrary number of times within a year is also possible, leading to the following definition:

Definition 2.1.6 (k-times-per-year compounded spot interest rate)**.** *The k-times-per-year compounded spot interest rate defined by*

$$Y^k(t,T) := \frac{k}{P(t,T)^{\frac{1}{k(T-t)}}} - k,$$

is the constant interest rate (referred to a one-year period) which, when investing an amount of $P(t,T)$ at time t and reinvesting it k times a year, yields one unit of money at maturity T.

Analogously to the annual compounding case, rearranging the definition yields

$$P(t,T)\left(1+\frac{Y^k(t,T)}{k}\right)^{k(T-t)} = 1. \tag{2.1.4}$$

Definition 2.1.7 (Continuously-compounded spot interest rate)**.** *The constant interest rate, which yields a unit of money at maturity T when investing an amount of $P(t,T)$ at time t and compounding continuously, is called* *continuously-compounded spot interest rate and denoted by* $R(t,T)$*. Thus,*

$$R(t,T) := -\frac{\ln P(t,T)}{T-t}.$$

Also in this case,

$$e^{R(t,T)(T-t)}P(t,T) = 1, \tag{2.1.5}$$

shows the motivation behind the definition. The origin of the term continuously compounded lies in the fact that compounding "infinitely" often, i.e., letting k tend to infinity in $Y^k(t,T)$, yields the continuously compounded interest rate. The following calculation shows this:

$$\begin{aligned}
\lim_{k\to\infty} Y^k(t,T) &= \lim_{k\to\infty} \frac{k}{P(t,T)^{\frac{1}{k(T-t)}}} - k \\
&= \lim_{k\to\infty} \frac{k\left(1 - P(t,T)^{\frac{1}{k(T-t)}}\right)}{\underbrace{P(t,T)^{\frac{1}{k(T-t)}}}_{\longrightarrow 1}} \\
&= \lim_{k\to\infty} \frac{\left(1 - \exp\left(k^{-1}\ln\left(P(t,T)^{\frac{1}{(T-t)}}\right)\right)\right)}{k^{-1}} \\
&= \lim_{k\to\infty} \frac{\exp\left(k^{-1}\ln\left(P(t,T)^{\frac{1}{(T-t)}}\right)\right)\ln\left(P(t,T)^{\frac{1}{(T-t)}}\right)k^{-2}}{-k^{-2}} \\
&= -\frac{\ln(P(t,T))}{T-t} = R(t,T),
\end{aligned}$$

where L'Hôpital's rule was used. It would be interesting to visualize the interest rates implied by the T-bond prices as a function of the maturities $T \in [t, T^*]$, with T^* being the time horizon. The following definition does just that:

Definition 2.1.8 (Zero-coupon curve)**.** *For fixed current time t the graph of the function*

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t+1\,(years) \\ Y(t,T) & T \geq t+1\,(years) \end{cases}$$

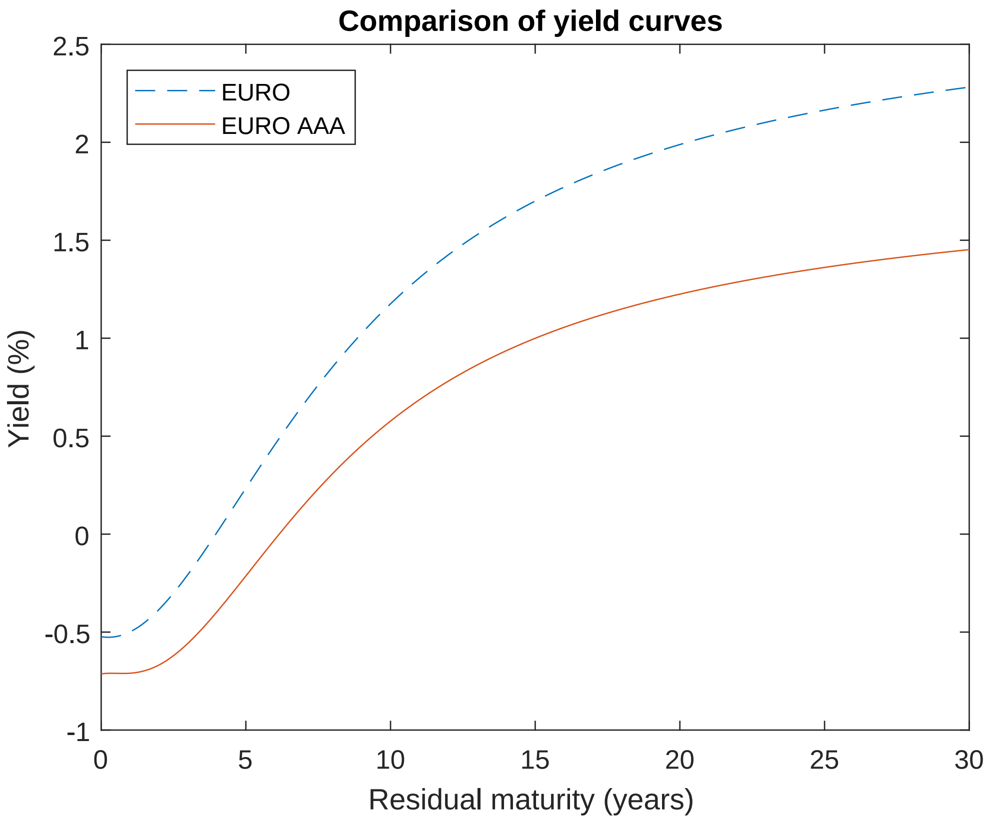


Figure 2.1: Comparison of euro zero-coupon yield curves estimated by the ECB on July 31st, 2017

is called zero-coupon curve, yield curve or term structure of interest rates.

I.e., for maturities less than a year simply-compounded interest rates are used, else annually-compounded ones. Figure 2.1 shows two zero-coupon yield curves estimated by the European Central Bank[3]: The dashed line is a yield curve average over all euro area central government bonds, the solid line only considers government bonds of euro countries rated AAA from the rating agency Fitch, which consisted of the Netherlands, Luxembourg and Germany at the time. As already mentioned above, negative interest rates are a very common sight nowadays. Figure 2.1 confirms this; AAA government bonds with maturities up to 6 years show negative yields.

[3]Source: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`

Remark: The notion of a risk-free rate is easily introduced in theory, yet needs some additional remarks in practice. A quick summary, based on [10] and [17], of current and former market practices for proxies of risk-free rates will now be stated: An intuitive proxy for the risk-free rate is the yield curve of high-rated government bonds. An example is the AAA-rated yield curve published daily by the ECB, cf. Figure 2.1 above. A flaw of this approach, especially in times of turmoil on the market, is that there might be rather large differences in the yield curves within the group of AAA rated countries, because potential downgrades might not yet have been executed by the rating agencies. Those downgrades can then have quite significant effects on the AAA curve. However, in the current market situation, which is very stable and where only 3 countries are rated AAA, the AAA yield curve is a very viable option.

In pre-crisis times, the market standard for risk-free rates were yield curves stripped from the market through quotes of derivatives with interbank offered rates (IBOR) as an underlying. In the euro area, the benchmark IBOR rate is the EURIBOR. With the start of the global financial crisis at the end of 2008, things changed. As banks grew more sceptical to lend money to other banks because of credit concerns, IBOR rates started to increase. Spreads between IBOR rates with different tenors arose, which made them unsuitable as a proxy for risk-free rates. This fact led to a different proxy for the risk-free rate, which has now become market standard. It is bootstrapped from the market with the aid of so-called OIS rates. OIS stands for Overnight Index Swap, which is an interest rate swap with an average of overnight rates as an underlying. In the euro area, the EONIA (Euro OverNight Index Average) is taken as an underlying for OIS. Although the EONIA is also an interbank rate, due to its overnight nature, the credit spreads incorporated in those rates are less than in EURIBOR rates, which have longer tenors. For more information why OIS-discounting is used in practice, see [17].

2.1.3 Forward interest rates

This section will deal with forward interest rates. A forward interest rate involves the current time t, and future times T, called expiry time, and S, called maturity, with $T \leq S$. It is the rate for which one agrees to lend/borrow at time t for the future time interval $[T, S]$. To guarantee absence of arbitrage, the forward interest rates have to be consistent with the current term structure of discount factors defined in the previous chapter, i.e., the current bond prices. The discount factors depend on the kind of spot rate used (simply-, annually-, continuously-compounded), therefore yielding also different forward rates. The simply-compounded forward rate will be considered first.

Definition 2.1.9 (Simply-compounded forward interest rate)**.** *The simply-compounded forward interest rate is the rate prevailing at time t for lending and borrowing within the time interval $[T, S]$ when using simple compounding and is defined by*

$$F(t, T, S) := \frac{1}{S - T} \left(\frac{P(t, T)}{P(t, S)} - 1 \right). \tag{2.1.6}$$

This definition can be derived by the assumption of absence of arbitrage. The value of an initial amount invested at time t for the time interval $[0, T]$ and reinvested at time T for the interval $[T, S]$ has to equal the value of the same initial amount invested at time t for the time interval $[0, S]$. Mathematically speaking, this means

$$\begin{aligned}
&(1 + (T - t)L(t, T))(1 + (S - T)F(t, T, S)) = (1 + (S - t)L(t, S)) \\
&\qquad\qquad \Leftrightarrow F(t, T, S) = \frac{1}{S - T} \left(\frac{1 + (S - t)L(t, S)}{1 + (T - t)L(t, T)} - 1 \right).
\end{aligned}$$

Rearranging (2.1.2) and plugging it into the above expression yields

$$F(t, T, S) = \frac{1}{S - T} \left(\frac{P(t, T)}{P(t, S)} - 1 \right), \tag{2.1.7}$$

which matches the definition.

Remark: A forward rate agreement (FRA) at time t is a contract including the following parameters: A nominal value N, a fixed rate K, a floating rate L, both simply-compounded, and future dates T, S with $T < S$. K is fixed at time t, whereas the floating rate is reset in T. The "payer" of a FRA agrees to pay an interest-rate payment on the nominal value for the time interval $[T, S]$ at time S based on the fixed rate K, whereas the "receiver" agrees in return to do the same for the floating rate. From the receivers viewpoint, the payoff thus equals $N(S-T)(K - L(T,S))$. (2.1.6) can also be derived from the fact that in order for the FRA to be a fair contract at time t, i.e. having value zero, the fixed rate K needs to equal $F(t, T, S)$, since arbitrage opportunities would arise otherwise. It is worth mentioning that since $L(T, S)$ is already known at time T, it is common practice that the interest rate payments are already exchanged at the reset date T by discounting the time S value of the payments by a discount factor agreed-upon before.

The definitions for the annually- and continuously compounded cases are as follows:

Definition 2.1.10 (Annually- and continuously-compounded forward interest rate)**.** *The annually- and continuously-compounded forward interest rates are defined by*

$$Y(t, T, S) := \sqrt[S-T]{\frac{P(t,T)}{P(t,S)}} - 1.$$

and

$$R(t, T, S) := \frac{1}{S-T}\left(\ln(P(t,T)) - \ln(P(t,S))\right).$$

respectively.

Both definitions can be derived in the same way as in the simple compounding case using (2.1.3) and (2.1.5) respectively. Clearly, $F(t, t, T) = L(t, T)$ holds, as well as the analogous statements in the annual and continuous compounding cases. If one lets the maturity tend to the expiry, a notion called the instantaneous forward rate arises, which can be seen as the short rate for a future time T prevailing at time t.

Definition 2.1.11 (Instantaneous forward interest rate)**.** *The interest rate defined by*

$$f(t,T) := \lim_{S \to T^+} F(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T}$$

is called instantaneous forward interest rate for maturity T. For the second equality to hold, it is assumed that $P(t,T)$ is sufficiently smooth.

The second equality in the definition is verified as follows:

$$\begin{aligned}
\lim_{S \to T^+} F(t,T,S) &= \lim_{S \to T^+} \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \\
&= - \lim_{S \to T^+} \frac{1}{P(t,S)}\left(\frac{P(t,S) - P(t,T)}{S-T}\right) \\
&= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{\partial \ln P(t,T)}{\partial T}.
\end{aligned}$$

Rearranging the definition yields

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

2.2 Interest rate derivatives

Interest rates can be used as an underlying for different kinds of derivatives. In this chapter, interest rate swaps and interest rate options are going to be presented.

2.2.1 Interest rate swaps

An interest rate swap (IRS) is a contract consisting of a series of payment exchanges between a fixed and a floating leg on a nominal value N

at prespecified points in time $\mathcal{T} := \{T_\alpha, \cdots, T_\beta\}$. At each *settlement date* $T_i, i \in \{\alpha+1, \cdots, \beta\}$, the so-called "payer" of the IRS pays an amount according to a fixed rate K and the "receiver" pays according to a floating rate L, determined at the corresponding *reset date* T_{i-1} for the i^{th} *accrual period* $[T_{i-1}, T_i]$. In other words, the payments $N\tau_i K$ and $N\tau_i L(T_{i-1}, T_i)$ are exchanged at time T_i, where $\tau_i := T_i - T_{i-1}$ and $\tau := \{\tau_{\alpha+1}, \cdots, \tau_\beta\}$. The IRS can therefore be seen as a portfolio of forward rate agreements. Consequently, the payoff at time T_i of a "Receiver IRS" (RFS) is given by

$$N\tau_i(K - L(T_{i-1}, T_i))$$

and therefore

$$\begin{aligned}
\mathbf{RFS}(t, \mathcal{T}, \tau, N, K) &= N \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i(K - F(t, T_{i-1}, T_i)) \\
&= N \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i \left(K - \frac{1}{\tau_i}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right)\right) \\
&= -NP(t, T_\alpha) + NP(t, T_\beta) + NK \sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i
\end{aligned}$$

is the value at time t. To not create arbitrage, the RFS needs to be fair, i.e., the value must equal zero. This gives rise to another definition:

Definition 2.2.1. *The particular interest rate for the fixed leg, which makes an IRS a fair contract at time t, i.e., the particular K for which* $\boldsymbol{RFS}(t, \mathcal{T}, \tau, N, K) = 0$*, is called forward swap rate* $S_{\alpha,\beta}(t)$ *and defined by*

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} P(t, T_i)\tau_i}.$$

An IRS can be used to lock in interest rates. Assume, for example, that someone pays interest on a loan according to a floating rate. By entering a payer IRS, one always pays interest according to the fixed rate, as the interest payments of the loan are met by the received floating leg payment.
IRS of above description are said to be settled *in arrears*, since the payment

exchanges occur at the settlement dates. In practice, IRS are also often settled *in advance*, i.e., reset and settlement dates coincide, and therefore the payments need to be discounted. The interest rate used to discount depends on the location of the market, with U.S. and European markets usually using the floating rate of the reset date $L(t, T_{i-1})$.

2.2.2 Interest rate options

In practice, the two most commonly used interest rate derivatives are both options, namely caps/floors and swap options (known as swaptions).

Caps and Floors: A **cap** is a contractual agreement, which involves the same parameters and payments as a payer IRS, with the difference that a payment exchange only occurs when the payoff is positive, i.e., if the floating rate at the settlement dates exceeds the previously agreed upon strike rate. The buyer's discounted payoff consequently equals

$$N \sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)^+.$$

Caps are used to protect oneself against rising interest rates when being floating rate indebted. Buying a cap leads to interest payments of $L(T_{i-1}, T_i) - (L(T_{i-1}, T_i) - K)^+ = min(L(T_{i-1}, T_i), K)$ at each settlement date T_i, so that the variable interest payments are "capped" by K.

A **floor** on the other hand is the equivalent to a cap for a receiver IRS. The seller agrees to make a payment if the floating rate at the settlement dates is below the strike rate. The buyer's discounted payoff at time t is therefore given by

$$N \sum_{i=\alpha+1}^{\beta} D(t, T_i)\tau_i(K - L(T_{i-1}, T_i))^+.$$

Analogously to the cap, the floor protects against decreasing interest rates. Caps/floors can be seen as a portfolio of $\beta - \alpha$ individual contracts, named **caplets/floorlets**, with discounted payoffs $D(t, T_i)\tau_i(L(T_{i-1}, T_i) - K)^+$ and

$D(t, T_i)\tau_i(K - L(T_{i-1}, T_i))^+$ respectively.
In practice, caps and floors are often priced by Black's Formula (for the derivation see Section 2.3).

$$\begin{aligned}
\mathbf{Cap}^{\mathbf{Black}}&(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&= N \sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, 1),^4 \qquad (2.2.1) \\
\mathbf{Flr}^{\mathbf{Black}}&(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&= N \sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, -1). \qquad (2.2.2)
\end{aligned}$$

Caplets and floorlets are just special cases of a cap/floor, where $\mathcal{T} = \{T, T + \tau^*\}$ and $\tau = \{\tau^*\}$, which means that the pricing formula in their cases reduces to

$$\begin{aligned}
\mathbf{Cpl}^{\mathbf{Black}}&(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&= NP(0, T + \tau^*)\tau^* Bl(K, F(0, T, T + \tau^*), \sigma_{\alpha,\beta}\sqrt{T}, 1), \qquad (2.2.3) \\
\mathbf{Fll}^{\mathbf{Black}}&(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&= NP(0, T + \tau^*)\tau^* Bl(K, F(0, T, T + \tau^*), \sigma_{\alpha,\beta}\sqrt{T}, -1).
\end{aligned}$$

In pre-crisis times, when interest rates were still strictly positive, the market commonly quoted caps/floors not by their price, but by their volatilities. Typically, the included volatilities related to $\alpha = 0$, $T_0 = 3M$ and $T_i = T_0 + i \cdot 3M$ for all i or, alternatively, 3-month spaced within the first year and $T_i = T_{i-1} + i \cdot 6M$ for reset dates exceeding one year. However, since negative interest rates for shorter maturities have appeared on the market in recent years, it is often not possible to use Black's formula. The reason for this is that certain forward rates also become negative, which is a contradiction to the Black-Scholes model, where the underlying is modelled as a geometric Brownian motion, which by definition is always strictly positive. Therefore, the caps/floors cannot be quoted by their volatilities in that case. Finally, a cap(let) is said to be *in the money* (ITM), if $K < S_{\alpha,\beta}(0)$, *at the money* (ATM) , if $K = S_{\alpha,\beta}(0)$ and *out of the money* (OTM), if $K > S_{\alpha,\beta}(0)$. In case of a floor(let), the inequalities are reversed.

Swaptions: A swap option, commonly known as swaption, entitles the buyer to enter an IRS at the swaption maturity. A payer (receiver) swaption contains the right to conclude a payer (receiver) IRS contract. Usually, the swaption maturity T_α and first reset date of the IRS coincide. Since a rational investor is only going to execute the swaption when the value of the underlying IRS at maturity (cf. section 2.2.1) is positive, the payoff of a (payer) swaption at $t \leq T_\alpha$ is given by

$$\begin{aligned} & ND(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(F(T_\alpha,T_{i-1},T_i)-K)\right)^+ \qquad (2.2.4) \\ &= ND(t,T_\alpha)\left(P(T_\alpha,T_\alpha)-P(T_\alpha,T_\beta)-K\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i\right)^+ \\ &= ND(t,T_\alpha)\left(\frac{P(T_\alpha,T_\alpha)-P(T_\alpha,T_\beta)}{\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i}-K\right)^+ \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i \\ &= ND(t,T_\alpha)\left(S_{\alpha,\beta}(T_\alpha)-K\right)^+ \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i \end{aligned}$$

[4] Black's Formula is given by

$$\begin{aligned} Bl(K,F,v,\omega) &= F\omega\Phi(\omega d_1(K,F,v)) - K\omega\Phi(\omega d_2(K,F,v)) \\ d_1(K,F,v) &= \frac{\ln\left(\frac{F}{K}\right)+\frac{v^2}{2}}{v} \\ d_2(K,F,v) &= \frac{\ln\left(\frac{F}{K}\right)-\frac{v^2}{2}}{v} \\ v_i &= \sigma_{\alpha,\beta}\sqrt{T_{i-1}}, \end{aligned}$$

where Φ denotes that CDF of the standard normal distribution and $\sigma_{\alpha,\beta}$ are quoted on the market.

Therefore, the time zero price of a payer swaption (PS) and receiver swaption (RS) according to Black's formula equal

$$\mathbf{PS}^{\mathbf{Black}}(0,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}) = NBl(K,S_{\alpha,\beta}(0),\sigma_{\alpha,\beta}\sqrt{T_\alpha},1)\sum_{i=\alpha+1}^{\beta}P(0,T_i)\tau_i,$$

$$\mathbf{RS}^{\mathbf{Black}}(0,\mathcal{T},\tau,N,K,\sigma_{\alpha,\beta}) = NBl(K,S_{\alpha,\beta}(0),\sigma_{\alpha,\beta}\sqrt{T_\alpha},-1)\sum_{i=\alpha+1}^{\beta}P(0,T_i)\tau_i.$$

Again, at time $t \leq T_\alpha$, a payer swaption is ITM, if $K < S_{\alpha,\beta}(t)$, ATM, if $K = S_{\alpha,\beta}(t)$ and OTM, if $K > S_{\alpha,\beta}(t)$. In case of a receiver swaption, the inequalities are reversed. As a final remark, one can mention that the price of a swaption is always lower than the price of the corresponding cap contract, since it is easily seen that

$$\left(\sum_{i=\alpha+1}^{\beta}P(T_\alpha,T_i)\tau_i(F(T_\alpha,T_{i-1},T_i)-K)\right)^+ \leq \sum_{i=\alpha+1}^{\beta}P(T_\alpha,T_i)\tau_i\left(F(T_\alpha,T_{i-1},T_i)-K\right)^+$$

holds.

2.3 No-arbitrage and pricing

2.3.1 Foundations

This section is based on [25, Chapter 12]. To get a deeper understanding in no-arbitrage theory and its consequences, the interested reader is referred to [25, Chapter 10] and [5, Chapter 2].

Let $(\Omega,\mathcal{A},\mathcal{F},\mathbb{P})$ again be a filtered probability space, where $\mathbb{P}$ is called the actual probability measure. In this case, $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ is the $\mathbb{P}$-completed version of the filtration generated by an underlying Brownian motion and T^* denotes some fixed end point in the considered time horizon. Let the short rate process $(r_t)_{t\in[0,T^*]}$ and the bank-account process $(B(t))_{t\in[0,T^*]}$ (cf. Definition 2.1.1) be adapted processes w.r.t. the same underlying probability space. Then the following notion of no-arbitrage can be defined.

Definition 2.3.1. *If for a family of processes $P(t,T)$, $t \leq T \leq T^*$ with $P(T,T) = 1$, $\forall T \leq T^*$, there exists a measure $\mathbb{P}^*$ equivalent* [5] *to $\mathbb{P}$, such that the relative bond price process*

$$Z^*(t,T) = \frac{P(t,T)}{B(t)}, \tag{2.3.1}$$

is a martingale under $\mathbb{P}^*$ for all $T \leq T^*$, then $P(t,T)$ is called an arbitrage-free family of bond prices w.r.t. $(r_t)_{t\in[0,T^*]}$. $\mathbb{P}^*$ is called a martingale measure for the family $P(t,T)$.

At first sight, it might not be clear where this definition comes from. It becomes clear when considering the following:

$$\begin{aligned} Z^*(t,T) &= \mathbb{E}_{\mathbb{P}^*}\left[Z^*(T,T)\,|\,\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}^*}\left[\frac{P(T,T)}{B(T)}\middle|\mathcal{F}_t\right] \\ \Leftrightarrow P(t,T) &= B(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{B(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}^*}\left[D(t,T)\,|\,\mathcal{F}_t\right]. \end{aligned}$$

This means that the bond price is the expectation of the discount factor under a certain equivalent measure, which is called *risk-neutral* or *equivalent martingale* measure. It will be used extensively later on to price bonds and derivatives. It can be shown that a market is arbitrage-free[6] if and only if there exists an equivalent martingale measure.
Let it now be assumed, that the short rate dynamics follow a one-dimensional diffusion process, i.e.,

$$dr_t = \mu_t dt + \sigma_t dW_t, \quad r_0 > 0, \tag{2.3.2}$$

under the actual probability measure $\mathbb{P}$. μ and σ are adapted processes and are implicitly assumed to fulfil the necessary conditions, such that a strong solution to the (2.3.2) exists[7]. W is the underlying Brownian motion and is

[5] Two probability measures are called equivalent if their respective sets of null sets coincide.

[6] Arbitrage can intuitively be described as the possibility of making riskless profit. For a mathematical definition see [25, Chapter 10.1] or [12, Chapter 4.3].

[7] See the Appendix of this thesis for details on those conditions.

here assumed to be one-dimensional. However, the following results are also valid if W is a d-dimensional Brownian motion and σ is a $\mathbb{R}^d$-valued adapted process. Using the natural filtration of the underlying Brownian motion has several advantages. Given a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, it can be shown that the Radon-Nikodym density equals

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \eta_{T^*}, \\ \eta_t := \mathcal{E}_t\left(\int_0^{\cdot} \lambda_u dW_u\right) &:= \exp\left(\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right), \end{aligned} \tag{2.3.3}$$

for some predictable, real-valued process λ. On the other hand, starting from an adapted real-valued process λ fulfilling the suitable integrability conditions (e.g. Novikov condition), (2.3.3) defines an equivalent measure $\mathbb{P}^\lambda$ and since the process η_t can be shown to be a martingale w.r.t. the natural filtration, Girsanov's theorem implies that

$$W_t^\lambda = W_t - \int_0^t \lambda_u du, \tag{2.3.4}$$

is a Brownian motion w.r.t. $\mathbb{P}^\lambda$. These observations lead to the following proposition. The formulation and the proof are based on [25, Chapter 10], with the proof being carried out in greater detail here.

Proposition 2.3.2. *Let the short-rate dynamics under $\mathbb{P}$ be as in* (2.3.2)*. Let the arbitrage-free family of bond prices $P(t,T)$ and the corresponding martingale measure $\mathbb{P}^*$ be as in Definition 2.3.1. Let λ be the predictable process fulfilling* (2.3.3) *for $\frac{d\mathbb{P}^*}{d\mathbb{P}}$, implying $\mathbb{P}^* = \mathbb{P}^\lambda$. Then:*

(i) The dynamics of r_t under $\mathbb{P}^\lambda$ are

$$dr_t = (\mu_t + \sigma_t\lambda_t)dt + \sigma_t dW_t^\lambda.$$

(ii) There exists a predictable, real-valued process $b^\lambda(t,T)$ which satisfies

$$dP(t,T) = P(t,T)(r_t dt + b^\lambda(t,T)dW_t^\lambda).$$

Proof. (i) immediately follows from plugging in (2.3.4) in (2.3.2).

(ii): Since $Z^*(t,T)$ is martingale under $\mathbb{P}^\lambda$, the conditional Bayes' rule implies that the process $M_t = Z^*(t,T)\eta_t$ is a martingale under $\mathbb{P}$ w.r.t. to $\mathcal{F}$. It can be shown (see [25, Appendix B.2]) that there exists a predictable process γ, such that

$$M_t = \mathbb{E}[M_t] + \int_0^t \gamma_u dW_u = \mathbb{E}[Z^*(t,T)\eta_t] + \int_0^t \gamma_u dW_u = Z^*(0,T) + \int_0^t \gamma_u dW_u,$$

which implies that M is continuous. Observe that

$$dZ^*(t,T) = d(M_t\eta_t^{-1}) = \eta_t^{-1}dM_t + M_t d(\eta_t^{-1}) + d[M_t, \eta_t^{-1}].$$

Calculating each summand separately using Itô's formula yields

$$\begin{aligned}
dM_t &= \gamma_t dW_t,\\
d\eta_t^{-1} &= \eta_t^{-1}\frac{1}{2}\lambda_t^2 dt - \eta_t^{-1} d\left(\int_0^t \lambda_u dW_u\right) + \frac{1}{2}\eta_t^{-1} d\underbrace{\left[\int_0^t \lambda_u dW_u\right]}_{\int_0^t \lambda_u^2 du}\\
&= \eta_t^{-1}\frac{1}{2}\lambda_t^2 dt - \eta_t^{-1}\lambda_t dW_t + \frac{1}{2}\eta_t^{-1}\lambda_t^2 dt\\
&= -\eta_t^{-1}\lambda_t(dW_t - \lambda_t dt) = -\eta_t^{-1}\lambda_t dW_t^\lambda,\\
d[M,\eta^{-1}]_t &= d\left[Z^*(0,T) + \int_0^t \gamma_u dW_u, \int_0^t -\eta_u^{-1}\lambda_u(dW_u - \lambda_u dt)\right]\\
&= d\left[\int_0^t \gamma_u dW_u, \int_0^t -\eta_u^{-1}\lambda_u dW_u\right]\\
&= d\left(-\int_0^t \gamma_u \eta_u^{-1}\lambda_u d\left[W,W\right]_u\right)\\
&= -\gamma_t\eta_t^{-1}\lambda_t dt.
\end{aligned}$$

Putting everything together leads to

$$\begin{aligned}
dZ^*(t,T) &= \eta_t^{-1}\gamma_t dW_t - M_t\eta_t^{-1}\lambda_t dW_t^\lambda - \gamma_t\eta_t^{-1}\lambda_t dt\\
&= \eta_t^{-1}\gamma_t(dW_t - \lambda_t dt) - M_t\eta_t^{-1}\lambda_t dW_t^\lambda\\
&= \eta_t^{-1}(\gamma_t - M_t\lambda_t) dW_t^\lambda.
\end{aligned}$$

Finally, the bond price dynamics are given by

$$\begin{aligned} dP(t,T) &= d(Z^*(t,T)B(t)) = Z^*(t,T)dB(t) + B(t)dZ^*(t,T) + d[\underbrace{Z^*(t,T), B(t)}_{=0}] \\ &= Z^*(t,T)B(t)r_t dt + B(t)n_t^{-1}(\gamma_t - M_t\lambda_t)dW_t^\lambda \\ &= P(t,T)r_t dt + P(t,T)(\frac{\gamma_t}{M_t} - \lambda_t)dW_t^\lambda \\ &= P(t,T)(r_t dt + (\frac{\gamma_t}{M_t} - \lambda_t)dW_t^\lambda) \\ &= P(t,T)(r_t dt + b^\lambda(t,T)dW_t^\lambda). \end{aligned}$$

Since γ, λ are predictable and M is continuous (and therefore in particular predictable), $b^\lambda(t,T)$ is also predictable, which concludes the proof. □

What do those results mean? Given a short rate process with dynamics as in (2.3.2), one can define a bond price by $P(t,T) = \mathbb{E}_{\mathbb{P}^\lambda}\left[D(t,T) \,\middle|\, \mathcal{F}_t^{W^\lambda}\right]$, where $\mathbb{P}^\lambda$ is an arbitrary measure equivalent to $\mathbb{P}$. A corollary from the previous proposition is, that bond price dynamics under $\mathbb{P}$ satisfy

$$dP(t,T) = P(t,T)((r_t - \lambda_t b^\lambda(t,T))dt + b^\lambda(t,T)dW_t^\lambda).$$

When examining this expression, it can be noticed that the drift term $r_t - \lambda_t b^\lambda(t,T)$ differs from the short term interest rate. The additional term is usually called market price of risk and originates from the argument, that since the bond is risky it should yield higher instantaneous return than a risk-free security, e.g., a savings account.

2.3.2 Pricing formulas

This section is based on [5, Chapter 2.6]. The setup consists again of a probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{Q})$, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration fulfilling the usual conditions and $\mathbb{Q}$ is the risk-neutral measure. It will be assumed that the time t price of any claim[8] with terminal payoff H_T is given by

$$\pi_t = \mathbb{E}\left[e^{-\int_t^T r_s ds} H_T \,\middle|\, \mathcal{F}_t\right]. \tag{2.3.5}$$

In particular, the price of a T-bond equals

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s ds} \,\middle|\, \mathcal{F}_t\right], \tag{2.3.6}$$

which will be a formula heavily used throughout this thesis. If the so-called T-forward measure $\mathbb{Q}^T$ is defined by the following Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T,T)B(0)}{P(0,T)B(T)} = \frac{e^{-\int_0^T r_s ds}}{P(0,T)} = \frac{D(0,T)}{P(0,T)}, \tag{2.3.7}$$

then it can be shown by the change-of-numeraire technique (see [5, Chapter 2.2]), that

$$\pi_t = P(t,T)\mathbb{E}^T\left[H_T \,|\, \mathcal{F}_t\right], \tag{2.3.8}$$

where $\mathbb{E}^T$ denotes the expectation under $\mathbb{Q}^T$. If one considers a European call option with strike K, maturity T and a S-bond as an underlying, its price at time $t \leq T \leq S \leq T^*$ obeys

$$\mathbf{ZBC}(t,T,S,K) = \mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right]. \tag{2.3.9}$$

The corresponding formula under $\mathbb{Q}^T$ reads

$$\mathbf{ZBC}(t,T,S,K) = P(t,T)\mathbb{E}^T\left[(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right], \tag{2.3.10}$$

which is very useful if the dynamics of the bond price under $\mathbb{Q}^T$ are known, as it will be the case later on. Extensions to arbitrary face values N of the

[8] A claim is any $\mathcal{F}_T$-measurable random variable H with $\mathbb{E}[H^2] < \infty$.

underlying S-bond are straightforward, namely

$$\begin{aligned}\mathbf{ZBC}(t,T,S,N,K) &= P(t,T)\mathbb{E}^T\left[(NP(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right]\\ &= NP(t,T)\mathbb{E}^T\left[(P(T,S)-\frac{K}{N})^+ \,\middle|\, \mathcal{F}_t\right] = N\mathbf{ZBC}(t,T,S,\frac{K}{N}).\end{aligned}$$

Formulas $\mathbf{ZBP}(t,T,S,N,K)$ for put options are defined analogously.
Consider a cap, where $\mathcal{T}=\{T_0,T_1,\cdots,T_n\}$ denotes the set of all relevant dates and $\tau=\{\tau_1,\cdots,\tau_n\}$ with $\tau_i=T_i-T_{i-1}$ are the involved time intervals. Let N be the cap nominal value and L the underlying simply compounded rate. Then, as already seen, the fair price at time $t\le T_{i-1}$ of the i-th caplet is given by

$$\begin{aligned}&\mathbf{Cpl}(t,T_{i-1},T_i,N,K)\\ &\quad= \mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}N\tau_i(L(T_{i-1},T_i)-K)^+ \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N\mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^{T_i} r_s ds}\tau_i(L(T_{i-1},T_i)-K)^+ \,\middle|\, \mathcal{F}_{T_{i-1}}\right] \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N\mathbb{E}\left[e^{-\int_t^{T_{i-1}} r_s ds}\tau_i(L(T_{i-1},T_i)-K)^+\mathbb{E}\left[e^{-\int_{T_{i-1}}^{T_i} r_s ds} \,\middle|\, \mathcal{F}_{T_{i-1}}\right] \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N\mathbb{E}\left[e^{-\int_t^{T_{i-1}} r_s ds}\tau_i(L(T_{i-1},T_i)-K)^+P(T_{i-1},T_i) \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

where the tower property of the conditional expectation was used. Plugging in (2.1.1), i.e. $L(t,T)=\frac{1}{T-t}\left(\frac{1}{P(t,T)}-1\right)$, yields

$$\begin{aligned}&\mathbf{Cpl}(t,T_{i-1},T_i,N,K)\\ &\quad= N\mathbb{E}\left[e^{-\int_t^{T_{i-1}} r_s ds}\left(\frac{1}{P(T_{i-1},T_i)}-1-K\tau_i\right)^+P(T_{i-1},T_i) \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N\mathbb{E}\left[e^{-\int_t^{T_{i-1}} r_s ds}\left(1-P(T_{i-1},T_i)(1+K\tau_i)\right)^+ \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N(1+K\tau_i)\mathbb{E}\left[e^{-\int_t^{T_{i-1}} r_s ds}\left(\frac{1}{(1+K\tau_i)}-P(T_{i-1},T_i)\right)^+ \,\middle|\, \mathcal{F}_t\right]\\ &\quad= N_i'\mathbf{ZBP}(t,T_{i-1},T_i,K_i') = \mathbf{ZBP}(t,T_{i-1},T_i,N_i',N), \qquad (2.3.11)\end{aligned}$$

where $N_i' = N(1 + K\tau_i)$ and $K_i' = \frac{1}{1+K\tau_i}$. The derivation of the formulas for floorlets is analogous and yields

$$\begin{aligned}\mathbf{Fll}(t, T_{i-1}, T_i, N, K) &= N_i'\mathbf{ZBC}(t, T_{i-1}, T_i, K_i') \\ &= \mathbf{ZBC}(t, T_{i-1}, T_i, N_i', N).\end{aligned} \tag{2.3.12}$$

Finally, cap and floor prices are just the sums of the underlying caplet/floorlet prices, i.e.,

$$\mathbf{Cap}(t, \mathcal{T}, \tau, N, K) = \sum_{i=1}^{n} N_i'\mathbf{ZBP}(t, T_{i-1}, T_i, K_i'), \tag{2.3.13}$$

$$\mathbf{Flr}(t, \mathcal{T}, \tau, N, K) = \sum_{i=1}^{n} N_i'\mathbf{ZBC}(t, T_{i-1}, T_i, K_i'). \tag{2.3.14}$$

2.3.3 Derivation of Black's formula

In this section Black's formula for prices of caplets will be derived. The case of floorlets is analogous. The derivation closely follows [5, Chapter 6.2]. Assume that the simply-compounded forward rate process $F(t, T_1, T_2)$ for future dates $T_1 < T_2$ follows, under the risk-neutral measure $\mathbb{Q}$, a driftless geometric Brownian motion with volatility $\sigma > 0$, i.e.,

$$dF(t, T_1, T_2) = \sigma F(t, T_1, T_2)dW_t,$$

where W_t as always denotes a Brownian motion. The well-known solution to this SDE is given by

$$F(t, T_1, T_2) = F(0, T_1, T_2) \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right), \tag{2.3.15}$$

which can be easily checked using Itô's formula. This means that $F(t, T_1, T_2)$ is a log-normally distributed random variable for every t. To obtain an explicit formula within this setup, the expectation

$$\mathbf{Cpl}(0, T_1, T_2, N, K) = \mathbb{E}\left[e^{-\int_0^{T_2} r_s ds}\tau N(L(T_1, T_2) - K)^+\right],$$

needs to be calculated ,where $\tau = T_2 - T_1$ and $F(T_1, T_1, T_2) = L(T_1, T_2)$. Since the discount factor within the expectation makes it impossible to compute this expectation directly, a change to the forward measure $\mathbb{Q}^{T_2}$ will be performed analogously to (2.3.7). It follows, as in (2.3.8), that

$$\mathbf{Cpl}(0, T_1, T_2, N, K) = P(0, T_2)\mathbb{E}^{\mathbb{Q}^{T_2}}\left[\tau N(L(T_1, T_2) - K)^+\right]$$

holds. The problem is that the dynamics of $F(t, T_1, T_2)$ under $\mathbb{Q}^{T_2}$ are not yet known. However, by definition

$$F(t, T_1, T_2) = \frac{1}{\tau}\left(\frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)}\right).$$

Consequently, the forward rate multiplied by $P(t, T_2)$ is a tradeable asset, because it is a multiple of the difference of two bonds. Therefore, $F(t, T_1, T_2)$ follows a martingale under $\mathbb{Q}^{T_2}$ (see [5], Prop. 2.5.1). It is well-known that diffusion processes are martingales if and only if they are driftless, since a non-zero drift contradicts the "fair game" property of martingales. Consequently, the dynamics of $F(t, T_1, T_2)$ equal (2.3.15), with the difference that the Brownian motion is now w.r.t. $\mathbb{Q}^{T_2}$. That in turns means, that $F(t, T_1, T_2)$ is also log-normally distributed under $\mathbb{Q}^{T_2}$ and therefore

$$\mathbb{E}^{\mathbb{Q}^{T_2}}\left[(L(T_1, T_2) - K)^+\right]$$

is the price of a call-option with strike K and maturity T_1, where the risk-free rate is zero and the volatility of the underlying is σ. It can therefore be computed by the usual Black-Scholes formula. Multiplication by $P(0, T_2)$, N and τ leads to (2.2.3).

Chapter 3

One-factor models

In this chapter, one-factor interest rate models will be presented, i.e., models driven by a single source of randomness. Let again $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{Q})$ be a filtered probability space, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration fulfilling the usual conditions and $\mathbb{Q}$ is the risk-neutral measure. For the moment, only time-homogeneous models are going to be considered, i.e., drift and diffusion coefficient are at most functions of the short rate and do not depend on time. The first two models, namely the Vasicek and the Cox-Ingersoll-Ross model (CIR), are in addition endogenous models, meaning that the current term structure is to be seen as an output rather than an input. Trying to fit the model to the initial market term structure, i.e., choosing parameters to match the model-implied term-structure curve $T \mapsto P(0,T)$ to the market curve $T \mapsto P^M(0,T)$, is not at all satisfactory. On the one hand, the number of parameters is too low, on the other hand, there are some term structure shapes (e.g. inverted), which cannot be reproduced by any choice of parameters. Despite their many drawbacks, those models are still presented as they do not only show the evolution of short-rate models through time and are therefore interesting in a historic sense, but also help to ease into this topic before more complicated models are considered, as they are very tractable analytically.

The last model which will be dealt with in this section, the CIR++ model, is an extension of the CIR model, which enables exact fitting of the current

market curve, while still preserving the analytical tractability of the other models.
In some of the proofs in this section, there will be references to proofs in the following Chapter 4 on two-factor-models, since there lies the main focus of this thesis.
The content and notation of this chapter are based on [5, Chapter 3] and [12, Chapter 5].

3.1 Affine-term-structure models

A model is called an affine-term-structure model (ATS), if the bond price admits the representation

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \quad \forall t \leq T,$$

where both A and B are suitably smooth functions. $P(T,T) = 1$ for all T implies $A(T,T) = 1$, $B(T,T) = 0$. The following proposition characterizes ATS models. The proposition and its proof closely follow [12, Chapter 5.3].

Proposition 3.1.1. *A short-rate model following, under the risk-neutral measure $\mathbb{Q}$, the dynamics*

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW(t),$$

is an ATS model if and only if

$$b(t,x) = \lambda(t)x + \eta(t), \quad \sigma^2(t,x) = \gamma(t)x + \delta(t) \tag{3.1.1}$$

with $\lambda, \eta, \gamma, \delta$ continuous, and A, B satisfy the system of ODEs

$$\begin{aligned} \partial_t B(t,T) + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 + 1 = 0, &\quad B(T,T) = 0 \\ \partial_t \ln A(t,T) - \eta(t)B(t,T) + \frac{1}{2}\delta(t)B(t,T)^2 = 0, &\quad A(T,T) = 1. \end{aligned} \tag{3.1.2}$$

for all $t \leq T$.

To proof this, a very famous result of stochastic calculus is needed, which is generally known as the Feynman-Kac formula. It makes a connection between partial differential equations and stochastic processes. The theorem and its proof follow the formulation of [12, Chapter 5.2]. Additional sources include [20, Chapter 4.4].

Theorem 3.1.2 (Feynman-Kac formula)**.** *Let the short-rate dynamics be as in Proposition 3.1.1. Let ϕ be continuous on a closed interval $I \subset \mathbb{R}$, which has a non-empty interior, and let $F(t,r) \in C^{1,2}((0,T) \times I)$ be a solution to*

$$\begin{aligned} \partial_t F(t,r) + b(t,r)\partial_r F(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_r^2 F(t,r) - rF(t,r) = 0, \\ F(T,r) = \phi(r). \end{aligned} \tag{3.1.3}$$

If $M(t) = F(t,r(t))e^{-\int_0^t r(u)du}$, $t \leq T$ is a martingale, then

$$F(t,r(t)) = \mathbb{E}\left[e^{-\int_t^T r(u)du}\phi(r(T)) \;\middle|\; F_t\right]$$

holds.

Proof. Using Itô's formula yields

$$\begin{aligned} dM(t) &= e^{-\int_0^t r(u)du} dF(t,r(t)) + F(t,r(t))d(e^{-\int_0^t r(u)du}) \\ &= e^{-\int_0^t r(u)du}\left(\partial_t F(t,r(t))dt + \partial_r F(t,r(t))dr(t) + \frac{1}{2}\partial_r^2 F(t,r(t)) \underbrace{d[r(t)]}_{\sigma^2(t,r(t))dt}\right) \\ &- F(t,r(t))e^{-\int_0^t r(u)du} r(t)dt \\ &= e^{-\int_0^t r(u)du}\left(\partial_t F(t,r(t)) + \partial_r F(t,r(t))b(t,r(t)) + \frac{1}{2}\partial_r^2 F(t,r(t))\sigma^2(t,r(t))\right. \\ &\left. - r(t)F(t,r(t))\right) dt + e^{-\int_0^t r(u)du}\partial_r F(t,r(t))\sigma^2(t,r(t))dW(t) \\ &= e^{-\int_0^t r(u)du}\partial_r F(t,r(t))\sigma^2(t,r(t))dW(t), \end{aligned}$$

where the last equality is due to $F(t,r(t))$ solving (3.1.3). This shows that $M(t)$ is always a local martingale, and under suitable conditions a true martingale[1] . Using this and the boundary condition of the partial differential equation leads to

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_0^T r(u)du}\phi(r(T))\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[M(T)\,|\,\mathcal{F}_t\right] = M(t) = F(t,r(t))e^{-\int_0^t r(u)du} \\
\Leftrightarrow\ &\mathbb{E}\left[-e^{\int_t^T r(u)du}\phi(r(T))\,\middle|\,\mathcal{F}_t\right] = F(t,r(t)).
\end{aligned}$$

□

This result is very powerful, because for a given claim with terminal payoff $\phi(r(T))$, the function $F(t,r(t))$ describes its price process. In particular, for a T-bond, i.e., $\phi \equiv 1$, the price can be represented by $P(t,T) = F(t,r(t),T)$. Solving the PDE is now one way of computing the price. However, a drawback is that a PDE has to be solved for every maturity T.
With the aid of the Feynman-Kac formula, Proposition 3.1.1 can be verified.

Proof Proposition 3.1.1. Since the bond price under the risk-neutral measure is given by

$$P(t,T) = \mathbb{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t] = A(t,T)e^{-B(t,T)r_t},$$

one can insert it into (3.1.3), which yields

$$\begin{aligned}
0 &= A(t,T)e^{-B(t,T)r}\partial_t(-B(t,T)r) - b(t,r)A(t,T)e^{-B(t,T)r}B(t,T) \\
&\quad + \frac{1}{2}\sigma^2(t,r)A(t,T)e^{-B(t,T)r}B(t,T)^2 - rA(t,T)e^{-B(t,T)r} + e^{-B(t,T)r}\partial_t A(t,T) \\
\Leftrightarrow 0 &= \partial_t A(t,T) + A(t,T)\partial_t(-B(t,T)r) - b(t,r)A(t,T)B(t,T) \\
&\quad + \frac{1}{2}\sigma^2(t,r)A(t,T)B(t,T)^2 - rA(t,T),
\end{aligned}$$

[1] Sufficient conditions for a local martingale to be a true martingale are:

- M is uniformly bounded
- $\mathbb{E}[\int_0^T |\partial_r F(t,r(t))e^{-\int_0^t r(u)du}\sigma(t,r(t))|^2 dt] < \infty$

which means an ATS is also characterized by

$$-\partial_t \ln A(t,T) + r(\partial_t B(t,T) + 1) = \frac{1}{2}\sigma^2(t,r)B(t,T)^2 - b(t,r)B(t,T). \tag{3.1.4}$$

Inserting equalities (3.1.1) and (3.1.2) in the above expressions yields a true statement, which proves the "if" part of the proposition. For the "only if" part, first assume, for fixed t, linear independence of $B(t,\cdot)$ and $B(t,\cdot)^2$. Then the matrix

$$M = \begin{pmatrix} B(t,T_1)^2 & -B(t,T_1) \\ B(t,T_2)^2 & -B(t,T_2) \end{pmatrix}$$

is invertible for some $T1 > T2 > t$. Rearranging (3.1.4) gives

$$\begin{pmatrix} \frac{1}{2}\sigma^2(t,r) \\ b(t,r) \end{pmatrix} = M^{-1}\left(- \begin{pmatrix} \partial_t \ln A(t,T_1) \\ \partial_t \ln A(t,T_2) \end{pmatrix} + \begin{pmatrix} \partial_t B(t,T_1) + 1 \\ \partial_t B(t,T_2) + 1 \end{pmatrix} r \right),$$

which shows that that drift and volatility are affine functions in r and can consequently be represented as in (3.1.1). Inserting this representation into (3.1.4) yields

$$\begin{aligned} \frac{1}{2}\delta(t)B(t,T)^2 - \eta(t)B(t,T) + \left(\frac{1}{2}\gamma(t)B(t,T)^2 - \lambda(t)B(t,T)\right) r \\ = -\partial_t \ln A(t,T) + r(\partial_t B(t,T) + 1). \end{aligned}$$

Comparing expressions containing r and expressions without r leads to (3.1.2). In the case of linear dependence, $B(t,\cdot) = c(t)B(t,\cdot)^2$ holds for some constant $c(t)$. For that relation to be true, $c(t) = B(t,\cdot)^{-1}$ would have to hold for all $T \geq t$. Since $B(t,t) = 0$, this is not possible, implying that $c(t) = 0$ and therefore $B(t,\cdot) \equiv 0$, which in turn means $\partial_t B(t,T) = -1$. Consequently, the union of all t, for which linear independence holds, constitutes an open and dense set in $\mathbb{R}$ and since a continuity assumption for $\lambda, \eta, \gamma, \delta$ was made, (3.1.1) and therefore (3.1.2) hold for all t, which concludes the proof. □

All models considered in this chapter are ATS models, even such that the Ricatti differential equation in (3.1.2) produces an analytical solution.

Another useful property of ATS models, which will be needed later on, is that the instantaneous forward rate can be written in a special form:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial \ln A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T} r(t). \tag{3.1.5}$$

3.2 The Vasicek Model

One of the first and most famous interest rate models is the Vasicek model (1977). Its instantaneous short-rate dynamics under the risk-neutral measure are given by

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t), \qquad r(0) = r_0, \tag{3.2.1}$$

with $r_0, k, \theta, \sigma > 0$. This means, that the short rate is modelled as an Ornstein-Uhlenbeck process with constant coefficients. Whenever $r(t) < \theta$, the drift becomes negative and whenever $r(t) > \theta$, it is positive, therefore $r(t)$ is always drifting towards θ. Hence, such a process is called mean-reverting. k is called the mean-reversion rate, i.e., the larger k, the faster the process will drift to θ.

Simple integration of (3.2.1), almost analogous to the proof of Proposition 4.1.1, yields for every $s \leq t$:

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)} dW(u).$$

Before distributional properties are going to be analysed, an important lemma will be stated, which will be used extensively throughout the thesis. The lemma is taken from [27, Chapter 10].

Lemma 3.2.1 (without proof)**.** *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ satisfy $\int_0^t f(s)^2 ds < \infty$ for all $t \geq 0$. Then the process $\{X_t := \int_0^t f(s) dW(s)\}_{t\geq 0}$ has continuous sample paths, independent increments and X_t is normally distributed with*

$$\mathbb{E}[X_t] = 0, \qquad Cov(X_t, X_s) = \int_0^{min(t,s)} f(u)^2 du.$$

Lemma 3.2.1 implies that $r(t)$ conditional on $\mathcal{F}_s$, the sigma-field generated by r up to time s, is normally distributed with

$$\begin{aligned}\mathbb{E}[r_t|\mathcal{F}_s] &= r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right),\\ Var[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}].\end{aligned}$$

Letting t tend to infinity in the above expression shows that θ is indeed the long-term mean. The normal distribution property of $r(t)$ also means, that there is positive probability for negative interest rates. This was considered one of the major drawbacks of the Vasicek model. However, as already mentioned, negative interest rates are not uncommon in the market anymore. The T-bond price in the Vasicek model is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$\begin{aligned}A(t,T) &= \exp\left(\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right)\\ B(t,T) &= \frac{1}{k}\left[1 - e^{-k(T-t)}\right].\end{aligned}$$

are the solutions of (3.1.2).

3.3 The Cox-Ingersoll-Ross model

The need to solve the (former) problem of a possibly negative interest rate led to the Cox-Ingersoll-Ross model (1985), which emerged from the Vasicek model by adding a multiplicative square-root factor to the diffusion coefficient, i.e., the short-rate evolves according to

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0,$$

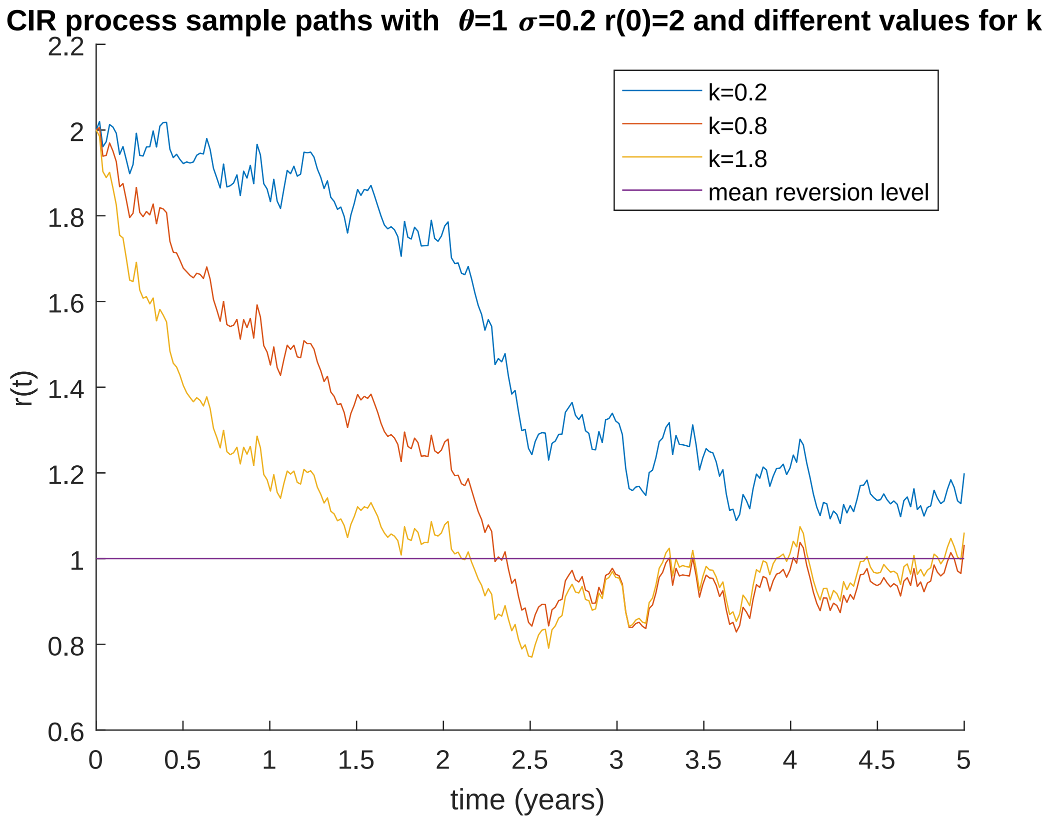


Figure 3.1: Sample paths of the interest rate in the CIR model with different mean-reversion rate

with $r_0, k, \theta, \sigma > 0$. The process is non-negative for arbitrary positive parameters and stays strictly positive for $2k\theta > \sigma^2$ (known as Feller-Condition). Existence and positivity of the solution to this particular stochastic differential equation will be proven in the Appendix.
Sample paths of a CIR process, generated in a distributionally exact way, are shown in Figure 3.1. One can clearly observe the effects of different mean-reversion rates, where higher values cause the process to approach the mean-reversion level faster.
Again, simple integration yields, for every $s \leq t$:

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)}\sqrt{r(u)}dW(u).$$

Using the expression and Lemma 3.2.1 above enables to calculate expected value and variance of $r(t)$ conditional on $\mathcal{F}_s$, which are given by

$$\begin{aligned}\mathbb{E}[r_t|\mathcal{F}_s] &= r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right),\\ Var[r_t|\mathcal{F}_s] &= r(s)\frac{\sigma^2}{k}(e^{-k(t-s)} - e^{-2k(t-s)}) + \theta\frac{\sigma^2}{2k}(1 - e^{-k(t-s)})^2.\end{aligned}$$

As far as the distributional properties of $r(t)$ are concerned, according to [8] it follows a non-central chi-squared distribution , i.e., the density function f is given by

$$f_{r(t)}(x) = c_t f_{\chi^2(v,\lambda_t)}(c_t x)^2 \tag{3.3.1}$$

where

$$\begin{aligned}c_t &= \frac{4k}{\sigma^2(1 - \exp(-kt))},\\ v &= \frac{4k\theta}{\sigma^2},\\ \lambda_t &= c_t r_0 \exp(-kt).\end{aligned}$$

The price of a T-bond in the CIR model again admits to the representation

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

[2] The density of a non-central chi-squared distribution with v degrees of freedom and non-centrality parameter λ and the chi-squared distribution with $v+2i$ degrees of freedom are respectively given by

$$\begin{aligned}f_{\chi^2(v,\lambda)}(z) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!} f_{\chi^2(v+2i)}(z),\\ f_{\chi^2(v+2i)}(z) &= \frac{\left(\frac{1}{2}\right)^{i+v/2}}{\Gamma(i+v/2)} z^{i-1+v/2}e^{-z/2}.\end{aligned}$$

where

$$\begin{aligned} A(t,T) &= \left[\frac{2h \exp\left(\frac{1}{2}(k+h)(T-t)\right)}{2h + (k+h)(\exp\{(T-t)h\} - 1)} \right]^{\frac{2k\theta}{\sigma^2}}, \\ B(t,T) &= \frac{2\left(\exp\{(T-t)h\} - 1\right)}{2h + (k+h)(\exp\{(T-t)h\} - 1)}, \end{aligned} \tag{3.3.2}$$

are the solutions of (3.1.2) and $h = \sqrt{k^2 + 2\sigma^2}$.

3.4 The Hull-White Model

The desire to be able to fit a model to the current yield curve led to an extension proposed by Hull and White in [16] in 1990, which in its general form follows, under the risk-neutral measure, the dynamics:

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)dW(t),$$

where ϑ, a, σ are deterministic functions. Since in the general setting analytical tractability is lost, the short rate process considered here evolves according to

$$dr(t) = [\vartheta(t) - ar(t)]dt + \sigma dW(t), \tag{3.4.1}$$

where $a, \sigma > 0$ are constants. ϑ is used to fit the model to the currently observed term structure of discount factors $P^M(0,T)$. Since the Hull-White model is an ATS model, by using (3.1.2) and (3.1.5) it is easily shown that

$$\vartheta(t) = \frac{\partial}{\partial T} f^M(0,T)|_{T=t} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

must hold, assuming $f^M(0,T) = -\frac{\partial \ln P(0,T)}{\partial T}$ is sufficiently smooth. For details see [12, Chapter 5.4.5]. Simple integration of (3.4.1), almost analogous to the proof of Proposition 4.1.1, and inserting the above expression for ϑ yields for every $s \leq t$:

$$
\begin{aligned}
r(t) &= r(s)e^{-a(t-s)} + \int_s^t \vartheta(u)e^{-a(t-u)}du + \sigma \int_s^t e^{-a(t-u)}dW(u) \\
&= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u),
\end{aligned}
$$

where $\alpha(t) := f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2$. Like in the Vasicek and CIR model, this representation can be used to derive the distributional properties of the short rate process. Here, $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with

$$
\begin{aligned}
\mathbb{E}[r_t|\mathcal{F}_s] &= r(s)e^{-a(t-s)} + \alpha(t) + \alpha(s)e^{-a(t-s)}, \\
Var[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2a}[1-e^{-2a(t-s)}].
\end{aligned}
$$

T-bonds can be priced with the formula

$$
P(t,T) = \mathbb{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t].
$$

To do this, let us first introduce the process $dx(t) = -ax(t)dt + \sigma dW(t)$, $x(0) = 0$. It is easy to show that $r(t) = x(t) + \alpha(t)$ holds for all t.

Lemma 3.4.1. *$\int_t^T r(u)du$ conditional on $\mathcal{F}_t$ is normally distributed with*

$$
\begin{aligned}
\mathbb{E}\left[\int_t^T r(u)du \,\middle|\, \mathcal{F}_t\right] =& B(t,T)[r(t)-\alpha(t)] + \ln\left[\frac{P^M(0,t)}{P^M(0,T)}\right] \\
&+ \frac{1}{2}[V(0,T) - V(0,t)] \qquad (3.4.2) \\
Var\left[\int_t^T r(u)du \,\middle|\, \mathcal{F}_t\right] =& V(t,T), \qquad (3.4.3)
\end{aligned}
$$

where

$$
\begin{aligned}
B(t,T) &= \frac{1}{a}[1-e^{-a(T-t)}] \\
V(t,T) &= \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right].
\end{aligned}
$$

Proof. The same computation as in the proof of Lemma 4.2.1 leads to

$$\int_t^T x(u)du = x(t)\frac{1-e^{-a(T-t)}}{a} + \frac{\sigma}{a}\int_t^T \left(1-e^{-a(T-u)}\right) dW(u).$$

Lemma 3.2.1 implies that $\int_t^T x(u)du$ is normally distributed with mean $B(t,T)[r(t)-\alpha(t)]$. Since $\alpha(t)$ is deterministic, $\int_t^T r(u)du$ is also normally distributed. Further,

$$\begin{aligned}
\int_t^T \alpha(u)du &= \int_t^T f^M(0,u) + \frac{\sigma^2}{2a^2}(1-e^{-au})^2 du \\
&= -\ln P^M(0,u)|_t^T + \frac{\sigma^2}{2a^2}\int_t^T (1-2e^{-au}+e^{-2au})du \\
&= \ln\left(\frac{P^M(0,t)}{P^M(0,T)}\right) + \frac{\sigma^2}{2a^2}\left[T-t+\left(\frac{2}{a}e^{-au}-\frac{1}{2a}e^{-2au}\right)\Bigg|_t^T\right] \\
&= \ln\left(\frac{P^M(0,t)}{P^M(0,T)}\right) + \frac{1}{2}[V(0,T)-V(0,t)].
\end{aligned}$$

Together, this yields

$$\begin{aligned}
\mathbb{E}\left[\int_t^T r(u)du \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T x(u)du \,\middle|\, \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T \alpha(u)du \,\middle|\, \mathcal{F}_t\right] \\
&= B(t,T)[r(t)-\alpha(t)] + \ln\left[\frac{P^M(0,t)}{P^M(0,T)}\right] + \frac{1}{2}[V(0,T)-V(0,t)]
\end{aligned}$$

Using Lemma 3.2.1 and the results of Lemma 4.2.1 once again yields

$$\begin{aligned}
Var\left[\int_t^T r(u)du \,\middle|\, \mathcal{F}_t\right] &= Var\left[\int_t^T x(u)du \,\middle|\, \mathcal{F}_t\right] \\
&= Var\left[\frac{\sigma}{a}\int_t^T \left(1-e^{-a(T-u)}\right) dW(u) \,\middle|\, \mathcal{F}_t\right] \\
&= \frac{\sigma^2}{a^2}\int_t^T \left(1-e^{-a(T-u)}\right)^2 du = V(t,T),
\end{aligned}$$

which concludes the proof. □

An analogous calculation to the one in the proof of Theorem 4.2.2 or solving

(3.1.2) leads to

$$
\begin{aligned}
P(t,T) &= A(t,T)e^{-B(t,T)r(t)} \\
A(t,T) &= \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right\},
\end{aligned}
$$

where $B(t,T)$ is as in Lemma 3.4.1. For the sake of completeness, it should be mentioned that this model allows again for negative interest rates.

3.5 CIR++ - the extended Cox-Ingersoll-Ross model

As already mentioned, one of the major drawbacks of the Vasicek and the CIR model is that the current term structure is endogenously given. The CIR++ model avoids that problem by adding a deterministic shift, i.e the short rate dynamics under the risk neutral measure $\mathbb{Q}$ are given by

$$
\begin{aligned}
r(t) &= x(t) + \varphi(t), \\
dx(t) &= k(\theta - x(t))dt + \sigma\sqrt{x(t)}dW(t), \qquad x(0) = x_0,
\end{aligned}
$$

with $x_0, k, \theta, \sigma > 0$.
Before one can compute the bond price within the CIR++ model, more general results need to be stated. Firstly, for the bond price $P(t,T)$ of a process given by $r(t) = x(t) + \varphi(t)$, the following holds:

$$
P(t,T) = \exp\left[-\int_t^T \varphi(s)ds\right] P^x(t,T,x(t)),
$$

where $P^x(t,T)$ is the bond price of a process which is governed by the same dynamics under a measure $\mathbb{Q}^x$ as $x(t)$ under $\mathbb{Q}$. Furthermore, it can be shown that the following statements are equivalent:

(i) The currently observed bond curve is fitted by the CIR++ model, i.e., $P^M(0,T) = P(0,T) \qquad \forall T \in [0,T^*]$.

(ii) $\varphi(t) = f^M(0,t) - f^x(0,t)$, where $f^M(t,T)$ and $f^x(t,T)$ are the instantaneous forward rates corresponding to the bond prices $P^M(t,T)$ and $P^x(t,T,x(t))$ respectively.

(iii) $\exp\left[-\int_t^T \varphi(s)ds\right] = \dfrac{P^M(0,T)}{P^M(0,t)} \dfrac{P^x(0,t,x(0))}{P^x(0,T,x(0))}$

For the proofs of the above statements see [5, Chapter 3.8].

Translated to our case, and assuming that the current market term structure is fitted, this leads to

$$\varphi(t) = \varphi^{CIR}(t) = f^M(0,t) - f^{CIR}(0,t),$$

where, because of (3.3.2) and (3.1.5),

$$f^{CIR}(0,t) = \frac{2k\theta(e^{th}-1)}{2h+(k+h)(e^{th}-1)} + x_0 \frac{4h^2 e^{th}}{(2h+(k+h)(e^{th}-1))^2}, \tag{3.5.1}$$

with $h = \sqrt{k^2 + 2\sigma^2}$. The bond price is thus given by

$$\begin{aligned} P(t,T) &= \bar{A}(t,T)e^{-B(t,T)r(t)}, \\ \bar{A}(t,T) &= \frac{P^M(0,T)A(0,t)\exp(-B(0,t)x_0)}{P^M(0,t)A(0,T)\exp(-B(0,T)x_0)} A(t,T)e^{B(t,T)\varphi^{CIR}}, \end{aligned} \tag{3.5.2}$$

where $B(t,T)$ and $A(t,T)$ are as in the standard CIR model. This also implies that the CIR++ is also an ATS model.

In all models presented in this chapter, caps/floors and swaptions have analytical pricing formulas. The interested reader is referred to [5, Chapter 2 and 3].

Chapter 4

G2++ - The two-factor Gaussian model

This chapter will introduce the G2++ model and carry out an in-depth analysis of its properties. It can be seen as a summary of [5, Chapter 4], where it was first proposed in this manner by Damiano Brigo & Fabio Mercurio. Therefore, the results and most of the proofs in this chapter will closely follow their approach.
As already mentioned at the beginning of this thesis, the motivation for choosing to put special emphasis on the G2++ model lies in the fact that it used in Germany for simulation of market scenarios in order to divide tariffs of retirement provision into risk-reward categories. The motivation for using two-factor models instead of one-factor models in general can be illustrated by the following example. Consider an arbitrary one factor model presented in the previous chapter. All of them were so-called ATS models, and therefore the continuously compounded spot rate prevailing at time t could be expressed as follows:

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t} r(t) =: a(t,T) + b(t,T) r(t).$$

Consider those rates at time t for two different maturities T_1 and T_2. Then,

$$
\begin{aligned}
&Corr(R(t,T_1), R(t,T_2)) \\
&\qquad = \frac{Cov(a(t,T_1) + b(t,T_1)r(t), a(t,T_2) + b(t,T_2)r(t))}{\sqrt{Var(a(t,T_1) + b(t,T_1)r(t))\, Var(a(t,T_2) + b(t,T_2)r(t))}} \\
&\qquad = \frac{b(t,T_1)b(t,T_2)Var(r(t))}{\sqrt{b(t,T_1)^2 Var(r(t)) b(t,T_2)^2 Var(r(t))}} = 1.
\end{aligned}
$$

This means, that the interest rates for all maturities are perfectly correlated. Consequently, if there is a shock to the initial point of the term-structure, the whole curve will be shifted in the same manner and direction. This is unrealistic, since interest rates usually feature non-perfect correlation. As long as a financial product only depends on the interest rate for a single maturity, this is not a problem. However, as soon as a payoff depends on two or more rates with different maturities, one-factor models are not able to reproduce a realistic correlation structure.

In the G2++ model, the rates can be expressed as an affine transformation of the two processes x and y, whose driving Brownian motions have instantaneous correlation ρ, i.e., $dW_1(t)W_2(t) = \rho dt$ and whose dynamics will be introduced shortly. The correlation between two rates with different maturities can then be shown to equal

$$
\begin{aligned}
&Corr(R(t,T_1), R(t,T_2)) \\
&\qquad = Corr\left(b^x(t,T_1)x(t) + b^y(t,T_1)y(t), b^x(t,T_2)x(t) + b^y(t,T_2)y(t)\right),
\end{aligned}
$$

which is in general not equal to 1 and depends on the correlation of x and y and therefore in part also on ρ. This adds a level of flexibility to the correlation structure of the model. As will be shown later, $\rho < 0$ allows for a humped shape in the volatility structure of the instantaneous forward rates, which according to Brigo & Mercurio is a desired feature in an interest rate model. This will be discussed in more detail in the last section of this chapter.

4.1 General setting and properties

Let $(r(t))_{t\geq 0}$ be the instantaneous short rate process. Let $(x(t))_{t\geq 0}$ and $(y(t))_{t\geq 0}$ also be stochastic processes and φ a real-valued deterministic function in time. The dynamics of $r(t)$ in the G2++ model under the risk-neutral measure $\mathbb{Q}$ are given by

$$\begin{aligned} r(t) &= x(t) + y(t) + \varphi(t), \quad r(0) = r_0 \\ dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0 \\ dy(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0, \end{aligned} \tag{4.1.1}$$

where $r_0, a, b, \sigma, \eta \in \mathbb{R}^+$ and (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ, i.e., $dW_1 dW_2 = \rho dt$, with $\rho \in [-1, 1]$. As an immediate consequence of this definition $\varphi(0) = r_0$ holds.

Proposition 4.1.1. *For all $s < t$, the following holds*

$$\begin{aligned} r(t) =& x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} \\ &+ \sigma \int_s^t e^{-a(t-u)} dW_1(u) + \eta \int_s^t e^{-b(t-u)} dW_2(u) + \varphi(t). \end{aligned} \tag{4.1.2}$$

In particular, for $s = 0$,

$$r(t) = \sigma \int_0^t e^{-a(t-u)} dW_1(u) + \eta \int_0^t e^{-b(t-u)} dW_2(u) + \varphi(t).$$

Proof. Consider the process $\left\{ e^{a(t-s)} x(t),\, t > s \right\}$. Using the known rules of stochastic integration,

$$\begin{aligned} d(e^{a(t-s)} x(t)) &= e^{a(t-s)} dx(t) + a e^{a(t-s)} x(t) dt \\ &= -e^{a(t-s)} a x(t) dt + e^{a(t-s)} \sigma dW_1(t) + a e^{a(t-s)} x(t) dt \\ &= e^{a(t-s)} \sigma dW_1(t). \end{aligned}$$

Writing the above expression in integral form yields

$$\begin{aligned} e^{a(t-s)}x(t) &= x(s) + \sigma \int_s^t e^{a(u-s)} dW_1(u) \\ \Leftrightarrow x(t) &= x(s)e^{-a(t-s)} + \sigma \int_s^t e^{a(u-s)} e^{-a(t-s)} dW_1(u) \\ &= x(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_1(u). \end{aligned}$$

Performing the analogous steps on $\left\{ e^{b(t-s)} y(t),\, t > s \right\}$ yields

$$y(t) = y(s)e^{-b(t-s)} + \eta \int_s^t e^{-b(t-u)} dW_2(u).$$

Combining both completes the proof. □

Let $\mathcal{F}_t$ be the sigma-field generated by the process (x, y) up to time t, then the following corollary can be stated.

Corollary 4.1.2. *The process* $r(t)$ *conditional on* $\mathcal{F}_s$ *is normally distributed, with mean and variance as follows:*

$$\mathbb{E}[r_t|\mathcal{F}_s] = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \varphi(t) \tag{4.1.3}$$

$$Var[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}] + \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}] + 2\rho\frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}]. \tag{4.1.4}$$

Proof. Because of Lemma 3.2.1, only the non-random terms in (4.1.2) do not vanish, which yields (4.1.3). Using the Itô isometry and the well-known identity $Var(X) = E(X^2) - E(X)^2$, which in our case reduces to $Var(X) = E(X^2)$, gives

$$\begin{aligned} Var[\sigma \int_s^t e^{-a(t-u)} dW_1(u)|\mathcal{F}_s] &= E[\sigma^2 \int_s^t e^{-2a(t-u)} du|\mathcal{F}_s] \\ &= \sigma^2 \int_s^t e^{-2a(t-u)} du = \sigma^2 \frac{1}{2a} e^{-2a(t-u)}\Big|_s^t \\ &= \sigma^2 \frac{1}{2a}[1 - e^{-2a(t-s)}]. \end{aligned}$$

Analogously,

$$Var[\eta \int_s^t e^{-b(t-u)} dW_2(u)|\mathcal{F}_s] = \eta^2 \frac{1}{2b}[1 - e^{-2b(t-s)}].$$

Furthermore,

$$\begin{aligned}
Cov[\sigma \int_s^t e^{-a(t-u)} dW_1(u), &\eta \int_s^t e^{-b(t-u)} dW_2(u)|\mathcal{F}_s] = \\
&= E[\sigma \int_s^t e^{-a(t-u)} dW_1(u) \cdot \eta \int_s^t e^{-b(t-u)} dW_2(u)|\mathcal{F}_s] \\
&= E[\sigma\eta \int_s^t e^{-a(t-u)} e^{-b(t-u)} dW_1 dW_2|\mathcal{F}_s] \\
&= E[\sigma\eta \int_s^t e^{-a(t-u)} e^{-b(t-u)} \rho dt|\mathcal{F}_s] \\
&= \rho\sigma\eta \int_s^t e^{-(a+b)(t-u)} dt \\
&= \rho\sigma\eta e^{-(a+b)(t-u)}\Big|_s^t \\
&= \rho\frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}].
\end{aligned}$$

Using the well-known formula $Var(X+Y) = Var(X)+Var(Y)+2Cov(X,Y)$ and the fact that non-random expressions have variance zero yields (4.1.4).

□

For simulation purposes, it is convenient to rewrite x and y in terms of independent Brownian motions $\bar{W}_1$ and $\bar{W}_2$ as follows:

$$\begin{aligned}
dx(t) &= -ax(t)dt + \sigma d\bar{W}_1(t), \\
dy(t) &= -by(t)dt + \eta\rho d\bar{W}_1(t) + \eta\sqrt{1-\rho^2} d\bar{W}_2(t),
\end{aligned}$$

where $W_1 = \bar{W}_1$ and $W_2 = \rho\bar{W}_1 + \sqrt{1-\rho^2}\bar{W}_2(t)$, which after integration looks like

$$\begin{aligned}
r(t) =& x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-u)} d\bar{W}_1(u) \\
&+ \eta\rho \int_s^t e^{-b(t-u)} d\bar{W}_1(u) + \eta\sqrt{1-\rho^2} \int_s^t e^{-b(t-u)} d\bar{W}_2(u) + \varphi(t).
\end{aligned}$$

4.2 Pricing a T-Bond

Now, one can price a zero-coupon bond $P(t,T)$, yielding unit value at maturity, by using the formula

$$P(t,T) = \mathbb{E}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t].$$

Before one can derive an explicit formula in the G2++ model, a lemma is needed. The lemma and the subsequent theorem closely follow [5, Chapter 4.2.2].

Lemma 4.2.1. *The random variable*

$$I(t,T) = \int_t^T [x(u) + y(u)]du$$

conditional on $\mathcal{F}_t$ is normally distributed, with

$$\begin{aligned}
\mathbb{E}[I(t,T)|\mathcal{F}_t] =& \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{1-e^{-b(T-t)}}{b}y(t) =: M(t,T) \qquad (4.2.1)\\
Var[I(t,T)|\mathcal{F}_t] =& \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]\\
&+ \frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] \qquad (4.2.2)\\
&+ 2\rho\frac{\sigma\eta}{ab}\left[T - t + \frac{e^{-a(T-t)}-1}{a} + \frac{e^{-b(T-t)}-1}{b} - \frac{e^{-(a+b)(T-t)}-1}{a+b}\right]\\
=:& V(t,T)
\end{aligned}$$

being the corresponding mean and variance.

Proof. Using the stochastic version of integration by parts, $d(X_tY_t) = X_tdY_t + Y_tdX_t + d[X,Y]_t$ for semimartingales X, Y, leads to

$$\int_t^T x(u)du = Tx(T) - tx(t) - \int_t^T udx(u) = \underbrace{\int_t^T (T-u)dx(u)}_{=:(I)} + (T-t)x(t),$$

since the quadratic variation part vanishes because $X_u = u$ is deterministic.

Inserting the definition of $dx(u)$ and $x(u)$ subsequently, one arrives at

$$(I) = \underbrace{-a\int_t^T (T-u)x(u)du}_{(II)} + \sigma\int_t^T (T-u)dW_1(u)$$

and

$$(II) = \underbrace{-ax(t)\int_t^T (T-u)e^{-a(u-t)}du}_{(III)} + \underbrace{(-a)\sigma\int_t^T (T-u)\int_t^u e^{-a(u-s)}dW_1(s)du}_{(IV)}.$$

Further, by usual integration by parts,

$$\begin{aligned}(III) &= x(t)(T-u)e^{-a(u-t)}\Big|_t^T + x(t)\int_t^T e^{-a(u-t)}du \\ &= -x(t)(T-t) + x(t)\frac{e^{-a(u-t)}}{-a}\Big|_t^T = -x(t)(T-t) - x(t)\frac{e^{-a(T-t)}-1}{a}.\end{aligned}$$

Then, by another application of stochastic integration by parts,

$$\begin{aligned}(IV) =& -a\sigma\int_t^T (T-u)e^{-au}\int_t^u e^{as}dW_1(s)du \\ =& -a\sigma\Bigg[\left(\int_t^u (T-v)e^{-av}dv\int_t^u e^{as}dW_1(s)\right)\Big|_t^T \\ & - \left(\int_t^T\int_t^u (T-v)e^{-av}dv\right)d\left(\int_t^u e^{as}dW_1(s)\right)\Bigg] \\ =& -a\sigma\Bigg[\left(\int_t^T (T-v)e^{-av}dv\right)\left(\int_t^T e^{au}dW_1(u)\right) \\ & - \int_t^T\left(\int_t^u (T-v)e^{-av}dv\right)e^{au}dW_1(u)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= - a\sigma \left[\left(\int_t^T \left(\int_t^T (T-v)e^{-av} dv \right) e^{au} dW_1(u) \right) \right. \\
&\quad \left. - \int_t^T \left(\int_t^u (T-v)e^{-av} dv \right) e^{au} dW_1(u) \right] \\
&= - a\sigma \left[\int_t^T \left(\int_u^T (T-v)e^{-av} dv \right) e^{au} dW_1(u) \right] \\
&= - a\sigma \left[\int_t^T \left(\frac{(T-u)e^{-au}}{a} + \frac{e^{-aT} - e^{-au}}{a^2} \right) e^{au} dW_1(u) \right] \\
&= - \sigma \left[\int_t^T \left((T-u) + \frac{e^{-a(T-u)} - 1}{a} \right) dW_1(u) \right].
\end{aligned}
$$

Altogether this yields, thanks to some cancellations,

$$
\begin{aligned}
\int_t^T x(u) du &= (III) + (IV) + \sigma \int_t^T (T-u) dW_1(u) + (T-t)x(t) \\
&= x(t) \frac{1 - e^{-a(T-t)}}{a} + \frac{\sigma}{a} \int_t^T \left(1 - e^{-a(T-u)} \right) dW_1(u) \qquad (4.2.3)
\end{aligned}
$$

The exact same calculation can be done for $y(u)$, leading to

$$
\int_t^T y(u) du = y(t) \frac{1 - e^{-b(T-t)}}{b} + \frac{\eta}{b} \int_t^T \left(1 - e^{-b(T-u)} \right) dW_2(u) \qquad (4.2.4)
$$

Lemma 3.2.1 once again implies both that (4.2.3) and (4.2.4) are normally distributed expressions and therefore (4.2.1) holds.
As far as the conditional variance is concerned, (4.2.3) and (4.2.4) combined

with the Itô isometry is the way to go, yielding

$$\begin{aligned}
Var[I(t,T)|\mathcal{F}_t] =& Var\left[\int_t^T x(u)du + \int_t^T y(u)du\middle|\mathcal{F}_t\right]\\
=& Var\left[x(t)\frac{1-e^{-a(T-t)}}{a} + \frac{\sigma}{a}\int_t^T \left(1-e^{-a(T-u)}\right) dW_1(u)\right.\\
&\left. + y(t)\frac{1-e^{-b(T-t)}}{b} + \frac{\eta}{b}\int_t^T \left(1-e^{-b(T-u)}\right) dW_2(u)|\mathcal{F}_t\right]\\
=& Var\left[\frac{\sigma}{a}\int_t^T \left(1-e^{-a(T-u)}\right) dW_1(u)\right.\\
&\left. + \frac{\eta}{b}\int_t^T \left(1-e^{-b(T-u)}\right) dW_2(u)|\mathcal{F}_t\right]\\
=& \frac{\sigma^2}{a^2}\int_t^T \left(1-e^{-a(T-u)}\right)^2 du + \frac{\eta^2}{b^2}\int_t^T \left(1-e^{-b(T-u)}\right)^2 du\\
&+ 2\rho\frac{\sigma\eta}{ab}\int_t^T \left(1-e^{-a(T-u)}\right)\left(1-e^{-b(T-u)}\right) du. \qquad (4.2.5)
\end{aligned}$$

Straightforward integration of (4.2.5) leads to the lengthy expression (4.2.2) and thus concludes the proof. □

With the aid of the preceding lemma, one can explicitly calculate the zero-coupon bond price within the G2++ model:

Theorem 4.2.2. *The price of the zero-coupon bond with unit face value $P(t,T)$ is given by*

$$P(t,T) = \exp\left(-\int_t^T \varphi(u)du - M(t,T) + \frac{1}{2}V(t,T)\right). \qquad (4.2.6)$$

Proof. Using the fact that φ is deterministic, the distributional properties of $I(t,T)$ stated in Lemma 4.2.1 and the well-known form of the moment-generating function of a normally distributed random variable, namely $M_X(t) =$

$\exp(\mathbb{E}[X]t + \frac{t^2}{2}Var[X])$, one gets

$$\begin{aligned}
P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r(u)du\right)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\exp\left(-\int_t^T x(u)+y(u)+\varphi(u)du\right)\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T x(u)+y(u)du\right)\exp\left(\int_t^T \varphi(u)du\right)\middle|\mathcal{F}_t\right] \\
&= e^{\int_t^T \varphi(u)du}\mathbb{E}\left[\exp\left(-\int_t^T x(u)+y(u)du\right)|\mathcal{F}_t\right] \\
&= e^{\int_t^T \varphi(u)du} M_{I(t,T)}(-1) = e^{\int_t^T \varphi(u)du}\exp\left(-M(t,T)+\frac{1}{2}V(t,T)\right) \\
&= \exp\left(-\int_t^T \varphi(u)du - M(t,T) + \frac{1}{2}V(t,T)\right).
\end{aligned}$$

□

Assume now that $T \mapsto P^M(0,T)$, i.e., the the term-structure of bond prices currently observed in the market, is a sufficiently smooth function in T, in a sense that the corresponding instantaneous forward rate $f^M(0,T) = -\frac{\partial}{\partial T}\ln(P^M(0,T))$ is well-defined. Then, one can state the following corollary to Theorem 4.2.2.

Corollary 4.2.3. *The following statements are equivalent:*

(i) The currently observed term-structure of discount factors is fitted by the G2++ model, i.e., $P^M(0,T) = P(0,T) \qquad \forall T$,

(ii)

$$\begin{aligned}
\varphi(T) =& f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 + \frac{\eta^2}{2b^2}\left(1-e^{-bT}\right)^2 \\
&+ \rho\frac{\sigma\eta}{ab}\left(1-e^{-aT}\right)\left(1-e^{-bT}\right) \qquad \forall T,
\end{aligned}$$

(iii)

$$\exp\left(-\int_t^T \varphi(u)du\right) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left(-\frac{1}{2}[V(0,T)-V(0,t)]\right) \qquad \forall T.$$

Proof. (i)⇔(ii): Since $x(0) = y(0) = 0$ by assumption, also $M(0,T) = 0$ holds. Plugging in the bond price formula from Theorem 4.2.2 in (i) yields for all T:

$$
\begin{aligned}
P^M(0,T) =& \exp\left(-\int_0^T \varphi(u)du + \frac{1}{2}V(0,T)\right) \qquad (*)\\
\Leftrightarrow \frac{\partial}{\partial T}\ln(P^M(0,T)) =& \frac{\partial}{\partial T}\left(-\int_0^T \varphi(u)du + \frac{1}{2}V(0,T)\right)\\
\Leftrightarrow \varphi(T) \overset{(**)}{=}& f^M(0,T)\\
&+ \frac{1}{2}\frac{\partial}{\partial T}\left[2\rho\frac{\sigma\eta}{ab}\int_0^T \left(1-e^{-a(T-u)}\right)\left(1-e^{-b(T-u)}\right)du\right.\\
&\left.+ \frac{\sigma^2}{a^2}\int_0^T \left(1-e^{-a(T-u)}\right)^2 du + \frac{\eta^2}{b^2}\int_0^T \left(1-e^{-b(T-u)}\right)^2 du\right],
\end{aligned}
$$

where at (**) identity (4.2.5) was used. Applying the Leibniz integral rule now leads to (ii).
(i)⇔(iii):

$$
\begin{aligned}
\exp\left(-\int_t^T \varphi(u)du\right) &= \exp\left(-\int_0^T \varphi(u)du\right)\exp\left(\int_0^t \varphi(u)du\right)\\
&\overset{(*)}{=} P^M(0,T)\exp\left(-\frac{1}{2}V(0,T)\right)\left(P^M(0,t)\exp\left(-\frac{1}{2}V(0,t)\right)\right)^{-1}\\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left(-\frac{1}{2}V(0,T)+\frac{1}{2}V(0,t)\right),
\end{aligned}
$$

which is equivalent to (iii). (ii)⇔(iii) follows immediately from (i)⇔(ii) and (i)⇔(iii). □

And immediate consequence of Corollary 4.2.3 is that

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T \varphi(u)du\right)\exp\left(-M(t,T)+\frac{1}{2}V(t,T)\right)\\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left(-\frac{1}{2}[V(0,T)-V(0,t)]\right)\exp\left(-M(t,T)+\frac{1}{2}V(t,T)\right)\\
&= \frac{P^M(0,T)}{P^M(0,t)}\exp\left(\frac{1}{2}[V(t,T)-V(0,T)+V(0,t)]-M(t,T)\right).
\end{aligned}
\tag{4.2.7}
$$

This expression is interesting because it shows that in order to calculate the

bond prices, one only needs to know the market bond prices $P^M(0,T)$ at the desired maturities. On the one hand, this keeps interpolation limited, and on the other hand, it does not involve deriving the function φ. This is important, because that would, according to Corollary 4.2.3, involve deriving the current instantaneous forward observed on the market. That in turn would require the derivative of the function $P^M(0,T)$, which for non-observable maturities needs to be interpolated, and thus may lead to approximations to a certain extent. However, if a financial product depends on the entire or parts of the path of the interest rate $r(t)$, calculating φ is still inevitable.
Assuming the G2++ model is fitted to the current market bond price curve, one can explicitly state the expected instantaneous short rate at time t. Applying (4.1.3) and (4.1.4) for $s = 0$ and (*) yields

$$\begin{aligned}
\mu_r(t) :=& E[r(t)] = f^M(0,t) + \frac{\sigma^2}{2}a^2\left(1-e^{-at}\right)^2 + \frac{\eta^2}{2b^2}\left(1-e^{-bt}\right)^2 \\
&+ \rho\frac{\sigma\eta}{ab}\left(1-e^{-at}\right)\left(1-e^{-bt}\right) \\
\sigma_r^2(t) :=& Var[r(t)] = \frac{\sigma^2}{2a}[1-e^{-2at}] + \frac{\eta^2}{2b}[1-e^{-2bt}] + 2\rho\frac{\sigma\eta}{a+b}[1-e^{-(a+b)t}].
\end{aligned}$$

4.3 Pricing derivatives within the G2++ Model

Recall that the interest rate dynamics within the G2++ model are as follows:

$$\begin{aligned}
r(t) &= x(t) + y(t) + \varphi(t), \quad r(0) = r_0 \\
dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0 \\
dy(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0,
\end{aligned}$$

where $r_0, a, b, \sigma, \eta \in \mathbb{R}^+$ and (W_1, W_2) is a two-dimensional Brownian Motion with instantaneous correlation ρ, i.e., $dW_1 dW_2 = \rho dt$, with $\rho \in [-1, 1]$. The deterministic function φ is used to fit the model to the initial term-structure of discount factors $T \mapsto P^M(0,T)$.
In this section the price for a European option with maturity T and strike price K with an underlying S-bond and subsequently the prices of caplets/caps

will be derived. The price of the former at time t and $S > T$ in case of a call is given by (cf. Chapter 2.3)

$$\mathbf{ZBC}(t,T,S,K) = \mathbb{E}\left[e^{-\int_t^T r(s)ds}(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right].$$

To obtain an explicit expression for this expectation, a change of measure is needed. For fixed T, the T-forward measure $\mathbb{Q}^T$ is introduced by setting the Radon-Nikodym derivative to (cf. (2.3.7))

$$\begin{aligned}\frac{d\mathbb{Q}^T}{d\mathbb{Q}} &:= \frac{B(0)P(T,T)}{B(T)P(0,T)} \\ &= \frac{\exp\left(-\int_0^T \varphi(u)du - \int_0^T x(u)+y(u)du\right)}{P(0,T)} \\ &= \exp\left(-\frac{1}{2}V(0,T) - \int_0^T x(u)+y(u)du\right).\end{aligned}$$

One can show (for details see [5, Lemma 4.2.2]) that the processes $x(t)$ and $y(t)$ under $\mathbb{Q}^T$ are given by

$$\begin{aligned}x(t) &= x(s)e^{-a(t-s)} - M_x^T(s,t) + \sigma\int_s^t e^{-a(t-u)}dW_1^T(u) \\ y(t) &= y(s)e^{-b(t-s)} - M_y^T(s,t) + \eta\int_s^t e^{-b(t-u)}dW_2^T(u),\end{aligned}$$

for $s \le t \le T$, where W_1^T and W_2^T are Brownian motions under $\mathbb{Q}^T$ with $d[W_1^T, W_2^T]_t = \rho dt$ and $M_x^T(s,t)$ and $M_y^T(s,t)$[1] are deterministic expressions depending on the model parameters. This means, that $x(t)$ and $y(t)$ conditional on $\mathcal{F}_s$ are normally distributed and consequently, $r(t)$ follows also a normal distribution. Knowing this, the following theorem can be stated (both theorem and proof closely follow [5, Chapter 4.2.4])

1

$$M_x^T(s,t) = \left(\frac{\sigma^2}{a^2} + \rho\frac{\sigma\eta}{ab}\right)\left(1 - e^{-a(t-s)}\right) - \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) - \frac{\rho\sigma\eta}{b(a+b)}\left(e^{-b(T-t)} - e^{-bT-at+(a+b)s}\right)$$

$$M_y^T(s,t) = \left(\frac{\eta^2}{b^2} + \rho\frac{\sigma\eta}{ab}\right)\left(1 - e^{-b(t-s)}\right) - \frac{\eta^2}{2b^2}\left(e^{-b(T-t)} - e^{-b(T+t-2s)}\right) - \frac{\rho\sigma\eta}{b(a+b)}\left(e^{-a(T-t)} - e^{-aT-bt+(a+b)s}\right)$$

Theorem 4.3.1. *A European call option with maturity T, strike K and an underlying S-bond with unit face value satisfies the following pricing formula within the G2++ model:*

$$\begin{aligned}\boldsymbol{ZBC}(t,T,S,K) =& P(t,S)\Phi\left(\frac{\ln\left(\frac{P(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right)\\ &- P(t,T)K\Phi\left(\frac{\ln\left(\frac{P(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right),\end{aligned}$$

where

$$\begin{aligned}\Sigma(t,T,S)^2 =& \frac{\sigma^2}{2a^3}\left(1-e^{-a(S-T)}\right)^2\left(1-e^{-2a(T-t)}\right) + \frac{\eta^2}{2b^3}\left(1-e^{-b(S-T)}\right)^2\left(1-e^{-2b(T-t)}\right)\\ &+ 2\rho\frac{\sigma\eta}{ab(a+b)}\left(1-e^{-a(S-T)}\right)\left(1-e^{-b(S-T)}\right)\left(1-e^{-(a+b)(T-t)}\right).\end{aligned}$$

Proof. It was shown earlier in (2.3.10) that under $\mathbb{Q}^T$ the considered call option can be priced by

$$\mathbf{ZBC}(t,T,S,K) = P(t,T)\mathbb{E}^T\left[(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right]. \tag{4.3.1}$$

Applying the logarithm to (4.2.7) yields

$$\begin{aligned}\ln P(T,S) = \ln\left(\frac{P^M(0,S)}{P^M(0,T)}\right) + \frac{1}{2}[V(T,S) - V(0,S) + V(0,T)]\\ - \frac{1-e^{-a(S-T)}}{a}x(T) - \frac{1-e^{-b(S-T)}}{b}y(T).\end{aligned}$$

Due to the above mentioned fact that $x(t)$ and $y(t)$ are normally distributed under $\mathbb{Q}^T$, the same holds for $\ln P(T,S)$ conditional on $\mathcal{F}_t$, namely with mean

$$\begin{aligned}M_p = \ln\left(\frac{P^M(0,S)}{P^M(0,T)}\right) + \frac{1}{2}[V(T,S) - V(0,S) + V(0,T)]\\ - \frac{1-e^{-a(S-T)}}{a}\mathbb{E}^T[x(T)|\mathcal{F}_t] - \frac{1-e^{-b(S-T)}}{b}\mathbb{E}^T[y(T)|\mathcal{F}_t]\end{aligned}$$

and variance $\Sigma(t,T,S)^2$. The expression for $\Sigma(t,T,S)^2$ can be derived by a calculation almost analogous to the one in the proof of Corollary 4.1.2. The expectation in (4.3.1) thus equals

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\Sigma(t,T,S)}(e^z - K)^+ e^{-\frac{1}{2}\frac{(z-M_p)^2}{\Sigma(t,T,S)^2}} dz \\
&= \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi}\Sigma(t,T,S)}(e^z - K) e^{-\frac{1}{2}\frac{(z-M_p)^2}{\Sigma(t,T,S)^2}} dz \\
&= \int_{\frac{\ln(K)-M_p}{\Sigma(t,T,S)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{M_p + v\Sigma(t,T,S)} e^{-\frac{1}{2}v^2} dv - K \int_{\frac{\ln(K)-M_p}{\Sigma(t,T,S)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv \\
&= e^{M_p + \frac{1}{2}\Sigma(t,T,S)^2} \int_{\frac{\ln(K)-M_p}{\Sigma(t,T,S)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v-\Sigma(t,T,S))^2} dv - K \int_{\frac{\ln(K)-M_p}{\Sigma(t,T,S)}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv \\
&= e^{M_p + \frac{1}{2}\Sigma(t,T,S)^2} \left[\Phi(\infty) - \Phi\left(\frac{\ln(K) - M_p - \Sigma(t,T,S)^2}{\Sigma(t,T,S)}\right)\right] \\
&- K\left[\Phi(\infty) - \Phi\left(\frac{\ln(K) - M_p}{\Sigma(t,T,S)}\right)\right] \\
&= e^{M_p + \frac{1}{2}\Sigma(t,T,S)^2} \Phi\left(\frac{-\ln(K) + M_p + \Sigma(t,T,S)^2}{\Sigma(t,T,S)}\right) - K\Phi\left(\frac{-\ln(K) + M_p}{\Sigma(t,T,S)}\right).
\end{aligned}$$

Using the fact that $\frac{P(t,S)}{P(t,T)}$ is a martingale under $\mathbb{Q}_T$ and the moment-generating function of the normal distribution lead to

$$\frac{P(t,S)}{P(t,T)} = E^{\mathbb{Q}^T}\left[P(T,S) \,|\, \mathcal{F}_t\right] = E^{\mathbb{Q}^T}\left[e^{\ln P(T,S)} \,\middle|\, \mathcal{F}_t\right] = e^{M_p + \frac{1}{2}\Sigma(t,T,S)^2},$$

which is equivalent to

$$M_p = \ln\frac{P(t,S)}{P(t,T)} - \frac{1}{2}\Sigma(t,T,S)^2.$$

Putting everything together concludes the proof:

$$\begin{aligned}
\mathbf{ZBC}(t,T,S,K) &= P(t,T)\mathbb{E}^T\left[(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right] \\
=&P(t,T)\left[\frac{P(t,S)}{P(t,T)}\Phi\left(\frac{-\ln(K)+\ln\frac{P(t,S)}{P(t,T)}-\frac{1}{2}\Sigma(t,T,S)^2+\Sigma(t,T,S)^2}{\Sigma(t,T,S)}\right)\right. \\
&\left.-K\Phi\left(\frac{-\ln(K)+\ln\frac{P(t,S)}{P(t,T)}-\frac{1}{2}\Sigma(t,T,S)^2}{\Sigma(t,T,S)}\right)\right] \\
=&P(t,S)\Phi\left(\frac{\ln\left(\frac{P(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)}+\frac{1}{2}\Sigma(t,T,S)\right) \\
&-P(t,T)K\Phi\left(\frac{\ln\left(\frac{P(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)}-\frac{1}{2}\Sigma(t,T,S)\right).
\end{aligned}$$

□

Remark:[2] Because of the put-call parity $\mathbf{ZBC}(t,T,S,K) + KP(t,T) =$

[2] For an arbitrary face value N of the underlying S-bond the formulas generalize to

$$\begin{aligned}
\mathbf{ZBC}(t,T,S,N,K) =&NP(t,S)\Phi\left(\frac{\ln\left(\frac{NP(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)}+\frac{1}{2}\Sigma(t,T,S)\right) \\
&-P(t,T)K\Phi\left(\frac{\ln\left(\frac{NP(t,S)}{KP(t,T)}\right)}{\Sigma(t,T,S)}-\frac{1}{2}\Sigma(t,T,S)\right), \\
\mathbf{ZBP}(t,T,S,N,K) =&-NP(t,S)\Phi\left(\frac{\ln\left(\frac{KP(t,T)}{NP(t,S)}\right)}{\Sigma(t,T,S)}-\frac{1}{2}\Sigma(t,T,S)\right) \\
&+P(t,T)K\Phi\left(\frac{\ln\left(\frac{KP(t,T)}{NP(t,S)}\right)}{\Sigma(t,T,S)}+\frac{1}{2}\Sigma(t,T,S)\right)
\end{aligned}$$

$\mathbf{ZBP}(t,T,S,K)+P(t,S)$, the put price is given by

$$\begin{aligned}\mathbf{ZBP}(t,T,S,K) = &- P(t,S)\Phi\left(\frac{\ln\left(\frac{KP(t,T)}{P(t,S)}\right)}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right)\\ &+ P(t,T)K\Phi\left(\frac{\ln\left(\frac{KP(t,T)}{P(t,S)}\right)}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right).\end{aligned}$$

As derived in Chapter (2.3), the price of a caplet with reset date T_1, settlement date T_2, strike K, nominal N and underlying simply compounded rate $L(t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right)$ equals

$$\begin{aligned}\mathbf{Cpl}(t,T_1,T_2,N,X) &= \mathbf{ZBP}(t,T_1,T_2,N',N),\\ N' &= N(1+X(T_2-T_1)),\end{aligned}$$

which within the G2++ model translates to

$$\begin{aligned}\mathbf{Cpl}(t,T_1,T_2,N,X) = &- N'P(t,T_2)\Phi\left(\frac{\ln\left(\frac{NP(t,T)}{N'P(t,T_2)}\right)}{\Sigma(t,T_1,T_2)} - \frac{1}{2}\Sigma(t,T_1,T_2)\right)\\ &+ P(t,T_1)N\Phi\left(\frac{\ln\left(\frac{NP(t,T_1)}{N'P(t,T_2)}\right)}{\Sigma(t,T_1,T_2)} + \frac{1}{2}\Sigma(t,T_1,T_2)\right).\end{aligned}$$

As the cap price is the sum of the underlying caplet prices, the cap prices are given by

$$\begin{aligned}&\mathbf{Cap}(t,\mathcal{T},\tau,N,X) =\\ &\quad\sum_{i=1}^{n}\left[-N(1+X\tau_i)P(t,T_i)\Phi\left(\frac{\ln\left(\frac{P(t,T_{i-1})}{(1+X\tau_i)P(t,T_i)}\right)}{\Sigma(t,T_{i-1},T_i)} - \frac{1}{2}\Sigma(t,T_{i-1},T_I)\right)\right.\\ &\quad\left.+P(t,T_{i-1})N\Phi\left(\frac{\ln\left(\frac{P(t,T_{i-1})}{(1+X\tau_i)P(t,T_i)}\right)}{\Sigma(t,T_{i-1},T_i)} + \frac{1}{2}\Sigma(t,T_{i-1},T_I)\right)\right], \qquad (4.3.2)\end{aligned}$$

where $\mathcal{T} = \{T_0, T_1, \cdots, T_n\}$ denotes the set of all relevant dates and $\tau = \{\tau_1, \cdots, \tau_n\}$ with $\tau_i = T_i - T_{i-1}$ are the involved time intervals.
The corresponding floor/floorlet prices can be derived by formulas (2.3.14)

and (2.3.12). Swaption prices within the G2++ model are a little more work and are given by a numerically computable integral. For the exact expressions and their derivations, the interested reader is again referred to [5, Chapter 4.2].

4.4 Connection to the Hull-White two-factor model

In the chapter about one-factor models, the Hull-White extension of the Vasicek model was investigated. There also exists a corresponding two-factor model, which under the risk-neutral measure follows the dynamics

$$\begin{aligned}
dr(t) &= [\theta(t) + u(t) - \bar{a}r(t)]dt + \sigma_1 dW_1(t), \quad r(0) = r_0 \\
du(t) &= -\bar{b}u(t)dt + \sigma_2 dW_2(t), \quad u(0) = 0,
\end{aligned}$$

where the Brownian motions are correlated by $dW_1 dW_2 = \bar{\rho}dt$ and $r_0, \bar{a}, \bar{b}, \sigma_1, \sigma_2 > 0$ and $-1 \leq \bar{\rho} \leq 1$ are constants. This means, that the mean-reversion level is a stochastic process. The function θ is deterministic and used to fit the model to the current term structure. On first sight, the connection to the G2++ model is not clear. However, it can be shown that the two models are analogies of each other. We assume $a > b$, the case $b < a$ can be treated in the same way. Starting from the G2++ model, (4.1.1), one arrives at the two-factor Hull-White by setting the parameters as follows:

$$\begin{aligned}
\bar{a} &= a, & \bar{b} &= b, \\
\sigma_1 &= \sqrt{\sigma^2 + \eta^2 + 2\rho\sigma\eta}, & \sigma_2 &= \eta(a - b), \\
\bar{\rho} &= \frac{\sigma\rho + \eta}{\sqrt{\sigma^2 + \eta^2 + 2\rho\sigma\eta}}, & \theta(t) &= \frac{d\varphi(t)}{dt} + a\varphi(t).
\end{aligned}$$

On the other hand, starting from the two factor Hull-White model, one can obtain the G2++ model by the following choice of parameters:

$$a = \bar{a}, \qquad b = \bar{b},$$

$$\sigma = \sigma_3 := \sqrt{\sigma_1^2 + \frac{\sigma_2^2}{(\bar{a}-\bar{b})^2} + 2\bar{\rho}\frac{\sigma_1\sigma_2}{\bar{b}-\bar{a}}}, \qquad \eta = \sigma_4 := \frac{\sigma_2}{\bar{a}-\bar{b}},$$

$$\rho = \frac{\sigma_1\bar{\rho} - \sigma_4}{\sigma_3}, \qquad \varphi(t) = r_0 e^{-\bar{a}t} + \int_0^t \theta(v) e^{-\bar{a}(t-v)} dv.$$

For details on how to derive this, see [5, Chapter 4.2.5].

4.5 Volatility structures and calibration

In this section, the volatility structure corresponding to short rate models will be investigated. Volatility structures are of great importance for calibration of short rate models, in particular the G2++ model, which will be dealt with later on in this chapter. The particular volatilities considered in this chapter will be caplet and cap volatilities, although one could also consider swaption volatilities. It should be mentioned, that a lot of definitions and procedures in this section are dependent on the possibility of using Black's cap formula. However, in the current market environment, where negative interest rates are very common, Black's formula in its original form cannot be applied and one therefore needs to use different approaches.

Market cap volatilities are implicitly defined as the parameter $\sigma_{\alpha,\beta}$, which, plugged into Black's cap formula (2.2.1), yields the current market price. T_α is the first reset date, T_β the last settlement date. To recall, Black's cap formula is given by

$$
\begin{aligned}
&\mathbf{Cap}^{\mathbf{Black}}(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
&\qquad = N \sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, 1), \\
Bl(K, F, v, \omega) &= F\omega\Phi(\omega d_1(K, F, v)) - K\omega\Phi(\omega d_2(K, F, v)), \\
d_1(K, F, v) &= \frac{\ln\left(\frac{F}{K}\right) + \frac{v^2}{2}}{v}, \\
d_2(K, F, v) &= \frac{\ln\left(\frac{F}{K}\right) - \frac{v^2}{2}}{v}, \\
v_i &= \sigma_{\alpha,\beta}\sqrt{T_{i-1}},
\end{aligned}
$$

where Φ denotes that CDF of the standard normal distribution and $\sigma_{\alpha,\beta}$ are quoted on the market. It is worth noting, that it is assumed that all caplets contributing to the cap have the same volatility. When considering caplet volatilities, different volatilities are allowed, even when being part of the same cap. As far as caplet volatilities are concerned, the market also knows a further definition. Look at a caplet with reset date T and settlement date $T + \tau$, where τ usually equals three or six months, and underlying rate $F(t, T, T + \tau)$ (which is of the form $dF(t, T, T + \tau) = (\cdots)dt + \sigma(t, T, T + \tau)F(t, T, T + \tau)dW_t$), then the caplet volatility is given by

$$
v^2_{T-caplet} := \frac{1}{T}\int_0^T d[\ln F(t, T, T + \tau)] = \frac{1}{T}\int_0^T \sigma(t, T, T + \tau)^2 dt,
$$

where $\sigma(t, T, T + \tau)$ is the percentage instantaneous volatility of the underlying rate $F(t, T, T + \tau)$[3]. Since the $\sigma(t, T, T + \tau)$ are deterministic in

[3] The percentage instantaneous volatility of a process Y_t is defined as the quantity $\sigma(t)$ in

$$
dY_t = (\cdots)dt + \sigma(t)Y_t dW_t.
$$

The absolute instantaneous volatility of a process Y_t is defined as the quantity $\sigma(t)$ in

$$
dY_t = (\cdots)dt + \sigma(t)dW_t.
$$

Black's model, so are the caplet volatilities. According to Brigo & Mercurio [5], in practice, the term structure of volatilities defined by the function $T \longrightarrow v_{T-caplet}$ often has a humped shape, which a good model should be able to reproduce. If the above definition is simply translated to arbitrary short rate models, i.e.,

$$v_{T-caplet} := \sqrt{\frac{1}{T}\int_0^T \sigma(t,T,T+\tau)^2 dt},^4$$

problems can arise. In the G2++ model amongst others, $\sigma(t,T,T+\tau)$ are not deterministic, as one can see after some tedious calculations by using Itô's formula for the expression

$$d\ln F(t,T,T+\tau) = d\ln\left(\frac{P(t,T)}{P(t,T+\tau)} - 1\right),$$

and consequently $v_{T-caplet}$ are stochastic as well. Since stochastic volatilities are of little use for calibration, one needs to find a different approach. To make the volatilities deterministic, an implied volatility approach is used. Within the model, consider an at-the money caplet, i.e., the strike rate K equals the forward rate F, with reset date T and settlement date $T+\tau$. This is a function depending on the model parameters. The idea of calculating the price within the model $\mathbf{Cpl}^{MODEL}(0,T,T+\tau,F(0,T,T+\tau))$ and then inverting the Black formula for the same caplet with respect to the volatility, motivates the following definition:

Definition 4.5.1 (Model-implied T-caplet volatility). *The quantity $v^{MODEL}_{T-caplet}$, which solves the equation*

$$P(0,T+\tau)\tau F(0,T,T+\tau)\left(2\Phi\left(\frac{v^{MODEL}_{T-caplet}\sqrt{T}}{2}\right) - 1\right) = \boldsymbol{Cpl}^{MODEL}(0,T,T+\tau,F(0,T,T+\tau)),$$

where the left-hand side corresponds to the Black caplet formula

[4] $v_{T-caplet}$ defined like this is called model-intrinsic T-caplet volatility.

$\boldsymbol{Cpl}^{\boldsymbol{Black}}(0, \mathcal{T}, \tau, 1, F(0,T,T+\tau), v^{MODEL}_{T-caplet})$ *in* (2.2.3)*, is called model-implied T-caplet volatility.*

Remark: The slightly different structure of the left-hand side of the above equation and (2.2.3) is because of simplifications arising due to the caplet being at-the-money.
For a given model, the map $T \mapsto v^{MODEL}_{T-caplet}$ is called term structure of caplet volatilities. Similarly, though with a little more notation, implied cap volatilities can be defined. Let $\{T_\alpha, \cdots, T_{\beta-1}\}$ be reset dates and $\{T_{\alpha+1}, \cdots, T_\beta\}$ settlement dates. Define $\mathcal{T}_i = \{T_\alpha, \cdots, T_i\}$ and $\bar{\tau}_i = \{\tau_{\alpha+1}, \cdots, \tau_i\}$, where $\tau_j = T_j - T_{j-1}$. Then the implied $\mathcal{T}_i$-cap volatility is the solution $v^{MODEL}_{\mathcal{T}_i-cap}$ of the equation

$$\sum_{j=\alpha+1}^{i} P(0,T_j)\tau_j Bl\left(S_{\alpha,\beta}(0), F(0,T_{j-1},T_j), v^{MODEL}_{\mathcal{T}_i-cap}\sqrt{T_{j-1}}\right)$$
$$= \mathbf{Cap}^{MODEL}(0, \mathcal{T}_i, \bar{\tau}_i, S_{\alpha,\beta}(0)), \tag{4.5.1}$$

where the forward swap rate $S_{\alpha,\beta}(0)$ enters the equation because the cap is at-the-money. Analogously to above, the map $T \mapsto v^{MODEL}_{\mathcal{T}_i-cap}$ is called term structure of cap volatilities for a given model.
As far as the humped shape mentioned above is concerned, practice in a lot of instances shows a connection between the volatility term structure and the absolute instantaneous volatilities of instantaneous forward rates $\sigma_f(t,T)$ (defined as in footnote 3 with $Y_t = f(t,T)$). More specifically, according to [5], one can usually observe the following.

- If there are no humps in $T \mapsto \sigma_f(t,T)$, there are at most small humps in $T \mapsto v^{MODEL}_{T-caplet}$.
- If there are humps in $T \mapsto \sigma_f(t,T)$, large humps in $T \mapsto v^{MODEL}_{T-caplet}$ are possible.

In case of the G2++ model, we can explicitly compute $\sigma_f(t,T)$. To make computations more convenient later on, the following notation is introduced:

$$\begin{aligned} A(t,T) &:= \frac{P^M(0,T)}{P^M(0,t)} \exp\left(\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]\right), \\ B(z,t,T) &:= \frac{1-e^{-z(T-t)}}{z}. \end{aligned}$$

Then, the bond price formula (4.2.7) can be rewritten to

$$P(t,T) = A(t,T)\exp\left(-B(a,t,T)x(t) - B(b,t,T)y(t)\right)$$

and the instantaneous forward rate is given by

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{\partial}{\partial T}\ln(A(t,T)) + \frac{\partial}{\partial T}B(a,t,T)x(t) + \frac{\partial}{\partial T}B(b,t,T)y(t).$$

Using Itô's formula to derive above expression yields

$$\begin{aligned} df(t,T) &= -d\left(\frac{\partial}{\partial T}\ln(A(t,T))\right) + x(t)d\left(\frac{\partial}{\partial T}B(a,t,T)\right) + \frac{\partial}{\partial T}B(a,t,T)dx(t) \\ &\quad + y(t)d\left(\frac{\partial}{\partial T}B(b,t,T)\right) + \frac{\partial}{\partial T}B(b,t,T)dy(t) \\ &= (\cdots)^{5}dt + \frac{\partial}{\partial T}B(a,t,T)\sigma dW_1(t) + \frac{\partial}{\partial T}B(b,t,T)\eta dW_2(t), \end{aligned}$$

which leads to

$$\begin{aligned} &\sigma_f(t,T) = \\ &= \sqrt{\left(\frac{\partial}{\partial T}B(a,t,T)\sigma\right)^2 + \left(\frac{\partial}{\partial T}B(b,t,T)\eta\right)^2 + 2\rho\sigma\eta\frac{\partial}{\partial T}B(a,t,T)\frac{\partial}{\partial T}B(b,t,T)} \\ &= \sqrt{\sigma^2 e^{-2a(T-t)} + \eta^2 e^{-2a(T-t)} + 2\rho\sigma\eta e^{-(a+b)(T-t)}}. \end{aligned}$$

[5] It is easily seen that $(\cdots)$ corresponds to the expression

$$\begin{aligned} &-\frac{\partial^2}{\partial t\partial T}\ln(A(t,T)) + x(t)\frac{\partial^2}{\partial t\partial T}B(a,t,T) + y(t)\frac{\partial^2}{\partial t\partial T}B(b,t,T) \\ &- ax(t)\frac{\partial}{\partial T}B(a,t,T) - by(t)\frac{\partial}{\partial T}B(b,t,T) \end{aligned}$$

Looking closely at this expression, one can notice two things: On the one hand, for $\rho < 0$, a suitable choice of model parameters allows for a humped shape of $T \mapsto \sigma_f(t,T)$. On the other hand, for $\rho > 0$ the function $T \mapsto \sigma_f(t,T)$ does not display humps for any choice of parameters, since each term of the sum is decreasing for $T > t$ and therefore $T \mapsto \sigma_f(t,T)$ is decreasing for $T > t$ as well.

A very important question remains: How is the G2++ model calibrated to real market data? Since the model is already assumed to be fitted to the market bond curve by choosing φ in a particular way, further input parameters are needed. Using Black cap volatilities and their model implied counterparts introduced in this chapter is one possible option. Assume that the cap volatilities $v_{T_i}^M$ for maturities T_i, $i = 1, \cdots, n$ are quoted by the market. In this case, the maturities are the last settlement dates and the caplets constituting each cap have usually 6-month period, except for the one-year cap, which typically has a period of three months. The two main ways to calibrate an interest rate model to cap volatilities are the following:

(i) Minimize the sum of squares of the model and market cap price percentage differences, i.e., $\min_{\beta} \sum_{i=1}^n \left(\frac{Cap_{T_i}^{G2++} - Cap_{T_i}^M}{Cap_{T_i}^M} \right)^2$, where the market cap prices are calculated using Black's cap formula (2.2.1) and the ones of the G2++ model by (4.3.2). $\beta = (a, b, \sigma, \eta, \rho)$ is the parameter vector.

(ii) Minimize the sum of squares of the model and market cap volatility percentage differences, i.e., $\min_{\beta} \sum_{i=1}^n \left(\frac{v_{T_i}^{G2++} - v_{T_i}^M}{v_{T_i}^M} \right)^2$. In order to do that, the model-implied cap volatilities defined earlier in this section by (4.5.1) need to be calculated for every choice of parameters.

As already mentioned at the beginning of the chapter, in the current negative-interest rate environment, caps can often not be quoted by Black volatilities, since Black's cap formula is not applicable. However, if actual market prices can be directly observed, one can still use procedure (i) described above to calibrate the G2++ model, since there is no need to calculate the prices from Black volatilities in that case. Alternatively, one can calibrate the model with the help of swaption volatilities.

Cap volatilities in the above way are currently not quoted on the market due to negative yields for short maturities as seen in Figure 2.1 and it was unfortunately not possible to retrieve current prices for caps. Therefore the data of [5, Chapter 4.2.7] is going to be used. The data is very outdated and market conditions today are entirely different to the ones then. However, for the purpose of illustrating the calibration procedure, the data is still suitable. The cap volatilities and maturities are shown in Table 4.1.

Maturity (years)	Black volatilities	Model implied volatilities
1	0.152	0.15199
2	0.162	0.16216
3	0.164	0.16332
4	0.163	0.16295
5	0.1605	0.16127
7	0.1555	0.15551
10	0.1475	0.14741
15	0.135	0.13483
20	0.126	0.12594

Table 4.1: At-the-money Euro cap volatilities on February 13th, 2001 and the corresponding G2++ model implied volatilities

Squared cap price percentage differences were minimized, as in procedure (i) above. Concerning the actual implementation of the calibration, the Matlab function **lsqnonlin** was used for the minimization and the function **fzero** to calculate the implied volatilities. The parameters resulting from the calibration are as follows:

$$\begin{aligned}
&a = 0.655945396082945,\ b = 0.111131575629247,\\
&\sigma = 0.007520300732930,\ \eta = 0.013125262945609\\
&\rho = -0.965745104235053.
\end{aligned}$$

The calibrated parameters differ slightly from the ones obtained by Brigo & Mercurio in [5, Chapter 4.2.7], which can be seen below:

$$a = 0.5430,\ b = 0.0757,\ \sigma = 0.0058,\ \eta = 0.0117,\ \rho = -0.9914.$$

Two reasons could be responsible for this: Firstly, the yield curve was not ex-

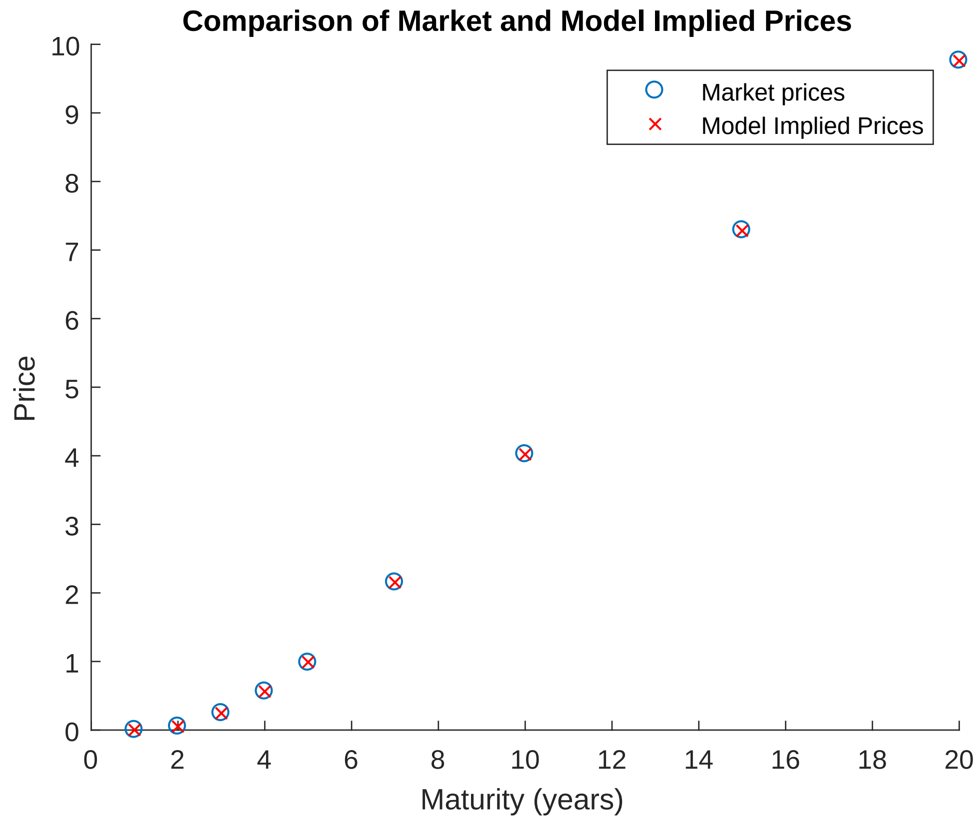


Figure 4.1: Market vs model-implied ATM cap prices (notional=100)

plicitly given, but could only be estimated by the graph they provided in [5, Chapter 1.3]. Secondly, they used a different and more sophisticated method to perform the minimization. But even with Matlab's in-built **lsqnonlin** function, which only performs local-minimization, the market prices and volatilities can be replicated very accurately, as can be seen in Figure 4.1 and Table 4.1.

Figure 4.2 shows sample paths of the calibrated G2++ process with the parameters above as well as the deterministic shift φ.

Remark: At the end of this chapter, a very important remark has to be made. The G2++ model was developed by Brigo & Mercurio before the global financial crisis. In those times, a single interest rate curve was used for discounting and for calculating forward rates when looking at interest rate

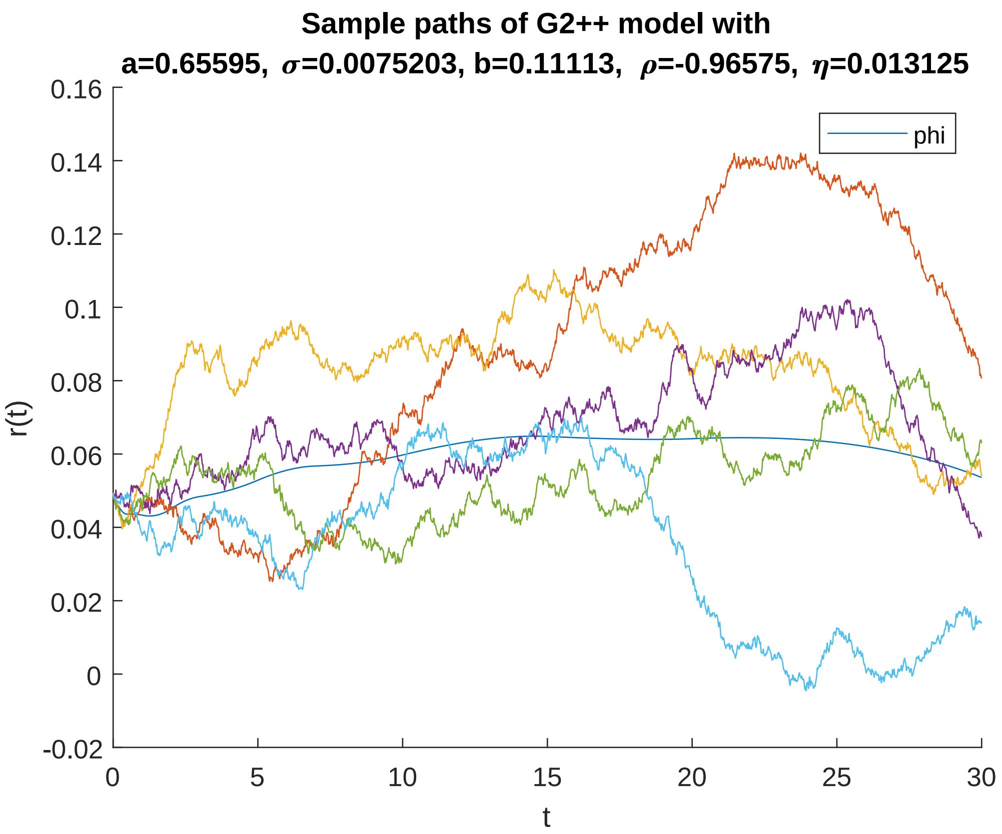


Figure 4.2: G2++ sample paths and deterministic shift

derivatives, since the so-called basis spread, i.e., spreads between IBOR rates of different tenors as well as OIS-rates, was negligible. However, the interest rate landscape changed in the course of the crisis, which led to significant spreads between EURIBOR rates with different tenors as well as EURIBOR-OIS spreads. Therefore, the market practice shifted towards a multi-curve framework. This means that additionally to a risk-free curve based on OIS-rates used for the discounting of future cash flows, (risky) yield curves based on the EURIBOR are stripped from the market for each tenor to derive the forward EURIBOR rates needed to valuate interest rate derivatives.

All that said, since a big part of this thesis focuses on the original G2++ model, and to the author's knowledge no extension of the G2++ model to the multi-curve framework with the same analytical tractability has yet been developed, the single-curve framework has still been adopted throughout this thesis. All the results, from pricing formulas of interest derivatives to calibration methods, are based on this assumption. Furthermore, if interest rate models are only used for market consistent valuation and not for trading, the single-curve framework might still be sufficient. Nevertheless, it should be brought to the reader's attention that in today's market environment, this presents a significant limitation to the model.

For the interest reader, a very good and easily-comprehensible overview of the differences of the single and multi-curve approach can be found in [3]. More on bootstrapping techniques within the multi- curve framework can be found in [1]. Further literature related to that topic include [2] and [14].

Chapter 5

Default risk

Up till now, it was assumed that the bond issuer is non-defaultable. This means that the principal is always paid at maturity, mathematically speaking $P(T,T) = 1$. A natural question which arises is: What if there is the risk of default of the bond issuer during the lifetime of a bond? This risk might be negligible for treasury bonds of economically sound countries. However, for corporate bonds this risk exists and needs to be addressed. In that case, the holder of the bond deserves compensation for this risk in form of a lower price, which is equivalent to a higher yield. This chapter will deal with this topic.

There are different approaches to this topic. One could assess credit risk by looking at the rating of a corporation, issued by rating agencies like Moody's, Fitch or Standard & Poors. The default probability for a special rating class could then be determined by dividing the number of corporations in that rating class which defaulted within a certain timespan by the cardinality of that rating class. One could also use the so-called structural approach, which goes back to Merton [23, 1974]. There, default is defined by the inability of a company to reimburse the bond holders, i.e., the liabilities exceeding the market value of its assets, at maturity T. Those two approaches are not going to be investigated in detail in this chapter, for further reading see for example [7, Chapter 3].

The emphasis in this chapter is going to be put on the so-called "intensity-based" approach, where the evolution of the default probabilities matters rather then the exact event of default. The main source of this chapter is [12, Chapter 12.3], and therefore the proofs to all the following results up to Proposition 5.1.7 closely follow the approach presented there. An almost identical approach with slight differences in some of the proofs is used in [22, Chapter 5]. Further sources include [15, Chapters 5,6] and [9].

5.1 Intensity-based approach

5.1.1 Theoretical basis

Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ be a fixed probability space, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration fulfilling the usual conditions. $\mathcal{F}_t$ is the sigma-field describing the information available on the market up to time t. Let the default time τ be a $\mathcal{F}_t$-stopping time. Then, the default indicator process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ is right-continuous and adapted to $\mathcal{F}_t$. Let $\mathcal{H}_t = \sigma(H_s | s \leq t)$ be the sigma-field generated by H. Further, assume there is a sub-filtration $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0} \subset \mathcal{F}$, such that $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. i.e., $\mathcal{G}_t$ contains all information except the default times. For example, $\mathcal{G}_t$ could be generated by a multi-dimensional driving process X_t. It is assumed that all default-free economic factors, such as the risk-free interest rate, are processes adapted to $\mathcal{G}$. From the definition of $\mathcal{G}$, intuition says that, given $\tau > t$, elements of $\mathcal{F}_t$ are observable in $\mathcal{G}_t$, which is formalized by the following lemma.

Lemma 5.1.1. *Let $A \in \mathcal{F}_t$. Then, there exist $B \in \mathcal{G}_t$, such that*

$$A \cap \{\tau > t\} = B \cap \{\tau > t\} \tag{5.1.1}$$

holds.

Proof. Define

$$\mathcal{F}_t^* = \{A \in \mathcal{F}_t | \exists B \in \mathcal{G}_t : A \cap \{\tau > t\} = B \cap \{\tau > t\}\}.$$

What needs to be shown is that $\mathcal{F}_t \subset \mathcal{F}_t^*$. $\mathcal{G}_t \subset \mathcal{F}_t^*$ follows from choosing

$B = A$. Since $A \in \mathcal{H}_t$ equals either $\emptyset$, Ω or a set of the form $\{\tau > s\}$ or $\{\tau \leq s\}$ for some $s \leq t$, the intersection $A \cap \{\tau > t\}$ equals either $\emptyset$ or $\{\tau > t\}$. Choosing B to be either $\emptyset$ or Ω implies $\mathcal{H}_t \subset \mathcal{F}_t^*$, which altogether means $(\mathcal{G}_t \cup \mathcal{H}_t) \subset \mathcal{F}_t^*$. Since $\mathcal{F}_t$ is the smallest sigma-field containing $\mathcal{G}_t$ and $\mathcal{H}_t$ and it is easily seen that $\mathcal{F}_t^*$ itself is a sigma-field , $\mathcal{F}_t \subset \mathcal{F}_t^*$ follows, which concludes the proof. $\square$

From now on, the following assumption will be made:
(A1): The probability, given $\mathcal{G}_t$, for no default up to time t, equals

$$\mathbb{P}[\tau > t|\mathcal{G}_t] = e^{-\int_0^t \lambda(s)ds},$$

for a non-negative $(\mathcal{G}_t)$-progressive process λ.
Since the sum of probabilities of complementary events conditional on the same sigma-field equals one, this assumption means that the conditional default probability $\mathbb{P}[\tau \leq t|\mathcal{G}_t] < 1$. Since $\mathbb{P}[\tau \leq t|\mathcal{F}_t] = H_t$, this means in particular that $\mathcal{G}_t \subsetneq \mathcal{F}_t$. Consequently, τ is not a stopping time for $(\mathcal{G}_t)$. The following lemma connects the conditional expectations of a random variable w.r.t. to $\mathcal{F}_t$ and $\mathcal{G}_t$.

Lemma 5.1.2. *Let Y be a non-negative random variable. Then, for all $t \geq 0$,*

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}}Y \,\middle|\, \mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}}Y \,\middle|\, \mathcal{G}_t\right].$$

Proof. Because of Lemma 5.1.1, for a fixed $A \in \mathcal{F}_t$ one can choose a $B \in \mathcal{G}_t$ with (5.1.1). This, combined with the definition of the conditional expectation, leads to

$$\begin{aligned}
\int_A \mathbb{1}_{\{\tau>t\}} Y \mathbb{P}[\tau > t|\mathcal{G}_t] d\mathbb{P} &= \int_{A\cap\{\tau>t\}} Y \mathbb{P}[\tau > t|\mathcal{G}_t] d\mathbb{P} \\
&= \int_{B\cap\{\tau>t\}} Y \mathbb{P}[\tau > t|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{1}_{\{\tau>t\}} Y \mathbb{P}[\tau > t|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{1}_{\{\tau>t\}} Y \mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y \mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{G}_t]|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{G}_t] \mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{G}_t]|\mathcal{G}_t] d\mathbb{P} \\
&= \int_B \mathbb{1}_{\{\tau>t\}} \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{G}_t] d\mathbb{P} \\
&= \int_A \mathbb{1}_{\{\tau>t\}} \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{G}_t] d\mathbb{P}.
\end{aligned}$$

Since τ is a $\mathcal{F}_t$-stopping time, $\{\tau > t\} = \{\tau \leq t\}^c \in \mathcal{F}_t$. Because $A \in \mathcal{F}_t$ was arbitrary, the definition of the conditional expectation yields

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y \mathbb{P}[\tau > t|\mathcal{G}_t]|\mathcal{F}_t] &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{G}_t] \\
\Leftrightarrow \qquad \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{F}_t] &= \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \lambda(s)ds} \mathbb{E}[\mathbb{1}_{\{\tau>t\}} Y|\mathcal{F}_t].
\end{aligned}$$

□

With the help of the previous lemma, one can express the conditional default probabilities w.r.t. $\mathcal{F}_t$, which is very important.

Lemma 5.1.3. *For $t \leq T$*

$$\begin{aligned}
\mathbb{P}[\tau > T|\mathcal{F}_t] &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}[e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t] \\
\mathbb{P}[t < \tau \leq T|\mathcal{F}_t] &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}[1 - e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t]
\end{aligned}$$

holds. Furthermore, $L(t) = \mathbb{1}_{\{\tau>t\}} e^{\int_0^t \lambda(s)ds}$ is a $\mathcal{F}_t$-martingale.

Proof. With the help of Lemma 5.1.2 and the relation $\mathbb{1}_{\{\tau>T\}}\mathbb{1}_{\{\tau>t\}} = \mathbb{1}_{\{\tau>T\}}$

$$\begin{aligned}
\mathbb{P}[\tau > T|\mathcal{F}_t] &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}\mathbb{1}_{\{\tau>t\}}\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_T]\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[\mathbb{P}[\tau>T|\mathcal{G}_T]\,|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[e^{-\int_0^T \lambda(s)ds}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t],
\end{aligned}$$

and since τ is a $\mathcal{F}_t$-stopping time

$$\begin{aligned}
\mathbb{P}[t < \tau \le T|\mathcal{F}_t] &= 1 - \mathbb{P}[\tau > T|\mathcal{F}_t] - \mathbb{P}[\tau \le t|\mathcal{F}_t] \\
&= 1 - \mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t] - (1 - \mathbb{1}_{\{\tau>t\}}) \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[1 - e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t\right].
\end{aligned}$$

To verify the martingale property, consider

$$\begin{aligned}
\mathbb{E}[L(T)|\mathcal{F}_t] &= \mathbb{E}[\mathbb{1}_{\{\tau>T\}}e^{\int_0^T \lambda(s)ds}|\mathcal{F}_t] \\
&= \mathbb{E}[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\tau>T\}}e^{\int_0^T \lambda(s)ds}|\mathcal{F}_t] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}e^{\int_0^T \lambda(s)ds}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(s)ds}\mathbb{E}[\underbrace{\mathbb{E}[\mathbb{1}_{\{\tau>T\}}|G_T]e^{\int_0^T \lambda(s)ds}}_{1}|\mathcal{G}_t] = L(t).
\end{aligned}$$

□

The previous lemma also gives an interpretation to the process λ. Replacing T with Δt in a first-order Taylor expansion gives $\mathbb{P}[t < \tau \le t + \Delta t|\mathcal{F}_t] = \mathbb{1}_{\{\tau>t\}}\lambda(t)\Delta t$, i.e., $\lambda(t)$ can be considered as the intensity of default within a small time interval $[t, t + \Delta t]$ given survival up to time t.

Before further assumptions are made, an interesting result is going to be stated:

Proposition 5.1.4. *The process*

$$N_t = H_t - \int_0^t \lambda(s)\mathbb{1}_{\{\tau>s\}}ds$$

is an $\mathcal{F}$-martingale,

Proof. Using Lemma 5.1.2, the tower property of conditional expectations and assumption **(A1)**, one can observe the following:

$$\begin{aligned}
\int_t^T \mathbb{E}[\lambda(s)\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_t]ds &= \int_t^T \mathbb{E}[\lambda(s)\mathbb{1}_{\{\tau>s\}}\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]ds \\
&= \int_t^T \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(u)du}\mathbb{E}[\lambda(s)\mathbb{1}_{\{\tau>s\}}|\mathcal{G}_t]ds \\
&= \int_t^T \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda(u)du}\mathbb{E}[\lambda(s)\overbrace{\mathbb{E}[\mathbb{1}_{\{\tau>s\}}|\mathcal{G}_s]}^{e^{-\int_0^s \lambda(u)du}}|\mathcal{G}_t]ds \\
&= \mathbb{1}_{\{\tau>t\}}\int_t^T \mathbb{E}[\lambda(s)e^{-\int_t^s \lambda(u)du}|\mathcal{G}_t]ds \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\int_t^T -\frac{\partial}{\partial s}e^{-\int_t^s \lambda(u)du}ds \,\middle|\, \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[1 - e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{G}_t\right] \qquad (5.1.2)
\end{aligned}$$

For $t \leq T$, using Lemmata 5.1.2 and 5.1.3 and the auxiliary calculations above leads to

$$\begin{aligned}
&\mathbb{E}[N_T|\mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau\leq T\}}|\mathcal{F}_t] - \int_0^t \lambda(s)\mathbb{1}_{\{\tau>s\}}ds - \mathbb{E}\left[\int_t^T \lambda(s)\mathbb{1}_{\{\tau>s\}}ds \,\middle|\, \mathcal{F}_t\right] \\
&= 1 - \mathbb{E}[\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t] - \int_0^t \lambda(s)\mathbb{1}_{\{\tau>s\}}ds - \int_t^T \mathbb{E}[\lambda(s)\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_t]ds \\
&= 1 - \mathbb{1}_{\{\tau>t\}}\mathbb{E}[e^{-\int_t^T \lambda(u)du}|\mathcal{G}_t] - \int_0^t \lambda(s)\mathbb{1}_{\{\tau>s\}}ds - \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[1 - e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{G}_t\right] \\
&= 1 - \mathbb{1}_{\{\tau>t\}} - \int_0^t \lambda(s)\mathbb{1}_{\{\tau>s\}}ds = N_t
\end{aligned}$$

□

This result opens another possibility to calculate the default probabilities of Lemma 5.1.3. Since the process H is obviously a $(\mathcal{F}_t)$-submartingale and uniformly integrable, the Doob-Meyer decomposition theorem says that there exists a unique, non-decreasing $(\mathcal{F}_t)$-predictable process A, called the compensator process, such that $N_t = H_t - A_t$. Since the previous lemma states exactly such a decomposition, the uniqueness of the compensator process yields $A_t = \int_0^t \lambda(s)ds$. Since N_t is a martingale, one gets

$$\begin{aligned} \mathbb{E}[N_T|\mathcal{F}_t] &= N_t \\ \Leftrightarrow \mathbb{E}[H_T|\mathcal{F}_t] - \mathbb{E}[A_T|\mathcal{F}_t] &= H_t - A_t \\ \Leftrightarrow \mathbb{P}(\tau \leq T|\mathcal{F}_t) &= \mathbb{1}_{\{\tau \leq t\}} + \mathbb{E}[A_T - A_t|\mathcal{F}_t]. \end{aligned}$$

Plugging in the special form of A in our case and using (5.1.2) eventually yields the equalities of Lemma 5.1.3.
In order to construct a default risk model later on, the following assumption will be made from now on:
(A2): For $t \geq 0$ and $\mathcal{G}_\infty = \sigma(\mathcal{G}_t : t \in \mathbb{R}_+)$, the following holds

$$\mathbb{P}[\tau > t|\mathcal{G}_\infty] = \mathbb{P}[\tau > t|\mathcal{G}_t].$$

The next lemma characterizes assumption **(A2)**:

Lemma 5.1.5. *Following statements are equivalent:*

(i) ***A2*** *holds.*
(ii) $\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{G}_t]$ *for all bounded, $\mathcal{G}_\infty$-measurable random variables X.*
(iii) If the process X is a $\mathcal{G}$-martingale, it is also an $\mathcal{F}$-martingale.

Proof. (i)$\Leftrightarrow$ (ii): Fix arbitrary $A \in \mathcal{G}_t$, $u \leq t$ and a bounded, $\mathcal{G}_\infty$-measurable random variable X and define:

$$
\begin{aligned}
I &:= \int_{A\cap\{\tau>u\}} X d\mathbb{P} = \int_A \mathbb{E}[X\mathbb{1}_{\{\tau>u\}}|\mathcal{G}_\infty]d\mathbb{P} \\
&= \int_A X\mathbb{E}[\mathbb{1}_{\{\tau>u\}}|\mathcal{G}_\infty]d\mathbb{P} = \int_A X\mathbb{P}[\{\tau>u\}|\mathcal{G}_\infty]d\mathbb{P}, \\
J &:= \int_{A\cap\{\tau>u\}} \mathbb{E}[X|\mathcal{G}_t]d\mathbb{P} = \int_A \mathbb{1}_{\{\tau>u\}}\mathbb{E}[X|\mathcal{G}_t]d\mathbb{P} \\
&= \int_A \mathbb{E}[\mathbb{1}_{\{\tau>u\}}\mathbb{E}[X|\mathcal{G}_t]|\mathcal{G}_t]d\mathbb{P} = \int_A \mathbb{E}[X|\mathcal{G}_t]\mathbb{E}[\mathbb{1}_{\{\tau>u\}}|\mathcal{G}_t]d\mathbb{P} \\
&= \int_A \mathbb{E}[X\mathbb{E}[\mathbb{1}_{\{\tau>u\}}|\mathcal{G}_t]|\mathcal{G}_t]d\mathbb{P} = \int_A X\mathbb{E}[\mathbb{1}_{\{\tau>u\}}|\mathcal{G}_t]d\mathbb{P} \\
&= \int_A X\mathbb{P}[\{\tau>u\}|\mathcal{G}_t]d\mathbb{P}.
\end{aligned}
$$

Assume (i) holds. Then, looking at the last expressions in the definition of I and J respectively yields $I = J$. Now assume (ii) holds. Because $A \cap \{\tau > u\} \in \mathcal{F}_t$, the definition of the conditional expectation implies $I = J$. Assume conversely $I = J$. Since sets of the form $A \cap \{\tau > u\}$ generate $\mathcal{F}_t$, equality $\int_{A\cap\{\tau>t\}} X d\mathbb{P} = \int_{A\cap\{\tau>t\}} \mathbb{E}[X|\mathcal{G}_t]d\mathbb{P}$ implies (ii). Setting $X \equiv 1$ implies (i).
(ii)⇔(iii): Assume (ii) holds. Let X be a $(\mathcal{G}_t)$-martingale. Then, X_t is $\mathcal{G}_t$-measurable and therefore also $\mathcal{G}_\infty$ and $\mathcal{F}_t$-measurable. Since for $t \le T$

$$
X_t = \mathbb{E}[X_T|\mathcal{G}_t] \overset{(ii)}{=} \mathbb{E}[X_T|\mathcal{F}_t],
$$

X is also an $(\mathcal{F}_t)$-martingale, concluding the "if" direction of the claim.
Now assume (iii) holds and let X be a bounded, $\mathcal{G}_\infty$-measurable random variable. Define $X_t := \mathbb{E}[X|\mathcal{G}_t]$. Since X is bounded, X_t is obviously a bounded, uniformly integrable $\mathcal{G}$-martingale. Then, due to the martingale convergence theorem, $\lim_{t\to\infty} X_t = X_\infty$ exists and is the unique, $\mathcal{G}_\infty$-measurable random variable with the property $X_t = \mathbb{E}[X_\infty|\mathcal{G}_t]$. Since X also fulfils that property, $X = X_\infty$ holds. Because of (iii), X is also a bounded, uniformly integrable $\mathcal{F}$-martingale, so $\mathbb{E}[X|\mathcal{G}_t] = X_t = \mathbb{E}[X_\infty|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t]$, which proves (ii) and also concludes the entire proof. □

The following lemma motivates the construction of an intensity based-model.

Lemma 5.1.6. *Define* $\Lambda(t) = \int_0^t \lambda(s)ds$. *If* $\Lambda(\infty) = \infty$ *holds,* $\Lambda(\tau)$ *is an exponential random variable with parameter 1 independent of* $\mathcal{G}_\infty$.

Proof. Since λ is non-negative, Λ is non-decreasing and, since it is an integral, continuous. Consequently, $\Lambda(\mathbb{R}_+) = \mathbb{R}_+$, which means it is surjective and one can define the $\mathcal{G}_\infty$-measurable right-inverse by

$$g(s) = \inf\{t|\Lambda(t) > s\}.$$

Due to continuity and the non-decreasing property of Λ one gets $\Lambda(g(s)) = s$ and $\Lambda(t) > s \Leftrightarrow t > g(s)$. Consequently

$$\begin{aligned}\mathbb{P}[\Lambda(\tau) < s|\mathcal{G}_\infty] &= 1 - \mathbb{P}[\Lambda(\tau) > s|\mathcal{G}_\infty] = 1 - \mathbb{P}[\tau > g(s)|\mathcal{G}_\infty] \\ &\overset{\textbf{(A2)}}{=} 1 - \mathbb{P}[\tau > g(s)|\mathcal{G}_{g(s)}] \overset{\textbf{(A1)}}{=} 1 - e^{-\Lambda(g(s))} = 1 - e^{-s},\end{aligned}$$

which is exactly the cumulative distribution function of the exponential distribution with parameter 1. $\square$

5.1.2 Construction of a defaultable model

With all the necessary theoretical results in place, one can now construct a model which fulfils conditions **(A1)** and **(A2)**. Starting point is a sigma-field $\mathcal{F}$ and a filtration $(\mathcal{G}_t)_{t\geq 0}$, which satisfies the usual conditions, such that $\mathcal{G}_\infty \subset \mathcal{F}$. Furthermore, let there be a non-negative $\mathcal{G}$-progressive process λ with

$$\int_0^t \lambda(s)ds < \infty$$

almost surely for all $t \in \mathbb{R}_+$. Additionally, let Z be a exponential variable with parameter 1, i.e., $\mathbb{P}(Z > t) = e^{-t}$, which is independent of $\mathcal{G}_\infty$. Define a non-negative, possibly infinite, stopping time by

$$\tau := \inf\left\{t \,\middle|\, \int_0^t \lambda(s)ds \geq Z\right\}.$$

One can now show that **(A1)** and **(A2)** hold in one calculation. Z being independent of $\mathcal{G}_\infty$ yields

$$\mathbb{P}[\tau > t|\mathcal{G}_\infty] = \mathbb{P}\left[Z > \int_0^t \lambda(s)ds \,\middle|\, \mathcal{G}_\infty\right] = e^{-\int_0^t \lambda(s)ds}.$$

Applying the tower property of the conditional expectation eventually gives

$$\mathbb{P}[\tau > t|\mathcal{G}_t] = e^{-\int_0^t \lambda(s)ds},$$

hence **(A1)** and **(A2)** are satisfied. Finally, define $H_t = \mathbb{1}_{\{\tau \leq t\}}$, $\mathcal{H}_t = \sigma(H_s|s \leq t)$ and $\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t$.

5.1.3 Computing default probabilities and pricing with default risk

Up till now, all results were stated w.r.t. to the physical measure $\mathbb{P}$. As it is known, for pricing, a risk-neutral measure is needed. Since in general, assumptions **(A1)** and **(A2)** are not preserved under an equivalent change of measure (for details, see [12, Chapter 12.3.4]), one needs to explicitly assume that **(A1)** and **(A2)** are satisfied for a measure $\mathbb{Q} \sim \mathbb{P}$. Furthermore it is supposed that there exists a process r and a non-negative process $\lambda^{\mathbb{Q}}$, both being $(\mathcal{G}_t)$-progressive, which represent the interest rate and intensity respectively, with

$$\int_0^t |r(s)| + \lambda^{\mathbb{Q}}(s)ds < \infty \quad \mathbb{Q} - a.s..$$

Default probabilities

Recall the formula for the conditional default probability

$$\mathbb{Q}[\tau > T|\mathcal{F}_t] = \mathbb{1}_{\{\tau > T\}}\mathbb{E}[e^{-\int_t^T \lambda(s)ds}|\mathcal{G}_t].$$

One could decide to use one of the positive stochastic processes used for interest rate modelling, such as the CIR or CIR++ model, to describe the dynamics of the process $\lambda^{\mathbb{Q}}$, where the process $W(t)$, which drives the random shocks, is assumed to be a Brownian motion w.r.t. to $\mathcal{G}$. Then, the conditional expectation in the expression above is equal to the bond price, and is therefore explicitly given. In mathematical terms

$$\mathbb{Q}[\tau \leq T|\mathcal{F}_t] = \begin{cases} 1 - P_{MODEL}(t,T), & \tau > t, \\ 1, & else. \end{cases}$$

In the case of CIR and CIR++, $P_{MODEL}(t,T)$ corresponds to (3.3.2) and (3.5.2) respectively.
The goal is to determine the price of a defaultable bond under different recovery assumptions, namely zero recovery and recovery of market value. It is also possible to assume a partial recovery at default or at maturity of the bond, but these cases are not going to be dealt with in this thesis.

Pricing with zero recovery

Zero recovery means that, in the event of default, the bond becomes worthless and no payment at all is made. The payoff of this bond at maturity T is $\mathbb{1}_{\{\tau>T\}}$. Because of the results stated in Chapter 2.3, the no-arbitrage price at time t is given by

$$\bar{P}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\mathbb{1}_{\{\tau>T\}} \,\middle|\, F_t\right].$$

Using the theoretical framework established in the previous section, especially Lemma 5.1.2, the above expression can be simplified to

$$\begin{aligned} \bar{P}(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\mathbb{1}_{\{\tau>T\}}\mathbb{1}_{\{\tau>t\}} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda^{\mathbb{Q}}(s)ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\mathbb{1}_{\{\tau>T\}} \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda^{\mathbb{Q}}(s)ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}} \,\middle|\, \mathcal{G}_T\right] \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}e^{\int_0^t \lambda^{\mathbb{Q}}(s)ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}e^{-\int_0^T \lambda^{\mathbb{Q}}(s)ds} \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda^{\mathbb{Q}}(s)ds} \,\middle|\, \mathcal{G}_t\right]. \end{aligned} \tag{5.1.3}$$

This means that in the zero recovery case, a defaultable bond can be priced by discounting by the sum of the risk-free rate r and a **spread** $\lambda^{\mathbb{Q}}$. For certain dynamics of r and $\lambda^{\mathbb{Q}}$, one can just apply the theory developed in the chapters on interest rate models, which can be seen in the following example:

Proposition 5.1.7. *Assume the interest rate dynamics are given by a Cox-Ingersoll-Ross process, i.e.,*

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t), \qquad r(0) = r_0,$$

with $r_0, k, \theta, \sigma > 0$*, where* $W(t)$ *is a Brownian motion w.r.t. to* $(\mathbb{Q}, \mathcal{G}_t)$*. Let* $\lambda^{\mathbb{Q}} = c_0 + c_1 r(t)$ *be an affine function in* $r(t)$ *with non-negative constants* c_0, c_1*. Then the zero-recovery defaultable bond price at time* t *with maturity* T *is given by*

$$\bar{P}(t,T) = \mathbb{1}_{\{\tau>t\}}\bar{A}(t,T)e^{-\bar{B}(t,T)r(t)},$$

where

$$\begin{aligned}
\bar{A}(t,T) &= e^{-c_0(T-t)}\left(\frac{2h\exp\left(\frac{1}{2}(k+h)(T-t)\right)}{2h+(k+h)(\exp\{(T-t)h\}-1)}\right)^{\frac{2k\theta(1+c_1)}{\sigma^2}}, \\
\bar{B}(t,T) &= (1+c_1)\frac{2\left(\exp\{(T-t)h\}-1\right)}{2h+(k+h)(\exp\{(T-t)h\}-1)}, \\
h &= \sqrt{k^2+2(1+c_1)\sigma^2}.
\end{aligned}$$

Proof. Inserting the model into the general bond price formula yields

$$\begin{aligned}
\bar{P}(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)+\lambda^{\mathbb{Q}}(s)ds}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T c_0+(1+c_1)r(t)ds}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{-\int_t^T c_0 ds}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (1+c_1)r(t)ds}\,\middle|\,\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}e^{-c_0(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T \tilde{r}(t)ds}\,\middle|\,\mathcal{G}_t\right]. \quad (*)
\end{aligned}$$

Since

$$\begin{aligned} d(1+c_1)r(t) &= k((1+c_1)\theta - (1+c_1)r(t))dt + (1+c_1)\sigma\sqrt{r(t)}dW(t) \\ \Leftrightarrow d\bar{r}(t) &= k((1+c_1)\theta - \bar{r}(t))dt + \sqrt{(1+c_1)}\sigma\sqrt{\bar{r}(t)}dW(t), \end{aligned}$$

$\bar{r}(t)$ is also a CIR process, just with parameters $\bar{k} = k, \bar{\theta} = \theta(1+c_1), \bar{\sigma} = \sqrt{(1+c_1)}\sigma$, which means that the expectation in $(*)$ is nothing else than the default-free bond price of the CIR process $\bar{r}(t)$. Therefore, one can use the explicit formula (3.3.2), which immediately proofs the claim. □

Pricing with recovery of market value

In a lot of cases, even if a company defaults, a claim does not become completely worthless. Therefore, a pricing formula under the recovery of market value assumption will be derived in this section. The derivation is based on the works of Duffie & Singleton [9].

Consider a defaultable claim with terminal payoff X. Assume that upon default, the claim holder receives a payment X'. The price of that claim at time t, assuming default has not yet occurred, equals

$$V_t := \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}e^{-\int_t^T r(s)ds}X + \mathbb{1}_{\{\tau\leq T\}}e^{-\int_t^\tau r(s)ds}X' \,\middle|\, \mathcal{F}_t\right],$$

and is assumed to be continuous. Further, it will be postulated, that $X' = (1-L)V_{\tau-}$, i.e., upon default the claim pays a fraction of its market value just before default occured. $L \in [0,1]$ represents the fractional loss given default and $V_{\tau-} = \lim_{s\nearrow\tau} V_s$. Under mild conditions (see [9]),

$$V_t = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s + \lambda_s^{\mathbb{Q}} L ds\right)X \,\middle|\, \mathcal{F}_t\right] \tag{5.1.4}$$

holds. This means, that the claim is priced as in the default-free case, except that discounting happens with respect to an additional credit spread $\lambda_s^{\mathbb{Q}}L$ to the interest rate r_t. Consequently, after choosing suitable models for r and λ^Q, the same methods of simulation can be performed as in the default-free case to price contingent claims.

To proof this formula in a somewhat informal way, recall Proposition 5.1.4, which states that the dynamics of the default indicator process H_t can be written as

$$dH_t = \lambda_t^{\mathbb{Q}}(1 - H_t)dt + dN_t,$$

where N_t is a martingale with respect to $\mathcal{F}$ under the risk-neutral measure $\mathbb{Q}$. Let further $dV_t = \alpha_t dt + dM_t$ be the Doob-Meyer decomposition of V_t, where M_t is another martingale under $\mathbb{Q}$. A priori, α_t does not need to be absolutely continuous, but in the course of our reasoning it will be shown that indeed it is. Let G_t be the discounted gain process of the claim, i.e.,

$$G_t = \exp\left(-\int_0^t r_s ds\right) V_t(1 - H_t) + \int_0^t \exp\left(-\int_0^s r_u du\right)(1 - L)V_{s-}dH_s.$$

where the first expression represents the discounted price of the claim and the second stands for the discounted payoff if default occurs. Using Itô's formula yields:

$$\begin{aligned} dG_t = &- \exp\left(-\int_0^t r_s ds\right) r_t V_t(1 - H_t)dt + \exp\left(-\int_0^t r_s ds\right)(1 - H_t)dV_t \\ &+ \exp\left(-\int_0^t r_s ds\right) V_t d(1 - H_t) + \exp\left(-\int_0^t r_s ds\right)(1 - L)V_{t-}dH_t, \end{aligned}$$

since all the covariation/quadratic variation terms except $[V]_t$ are zero, because $1 - H_t$ and $\exp\left(-\int_0^t r_s ds\right)$ are of finite variation. However, $[V]_t$ is irrelevant, since $\frac{\partial^2}{\partial v^2} f(t, v, h) = \frac{\partial^2}{\partial v^2} \exp\left(-\int_0^t r_s ds\right) v(1 - h) = 0$. Inserting the expressions for dH_t and dV_t leads to

$$\begin{aligned} dG_t = &- \exp\left(-\int_0^t r_s ds\right)\Big[r_t V_t(1 - H_t) - (1 - H_t)\alpha_t + V_t(1 - H_t)\lambda_t^{\mathbb{Q}} \\ &-(1 - L)V_t(1 - H_t)\lambda_t^{\mathbb{Q}}\Big] dt \\ &+ \underbrace{\exp\left(-\int_0^t r_s ds\right)[(1 - H_t)dM_t - V_t dN_t + (1 - L)V_t dN_t]}_{dM_t^*}, \end{aligned}$$

where $V_{t-} = V_t$ holds because of the continuity assumption and M_t^* is a martingale under $\mathbb{Q}$, because it is the sum of stochastic integrals with respect to martingales, where the integrands are L^1-integrable. Since the gains process is a martingale and the Doob-Meyer decomposition is unique, the drift needs to be zero, i.e. $(r_t V_t - \alpha_t + L V_t \lambda_t^{\mathbb{Q}})(1 - H_t) = 0$. Since $t < \tau$, the second term equals one, which implies

$$\begin{aligned} 0 &= r_t V_t - \alpha_t + L\lambda_t^{\mathbb{Q}} \\ \alpha_t &= V_t(r_t + L\lambda_t^{\mathbb{Q}}). \end{aligned}$$

Since under the risk-neutral measure, drift and risk-free rate need to coincide, discounting for the defaultable claim has to be done by $r_t + L\lambda_t^{\mathbb{Q}}$, which mathematically speaking yields our conjecture (5.1.4).

5.2 Credit default swaps

One way of calibrating the intensity process $\lambda^{\mathbb{Q}}$ to the market, is to use credit default swap (CDS) data. Therefore, this chapter is devoted to this kind of financial contracts. The upcoming formulas are not going to be derived in great detail, so the interested reader is therefore referred to [6, Chapter 21.1, 21.3 , 22.3], on which also the notation and structure is based.

A credit default swap (CDS) is a derivative which protects against the the event of default. The protection buyer (A) agrees to make regular payments, so called premiums, to the protection seller (B), which in return agrees to make a single (deterministic) payment, which here will be called loss given default L, to the buyer in case of the default of a reference obligor (C). Under which conditions (C) is considered to be defaulted needs to be specified in the contract, and is definitely not limited to bankruptcy or liquidation. The regular payments are usually denoted by basis points with respect to the notional amount. Contrary to a credit insurance, (A) does not need to be a creditor of the (C). This means that a CDS can not only be used for hedging purposes, but also for speculation, which makes it potentially dangerous. Incidentally, these contracts contributed to the emergence of the

global financial crisis in 2008.
To put the definition above into mathematical terms, let T_a be the starting date of the contract and $T_{a+1}, \cdots, T_b$ the dates where premium payment rates R are made. The usual time span between two payment dates is three months. These financial obligations of (A) are called the premium leg of the CDS. In case of default at time $T_a < \tau \leq T_b$, premium payments stop and (B) is obliged to pay the amount L to (A). This is called the protection leg of the CDS. Let $T_{\beta(t)}$ be the next payment date following t, i.e., $t \in [T_{\beta(t)-1}, T_{\beta(t)})$. Then, as seen from (B), the discounted value process of the CDS at time t is given by

$$\begin{aligned}\Pi_{CDS_{a,b}}(t) := \mathbb{1}_{\{\tau>t\}} \Bigg(& e^{-\int_t^\tau r_s ds}(\tau - T_{\beta(t)-1}) R \mathbb{1}_{\{T_a<\tau<T_b\}} \\ & + \sum_{i=a+1}^{b} e^{-\int_t^{T_i} r_s ds} \alpha_i R \mathbb{1}_{\{\tau \geq T_i\}} - \mathbb{1}_{\{T_a<\tau\leq T_b\}} e^{-\int_t^\tau r_s ds} L \Bigg), \end{aligned} \tag{5.2.1}$$

where $\alpha_i = T_i - T_{i-1}$. For pricing purposes, recall that the stochastic framework still consists of a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{Q})$, where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ denotes the filtration of the complete market information, the filtration $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0} \subset \mathcal{F}$ contains all non default-related information and $\mathbb{Q}$ is the risk-neutral probability measure. Denote the price of a CDS at time t with input data as above with

$$CDS_{a,b}(t, R, L) := \mathbb{E}_{\mathbb{Q}}\left[\Pi_{CDS_{a,b}}(t) \,\middle|\, \mathcal{F}_t\right]$$

Lemma 5.1.2 together with (5.2.1) yields

$$\begin{aligned} CDS_{a,b}(t, R, L) =& \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{G}_t)} \mathbb{E}_{\mathbb{Q}}\left[\Pi_{CDS_{a,b}}(t) \,\middle|\, \mathcal{G}_t\right] \\ =& \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t|\mathcal{G}_t)} \mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{1}_{\{\tau>t\}} \Bigg(e^{-\int_t^\tau r_s ds}(\tau - T_{\beta(t)-1}) R \mathbb{1}_{\{T_a<\tau<T_b\}} \\ & + \sum_{i=a+1}^{b} e^{-\int_t^{T_i} r_s ds} \alpha_i R \mathbb{1}_{\{\tau \geq T_i\}} - \mathbb{1}_{\{T_a<\tau\leq T_b\}} e^{-\int_t^\tau r_s ds} L \Bigg) \Bigg| \mathcal{G}_t \Bigg] \end{aligned}$$

$$
\begin{aligned}
=&\frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau>t|\mathcal{G}_t)}\left(R\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>t\}}e^{-\int_t^\tau r_s ds}(\tau-T_{\beta(t)-1})\mathbb{1}_{\{T_a<\tau<T_b\}}\,\middle|\,\mathcal{G}_t\right]\right.\\
&+\sum_{i=a+1}^{b}\alpha_i R\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_i} r_s ds}\mathbb{1}_{\{\tau\geq T_i\}}\,\middle|\,\mathcal{G}_t\right]\\
&\left.-L\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{T_a<\tau\leq T_b\}}e^{-\int_t^\tau r_s ds}\,\middle|\,\mathcal{G}_t\right]\right).
\end{aligned}
$$

The CDS forward rate $R_{a,b}(t)$ is now defined as the rate which satisfies $CDS_{a,b}(t, R_{a,b}(t), L) = 0$. It is important to note that on the market, CDS contracts are quoted by those CDS forward rates. As the quotes represent the regular premium payments, the higher this rate is, the less creditworthy the market considers the particular company. Assuming now that interest rate and default are independent, one can derive valuation formulas for the premium and protection leg at time zero which just depend on R, L, the initial bond curve and the initial survival probabilities. The premium leg formula is given by

$$
\begin{aligned}
PremLeg_{a,b}(R, P(0,\cdot), \mathbb{Q}(\tau>\cdot)) =&R\left(-\int_{T_a}^{T_b} P(0,t)(\tau-T_{\beta(t)-1})d\{\mathbb{Q}(\tau>t)\}_t\right.\\
&\left.+\sum_{i=a+1}^{b} P(0,T_i)\alpha_i\mathbb{Q}(\tau>T_i)\right),
\end{aligned}
$$

where the integral is a Stieltjes integral in the survival probabilities (for the derivation see [6, Chapter 21.3] . The protection leg formula looks as follows:

$$
ProtLeg_{a,b}(L, P(0,\cdot), \mathbb{Q}(\tau>\cdot)) = -L\int_{T_a}^{T_b} P(0,t)d\{\mathbb{Q}(\tau>t)\}_t.
$$

For the exact derivation see [6, Chapter 21.3]. Given the CDS forward rates $R_{0,b}^M(0)$ observed on the market for different maturities T_b, it is possible to iteratively strip survival probabilities from the market by solving

$$
PremLeg_{0,b}(R_{0,b}^M(0), P(0,\cdot), \mathbb{Q}(\tau>\cdot)) = ProtLeg_{0,b}(L, P(0,\cdot), \mathbb{Q}(\tau>\cdot)). \tag{5.2.2}
$$

For example, if the available maturities are $\{1y, 3y, 5y\}$, first solve (5.2.2) for $T_b = 1y$ to get the implied survival probabilities for $t \leq 1y$. Then, insert them in (5.2.2) for $T_b = 2y$ to retrieve $\mathbb{Q}(\tau \geq t)$ for $1 < t \leq 2$ and so on. A priori, this method is model independent. In the given setting, assuming **(A1)**, $\mathbb{Q}(\tau > t) = e^{-\int_0^t \lambda_s^{\mathbb{Q}} ds}$ holds. This leads to the following special case of above formulas:

$$\begin{aligned}
CDS_{a,b}(0, R, L, \Gamma(\cdot)) =& PremLeg_{a,b}(R, P(0,\cdot), \Gamma(\cdot)) - ProtLeg_{a,b}(L, P(0,\cdot), \Gamma(\cdot)) \\
=& R\left(\int_{T_a}^{T_b} P(0,t)(T_{\beta(t)-1} - t) de^{-\Gamma(t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i e^{-\Gamma(T_i)}\right) \\
&+ L\left(\int_{T_a}^{T_b} P(0,t) de^{-\Gamma(t)}\right), \qquad (5.2.3)
\end{aligned}$$

where $\Gamma(t) = \int_0^t \lambda_s^{\mathbb{Q}} ds$. To be able to strip survival probabilities, a deterministic setup is in order. Let $\lambda_t^{\mathbb{Q}} = \lambda_i$, $t \in [T_{i-1}, T_i)$, $\lambda_i \in \mathbb{R}^+$ be piecewise constant (they could also be assumed to be piecewise linear). Plugging this in (5.2.3) yields

$$\begin{aligned}
&CDS_{a,b}(0, R, L, \Gamma(\cdot)) \\
&= R \sum_{i=a+1}^{b} \lambda_i \int_{T_{i-1}}^{T_i} \exp\left(-\sum_{j=1}^{i-1} \lambda_j (T_j - T_{j-1}) - \lambda_i (t - T_{i-1})\right) P(0,t)(t - T_{i-1}) dt \\
&+ R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i e^{-\Gamma(T_i)} \\
&- L \sum_{i=a+1}^{b} \lambda_i \int_{T_{i-1}}^{T_i} \exp\left(-\sum_{j=1}^{i-1} \lambda_j (T_j - T_{j-1}) - \lambda_i (t - T_{i-1})\right) P(0,t) dt.
\end{aligned}$$

As before, by inserting the observed CDS rates $R_{0,b}^M(0)$ in the formula above, the λ_i and therefore the survival probabilities can be iteratively stripped from the market. If, as usual, the time span between T_i is three months and maturities are available yearly, in a first step $CDS_{0,1}(0, R_{0,1}^M(0), L, \{\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4\}) = 0$ is solved, and in each further step, four new λ_i can be calculated.

5.3 Specific model choice and practical example

In this last section, some specific choices for modelling the interest rate and the intensity process are going to be made. At the end, a small practical example will illustrate some of the theoretical results.
The presented calibration procedure for the CIR++ model was proposed by [6, Chapter 22.7]. The same stochastic framework as in the above sections of this chapter is assumed, i.e., **(A1)** and **(A2)** are satisfied for a risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$. Furthermore, there exists an interest rate process r and a non-negative intensity process $\lambda^{\mathbb{Q}}$, both being $(\mathcal{G}_t)$-progressive, with

$$\int_0^t |r(s)| + \lambda^{\mathbb{Q}}(s) ds < \infty \quad \mathbb{Q} - a.s..$$

5.3.1 Model choice and calibration methods

The interest rate dynamics follow the G2++ model, i.e.

$$\begin{aligned} r(t) &= x(t) + y(t) + \varphi(t), \quad r(0) = r_0 \\ dx(t) &= -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0 \\ dy(t) &= -by(t)dt + \eta dW_2(t), \quad y(0) = 0, \end{aligned} \tag{5.3.1}$$

where $r_0, a, b, \sigma, \eta \in \mathbb{R}^+$ and (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation $\rho \in [-1, 1]$, i.e., $dW_1 dW_2 = \rho dt$. The deterministic function φ is used to fit the model to the current term-structure of discount factors observed on the market.
Since the intensity process $\lambda(t)$ needs to be positive, it will be modelled by a CIR++ process, i.e.,

$$\begin{aligned} \lambda(t) &= z(t) + \psi(t), \\ dz(t) &= k(\theta - z(t))dt + \nu\sqrt{z(t)}dW_3(t), \qquad z(0) = z_0, \end{aligned}$$

with $z_0, k, \theta, \nu > 0$. The process stays positive for $2k\theta > \nu^2$ and $\psi(t) \geq 0$. The interest rate process and intensity process can of course be correlated, for example by the instantaneous correlations $\rho_{1,3}, \rho_{2,3} \in [-1, 1]$ between the involved Brownian motions, i.e., $dW_i dW_3 = \rho_{i,3} dt, \ i = 1, 2$. This would result in an instantaneous correlation between r and λ of

$$\bar{\rho} = \frac{\sigma\rho_{1,3} + \eta\rho_{2,3}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho}}.$$

Since the method for stripping the survival probabilities requires the correlation between the interest rate and the intensity to be zero, this assumption is also necessary for this calibration procedure presented below. However, Brigo & Mercurio showed in [6, Chapter 22.7], that the influence of the correlation on CDS prices is negligible, so that one can first calibrate the model assuming that no correlation is present and then set it to a value of one's choice. Be it how it may, in the following $\rho_{1,3}, \rho_{2,3} = 0$ is going to be assumed, so that r and λ are independent.

The deterministic function ψ is going to be used to calibrate the CIR++ model to the market survival probabilities $\mathbb{Q}(\tau > t)_M = e^{-\Gamma^M(t)}$, where $\Gamma^M(t) = \int_0^t \lambda_s^M ds = \sum_{i=1}^{\beta(t)-1} (T_i - T_{i-1})\lambda_i^M + (t - T_{\beta(t)-1})\lambda_{\beta(t)}^M$. The piecewise constant intensities $\lambda_t^M = \lambda_i^M$, $t \in [T_{i-1}, T_i)$, $\lambda_i \in \mathbb{R}^+$ are stripped from the market by applying the procedure described at the end of the previous Chapter 5.2. To do this, $\mathbb{Q}(\tau > t)^{CIR} = e^{-\Gamma^M(t)}$ must hold. This leads to

$$\begin{aligned}
e^{-\Gamma^M(t)} &= \mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_0^t z(s) + \psi(s)ds)\right] \\
\Leftrightarrow e^{-\Gamma^M(t)} &= \exp(-\int_0^t \psi(s)ds)\mathbb{E}^{\mathbb{Q}}\left[\exp(-\int_0^t z(s)ds)\right] \\
\Leftrightarrow e^{-\Gamma^M(t)} &= \exp(-\int_0^t \psi(s)ds) P^{CIR}(0,t) \\
\Leftrightarrow \int_0^t \psi(s)ds &= \Gamma^M(t) + \ln P^{CIR}(0,t) \qquad (5.3.2) \\
\Leftrightarrow \psi(t) &= \lambda^M(t) - f^{CIR}(0,t), \qquad (5.3.3)
\end{aligned}$$

where $P^{CIR}(t,T) = A(t,T)e^{-B(t,T)z(t)}$ and $A(t,T), B(t,T)$ are given by (3.3.2) and $f^{CIR}(0,t)$[1] is given by (3.5.1).
Since the choice of the deterministic shift according to (5.3.3) ensures that the CIR++ model matches the survival probabilities stripped from the market, the parameter vector $\beta = \{k, \theta, \nu, z_0\}$ can still be chosen. When considering interest rate models, the parameters can be calibrated to the cap/swaption volatilities observed on the market. Applying that to intensity models, one could calibrate the vector β to some kind of CDS options. However, as pointed out in [7, Chapter 5], single-name CDS options are not liquid enough in the market, which makes them unsuitable for calibration purposes at the moment. Therefore, in [6, Chapter 22] the following, somewhat heuristic, approach is suggested: Choose β such that $\Psi(t) := \int_0^T \psi(s;\beta)ds$ is positive and increasing (to make sure that the CIR++ process stays positive), and such that it minimizes $\int_0^T \psi(s;\beta)^2 ds$. In that way, one can find the parameters for the underlying CIR process z^β, which is closest to the corresponding calibrated CIR++ process and therefore closest to the CDS data observed on the market. This is also the approach which will be used in the upcoming example. However, it is worth mentioning that there are other approaches: In [7, Chapter 5] for example, parameters are chosen such that the implied volatilities of hypothetical CDS options yield possibly reasonable values.
Finally, one can state the defaultable bond prices within this framework. Under the assumption of zero recovery and using (5.1.3), the defaultable T-bond

[1] In the given case:

$$
\begin{aligned}
A(t,T) &= \left[\frac{2h\exp\left(\frac{1}{2}(k+h)(T-t)\right)}{2h+(k+h)(\exp\{(T-t)h\}-1)}\right]^{\frac{2k\theta}{\nu^2}}, \\
B(t,T) &= \frac{2\left(\exp\{(T-t)h\}-1\right)}{2h+(k+h)(\exp\{(T-t)h\}-1)}, \\
f^{CIR}(0,t) &= \frac{2k\theta(e^{th}-1)}{2h+(k+h)(e^{th}-1)} + z_0\frac{4h^2e^{th}}{\left(2h+(k+h)(e^{th}-1)\right)^2} \\
h &= \sqrt{k^2+2\nu^2},
\end{aligned}
$$

price at $t = 0$ is given by

$$\begin{aligned}\bar{P}(0,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)+\lambda^{\mathbb{Q}}(s)ds}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T \lambda^{\mathbb{Q}}(s)ds}\right] \\ &= P^{G2++}(0,T)\mathbb{Q}(\tau > T). \end{aligned} \tag{5.3.4}$$

This means that to price a defaultable bond within this framework, one only needs to multiply the default-free bond price by the survival probability, which is a very intuitive result. Since the interest rate was calibrated to the initial term-structure and the intensity to the survival probabilities, (5.3.4) simplifies to

$$\bar{P}(0,T) = P^M(0,T)e^{-\Gamma^M(t)}.$$

If recovery of market value with fractional loss given default $L \in [0,1]$ is assumed, (5.1.4) with $X \equiv 1$ (bond pays 1 at maturity) leads to

$$\begin{aligned}\bar{P}^{RMV}(0,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\right]\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T L\lambda^{\mathbb{Q}}(s)ds}\right] \\ &= P^{G2++}(0,T)P^{CIR++}(0,T;\bar{\beta}),\end{aligned}$$

where the second factor of the result is the bond price in the CIR++ model with parameter vector $\bar{\beta} = \{k, L\theta, \sqrt{L}\nu, Lz_0\}$ and deterministic shift $\bar{\psi} = L\psi$. The formula for the CIR++ bond-price is given by (3.5.2).

Finally, it is worth mentioning, that the specific choice of the G2++ model for the interest rate and the CIR++ model for the intensity does not produce a known distribution for the expression $\int_t^T r(s) + \lambda^Q(s)ds$. Therefore, it is not possible to calculate expectations such as (5.1.3) or (5.1.4) explicitly if $r(t)$ and $\lambda^Q(t)$ are correlated. Consequently, if there is non-zero correlation, one needs to use simulation to determine the prices of contingent claims.

5.3.2 Market data and application

In this last section, current market data will be used to illustrate the stripping and calibration procedures from the previous section. Unfortunately, as already mentioned in Chapter 4.5, no current market data for cap prices could be acquired. Since it was desirable to at least fit the G2++ model to the initial yield curve seen in Figure 2.1, dummy-data had to be used for the cap prices. To not choose the data completely arbitrarily, the cap prices in Table 5.1, calculated from the cap volatilities and the yield curve in [11, Chapter 2], were taken.

Maturity (years)	Cap prices	Maturity (years)	Cap prices
1	0.02281	9	4.90252
2	0.11092	10	5.93530
3	0.30154	12	8.02445
4	0.69595	15	10.93077
5	1.29645	20	14.81939
6	2.04150	25	18.27075
7	2.92555	30	21.37741
8	3.87695		

Table 5.1: At-the-money euro cap prices calculated from data in [11, Chapter 2] with notional=100

The data was retrieved in August 2014, when the interest rates were not yet negative, but already really low, about 0.7 percentage points above the current yield curve for all maturities. The data might therefore be a little closer to the actual values than the data of Brigo & Mercurio. Nevertheless, Figure 5.4 still serves a purely illustratory purpose. However, the rest of the figures are consistent with the market, as the yield curve and the CDS data is actual market data.
Calibration to the dummy cap prices resulted in the following parameters:

$$
\begin{aligned}
&a = 0.796545671517210,\ b = 0.985476920923802,\\
&\sigma = 0.293508841864317,\ \eta = 0.345431709490712\\
&\rho = -0.998852525708897.
\end{aligned}
$$

The CDS data consists of senior[2] and subordinated CDS MID-quotes of the reinsurance company Munich Re. It was retrieved from the Thomson Reuters Datastream. MID-quotes are the mean of the bid and the ask CDS quotes. As mentioned in Chapter 5.2, when speaking of CDS-quotes, one always refers to the CDS forward rate $R_{0,b}(0)$, where b is the maturity, which in this case ranges from half a year to 30 years and can be seen in Table 5.2.

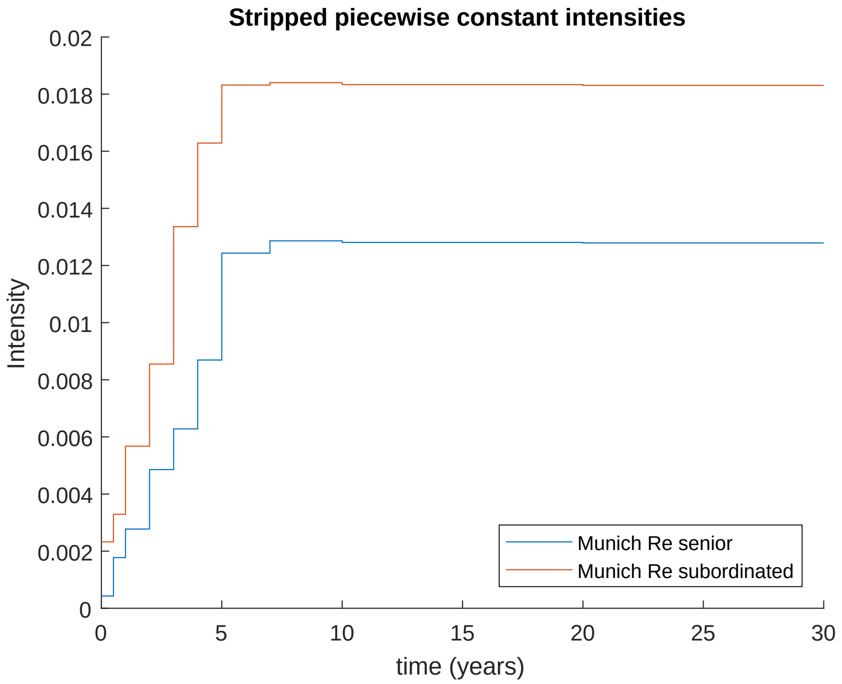
Maturity (years)	Munich Re senior	Munich Re subordinated
0.5	2.57	18.59
1	6.61	22.45
2	11.63	33.92
3	17.46999	45.39
4	22.50999	60.62999
5	28.37999	74.28
7	41.23999	94.09999
10	51.37999	108.79
20	62.53999	124.9
30	66.00999	129.84

Table 5.2: Senior and subordinated CDS MID-quotes (basis points) of Munich Re on July 31st, 2017 (Source: Thomson Reuters Datastream).

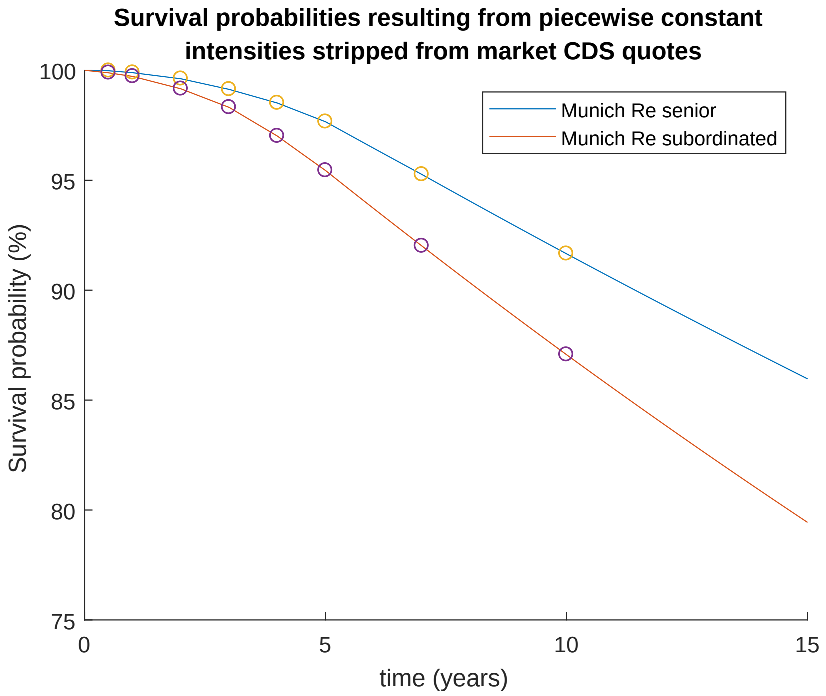
Since senior debt has priority over subordinated debt, senior CDS quotes are much lower. Survival probabilities can now be stripped from the given CDS quotes as described at the end of Chapter 5.2. The recovery rate, $Rec = 1-L$, for senior CDS is assumed to be 40 percent, which is also used by [6] and according to [19] is the standard value. The standard value for the recovery rate of subordinated CDS is 20 percent. The results are shown in Figure 5.1(a) and Figure 5.1(b).

The survival probabilities might seem low for a solid company like Munich Re and it might seem odd to have different survival probabilities for the same company. The reason for this is, that the event of default does not just comprise bankruptcy or liquidation, but all events of default specified in the CDS contract. Those contractual default events can range from bankruptcy

[2]In case of default, so-called senior debt has priority over so-called subordinated debt, i.e., it is paid first when a company is bankrupt or liquitated.



(a) Piecewise constant intensities stripped from CDS quotes on July 31st, 2017



(b) Survival probabilities resulting from piecewise constant intensities

Figure 5.1: MunichRe default analysis

to mere delayed repayment. Therefore, those survival probabilities should be interpreted as the likelihood of the occurrence of a default event within the specific CDS contract from which they were stripped.
The next step is to calibrate the data to a CIR++ process with the parameter vector $\beta = \{k, \theta, \nu, z_0\}$, such that $\Psi(t) := \int_0^T \psi(s;\beta)ds$ is positive and increasing, and such that it minimizes $\int_0^T \psi(s;\beta)^2 ds$, as described in Chapter 5.3.1. The parameters were calibrated to the senior CDS quotes using the Matlab function **fmincon**, which determines the minimum of a constrained non-linear multi-variable function, and resulted in the parameters

$$k = 0.051494646843855,\ \theta = 0.013409534075701,$$
$$\nu = 0.022655209530356,\ z_0 = 1.031332641842782 \times 10^{-4}.$$

It is worth noting, that in this case, the Feller condition is just barely satisfied. However, even if the calibration results violated the Feller condition, the resulting CIR++ process would still be strictly positive, because the calibration procedure makes sure that the shift is strictly positive. The corresponding shift ψ and integrated shift Ψ are shown in Figure 5.2 and a sample path of the resulting CIR++ process as well as its underlying CIR process in Figure 5.3. To avoid confusion, notice that those two figures feature different end points on the right of the time axis. As one can see, the CIR++ process reaches values up to $1,5\%$, which in the current interest rate environment can be considered quite a large spread.
Since both the G2++ process and the CIR++ process are now fully specified, one can compare the risk-free rate $r(t)$ with the interest rate with added spread $r(t) + \lambda^{\mathbb{Q}}(t)$, which is used for discounting in the defaultable case. Figure 5.4 shows a sample path of this scenario. As already mentioned, the spread is not insignificant, but the G2++ path is still not suitable for any kind of interpretation, since dummy cap prices were used for the calibration.

Using (5.3.4) yields a zero-recovery defaultable bond price term-structure, which is shown in Figure 5.5 compared to the risk-free bond curve obtained corresponding to the risk free term-structure. As expected, the defaultable

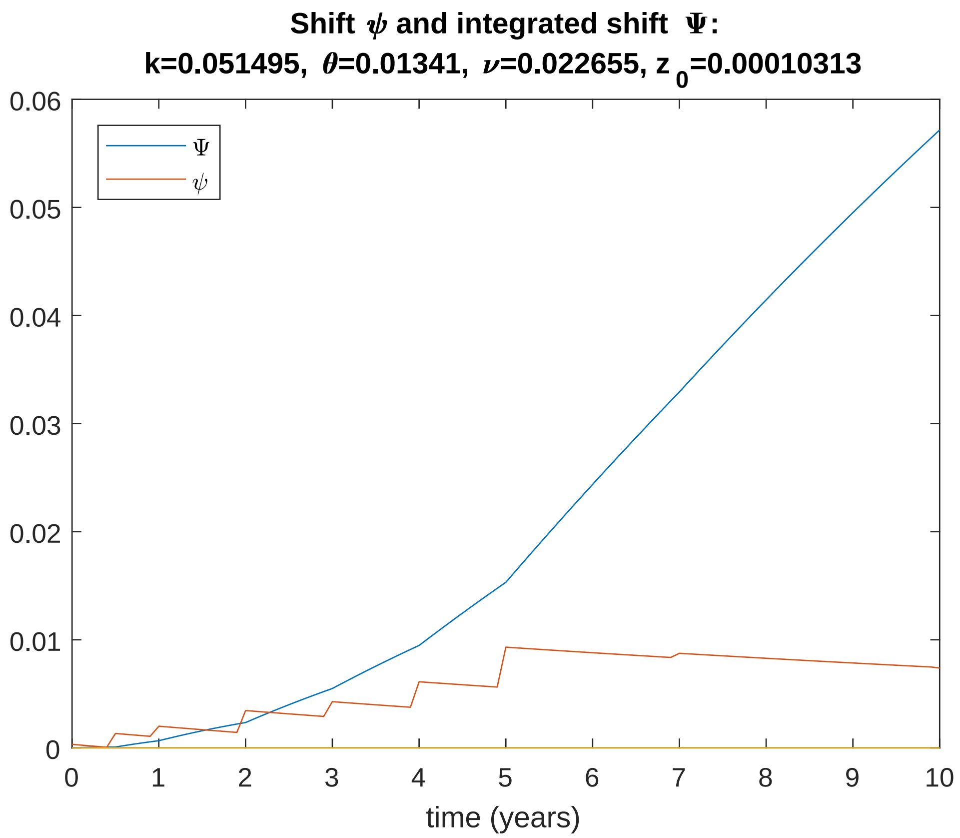


Figure 5.2: Shift and integrated shift of the calibrated CIR++ process

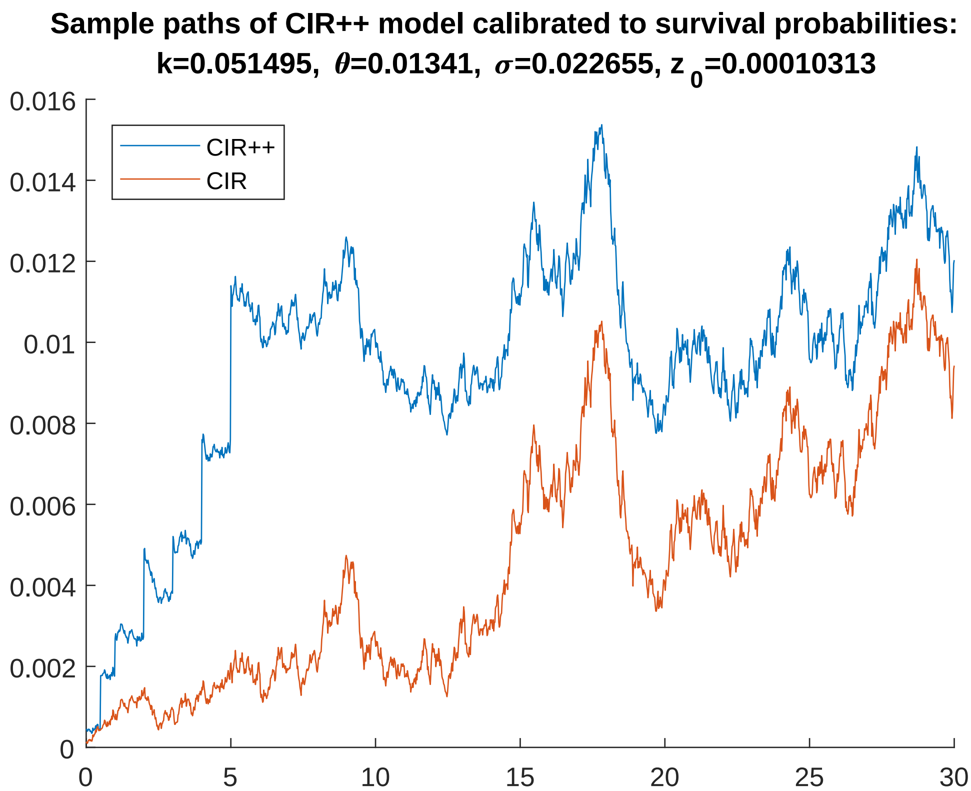


Figure 5.3: Calibrated CIR++ process and underlying CIR process

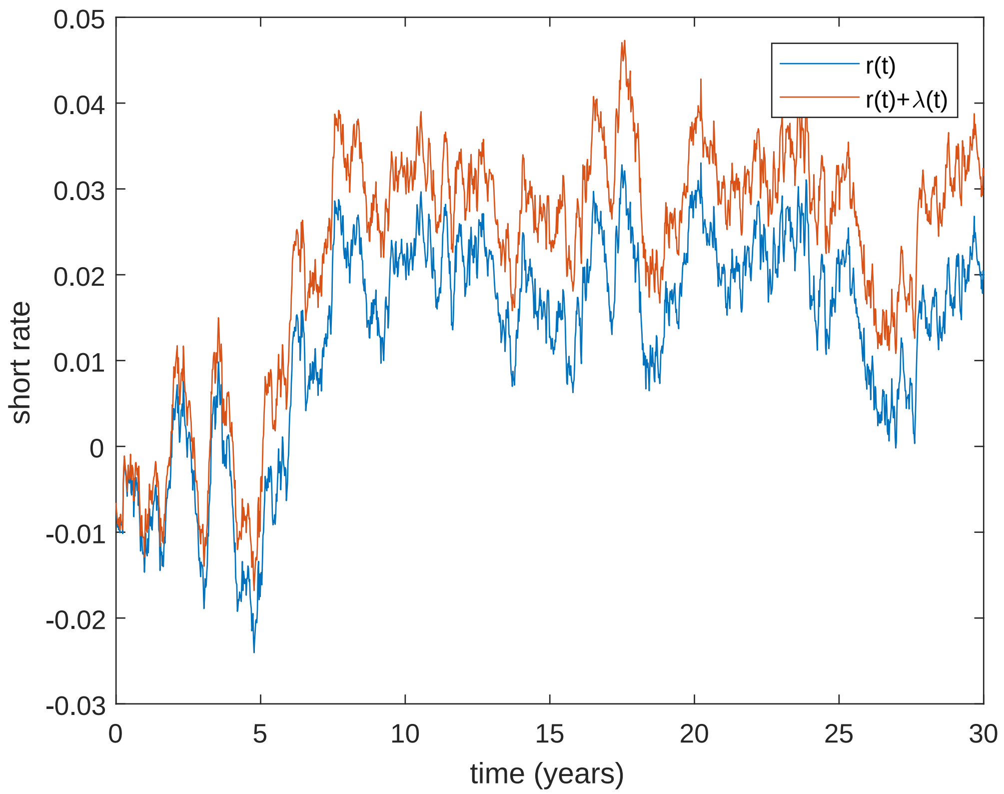


Figure 5.4: Risk-free rate vs. Risk-free rate with credit spread

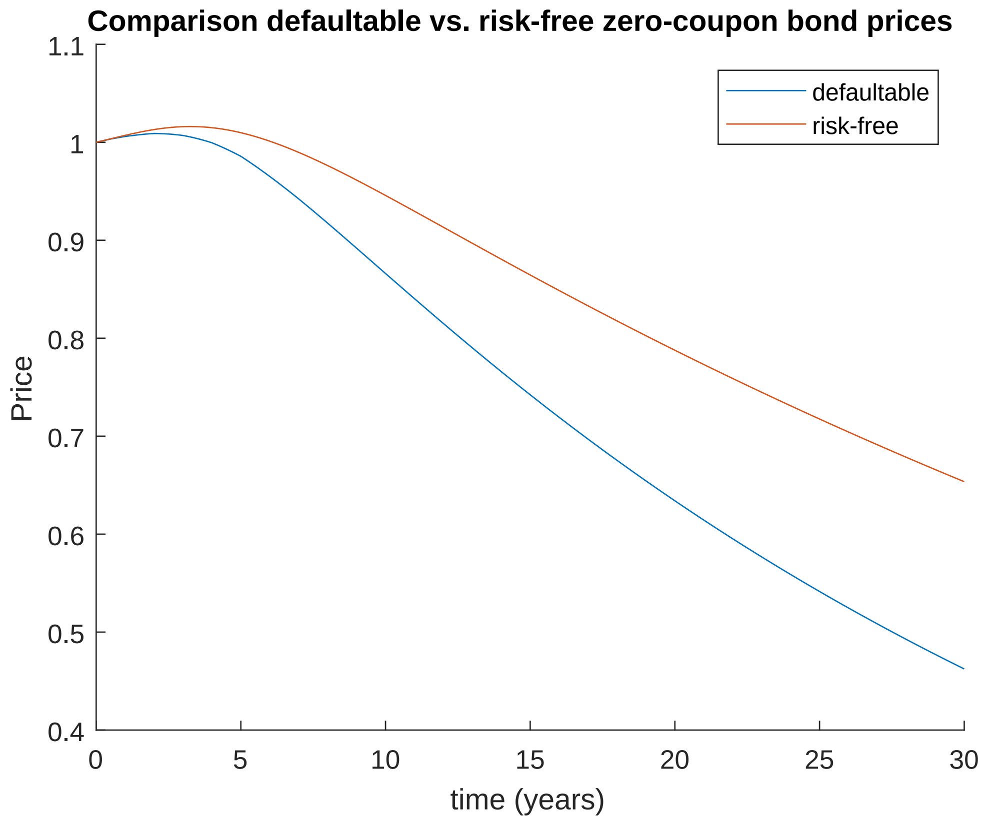


Figure 5.5: Defaultable (Munich re) vs risk-free zero-coupon bond price

bond prices are always lower than the default-free ones, since (survival) probabilities are always less than or equal to one. As bond prices correspond to discount factors, values greater than one might seem unusual. However, this is explained by the negative interest rates for shorter maturities in the yield curve seen in Figure 2.1.

Chapter 6

Conclusion

In the course of this thesis, the world of interest rate models was explored. The basics of interest rate theory and the historically important models of Vasicek, of Cox, Ingersoll & Ross, and of Hull& White as well as extensions to fit the currently observed term-structure, like the CIR++, were examined in a brief and compact fashion.

Due to the inability of one-factor models to address correlation between interest rates of different maturities, the two-factor G2++ model developed by Brigo & Mercurio in [5, 2001], consisting of two correlated stochastic processes and a deterministic shift to fit the current term-structure, was analysed in detail. Pricing formulas for interest rate derivatives were derived and methods for calibration to the market presented and implemented. One of the lessons learned in that context was, unfortunately, that it can prove to be very difficult to obtain the relevant market data in some cases.

Finally, the risk of default was incorporated in the form of an intensity-based approach, where a stochastic intensity process is assumed to drive the default probability. The stochastic setup within which the assumptions were made was fairly simple. The more interesting it was to see, that a lot of theory developed for interest rates turned out to be very useful also in the presence of default risk. For example, assuming zero recovery, the defaultable bond prices are simply the product of the risk-less bond prices and the survival probability of the bond issuer, if the interest rate and the intensity processes

are assumed to be uncorrelated, which is a very intuitive result.
Somewhat surprising was the fact that calibrating a model to the market is not always as straightforward as one might think. In the interest rate case, the calibration procedure makes perfect sense, since market data is used without any additional assumptions, even if different data then presented in this thesis needs to be used in the current market environment due to Black cap volatilities often not being quoted. However, in the intensity process case, where the CIR++ model was chosen, the calibration procedure seems somewhat heuristic. On the one hand, intensities need to be assumed to be deterministic (piecewise constant or piecewise linear) in order to even fit the stochastic intensity model to the CDS-quotes observed on the market. On the other hand, as there is a lack of liquidity in additional market data such as CDS-options, their is no real possibility to calibrate the parameters in a straightforward way, which leads to procedures seeming to be somewhat arbitrary. This poses the question, if this is all there is, or if the proprietary models developed and used by big financial institutions around the world contain more sophisticated approaches.

Appendix A

Stochastic differential equations

In the appendix some results about the existence and uniqueness of solutions of time-homogeneous diffusion processes are going to be stated. In the course of the constructive proof, it will also be verified that the Euler-Maruyama method has order of convergence $\frac{1}{2}$. Lastly, the positivity of a CIR process fulfilling the Feller condition is going to be shown. The appendix is closely following [26], though not all proofs are going to be carried out in full detail. For further reading on the topic [28, Chapter 5] is recommended, which uses the same arguments to proof uniqueness, but a different approach for existence.
Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\mathcal{F}$ satisfies the usual conditions and let W be a d-dimensional Brownian motion w.r.t. to this probability space. Consider the stochastic differential equation (SDE)

$$dX(t) = \mu(X_t)dt + \sigma(X_t)dW(t), \quad X_0 = Z. \tag{A.0.1}$$

$\mu : \mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : \mathbb{R}^k \to R^{k \times d}$ are continuous functions and Z is a k-dimensional, $\mathcal{F}_0$-measurable random vector. A process X, which is adapted to the natural filtration generated by W and fulfils

$$X(t) = Z + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW(s) \quad \forall t \geq 0, \tag{A.0.2}$$

is called strong solution of the SDE (A.0.1). The following theorem states necessary conditions for the existence and uniqueness of such a solution.

Theorem A.0.1. *If there is a scalar K such that μ and σ satisfy*

$$|\mu(x) - \mu(y)| < K|x - y|, \quad |\sigma(x) - \sigma(y)| < K|x - y| \quad \forall x, y \in \mathbb{R}^k, \tag{A.0.3}$$

i.e., μ and σ are Lipschitz, and $\mathbb{E}[|Z|^2] < \infty$ holds, then there exists a strong solution X to (A.0.1) *(up to stochastic indistinguishability). Additionally, $\mathbb{E}[\sup_{0\le s\le t} |X_s| < \infty]$ holds.*

Proof. Let $\pi^n = \{0 = t_0^n < \cdots < t_{k_n}^n = T\}$ be a partition of the interval $[0, T]$ with norm $||\pi^n|| = \sup_{1\le l\le k_n} |t_l^n - t_{l-1}^n|$. The Euler-Maruyama method for numerically solving SDEs is defined by

$$\begin{aligned} X_0^n &:= Z \\ X_t^n &:= X_{t_k^n}^n + \mu(X_{t_k^n}^n)(t - t_k^n) + \sigma(X_{t_k^n}^n)(W_t - W_{t_k^n}), \quad (t_k^n < t \le t_{k+1}^n). \end{aligned} \tag{A.0.4}$$

Since all the expressions in the definition are continuous and adapted, X^n is as well. For $t \in [t_k^n, t_{k+1}^n)$, let $\eta^n(t) := t_k^n$ be the left end point of the interval. Then, by definition of the stochastic integral, (A.0.4) can be rewritten as

$$X_t^n := Z + \int_0^t \mu(X_{\eta^n(u)}^n)du + \int_0^t \sigma(X_{\eta^n(u)}^n)dW(u), \quad 0 \le t \le T.$$

Let $L_T^2 := \{X|X$ is adapted, continuous and $||X||_T < \infty\}$ and $||X||_T := \sqrt{\mathbb{E}[\sup_{t\le T} |X_t|^2]}$. It can be shown that $(L_T^2, ||X||_T)$ is a complete normed space, so that it is sufficient to prove that $(X^n)_{n\ge1}$ is a Cauchy sequence, which converges to a solution of (A.0.2) as $||\pi^n||$ goes to zero. The following three inequalities are needed:

$$||X^n||_T \le C_1(T), \tag{A.0.5}$$

$$\mathbb{E}\left[|X_t^n - X_{\eta^n(t)}^n|^2\right] \le C_2(T)||\pi^n|| \quad \forall t \le T, \tag{A.0.6}$$

$$||X^n - X^m||_T \le C_3(T)\sqrt{(||\pi^n|| + ||\pi^m||)}, \tag{A.0.7}$$

where $C_k(T), k = 1, 2, 3$ are constants only depending on T, μ, σ and Z. The details of their derivations are omitted, but it should be mentioned that the ingredients include Cauchy's inequality, Doob's martingale inequality, the Itô isometry, Gronwall's Lemma[1] as well as the fact that global Lipschitz continuity implies at most linear growth. (A.0.5) implies that X^n is in L_2^T and (A.0.7) that it is a Cauchy sequence w.r.t. the norm $||X||_T$. Since $(L_T^2, ||X||_T)$ is complete, X^n converges to a limit process $X \in L_T^2$. It is going to be shown that X fulfils SDE (A.0.2). The same ingredients as above as well as the inequality $(x+y)^2 \leq 2(x^2+y^2)$ are used to justify the following steps:

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq T} |X_t - Z - \int_0^t \mu(X_s)ds - \int_0^t \sigma(X_s)dW(s)|^2\right] \\
&\leq \mathbb{E}\left[\sup_{t\leq T}\left(|X_t - X^n| + |X^n + Z - \int_0^t \mu(X_s)ds - \int_0^t \sigma(X_s)dW(s)|\right)^2\right] \\
&\leq \mathbb{E}\left[\sup_{t\leq T} 2|X_t - X^n|^2 + 2|X^n + Z - \int_0^t \mu(X_s)ds - \int_0^t \sigma(X_s)dW(s)|^2\right] \\
&\leq 2||X - X^n||_T^2 + 2\mathbb{E}\left[\sup_{t\leq T} |\int_0^t \mu(X^n_{\eta^n(s)}) - \mu(X_s)ds + \int_0^t \sigma(X^n_{\eta^n(s)}) - \sigma(X_s)dW(s)|^2\right] \\
&\leq 2||X - X^n||_T^2 + 4\mathbb{E}\left[\sup_{t\leq T} |\int_0^t \mu(X^n_{\eta^n(s)}) - \mu(X_s)ds|^2 + |\int_0^t \sigma(X^n_{\eta^n(s)}) - \sigma(X_s)dW(s)|^2\right] \\
&\leq 2||X - X^n||_T^2 + 4\left(\mathbb{E}\left[T\int_0^T |\mu(X^n_{\eta^n(s)}) - \mu(X_s)|^2 ds\right] + 4\mathbb{E}\left[\int_0^T |\sigma(X^n_{\eta^n(s)}) - \sigma(X_s)|^2 ds\right]\right) \\
&\leq 2||X - X^n||_T^2 + 4(T+4)K^2\int_0^T \mathbb{E}\left[|X^n_{\eta^n(s)} - X_s|^2\right] ds \\
&\leq 2||X - X^n||_T^2 + 4(T+4)K^2 T C_2(T)||\pi^n||.
\end{aligned}
$$

Because of (A.0.6) and X^n converging to X, the right-hand side goes to zero as $||\pi^n||$ goes to zero, which concludes the proof of the existence of a strong solution. Letting $||\pi^m||$ go to zero in (A.0.7) shows that the Euler-Maruyama method converges with order $\frac{1}{2}$.

Uniqueness is shown in a similar way. Assume X and X' to be two different

[1]Gronwall's Lemma states: If $f : [0, T] \to \mathbb{R}$ is continuous and $f(t) \leq a + b\int_0^t f(u)du$ holds for some $a, b \in \mathbb{R}$ and all $t \leq T$, then $f(t) \leq ae^{bt}$ also holds.

solutions to (A.0.2). Then, by the same methods as above,

$$||X - X'||_t^2 \leq (2T + 8)K^2 \int_0^t ||X - X'||_u^2 du,$$

holds. Applying Gronwall's Lemma yields $||X - X'||_T = 0$ and therefore $X = X'$, which proves the uniqueness of the solution. □

Consider now a CIR-process

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{|r(t)|}dW(t), \qquad r(0) = r_0,$$

with $r_0, k, \theta, \sigma > 0$, where $W(t)$ is a Brownian motion w.r.t. to $(\mathbb{P}, \mathcal{F}_t)$. Since $\mu(r) = k(\theta - r)$ is Lipschitz continuous and $\sigma(r) = \sigma\sqrt{|r|}$ fulfils $|\sigma(x) - \sigma(y)| \leq K \max(|x - y|^{1/2}, |x - y|)$, there exists a unique solution to the SDE above (this result can be proven in a similar, yet slightly more complicated way than A.0.1). To show non-negativity of the solution as well as positivity in case of $2k\theta > \sigma^2$, some further results of stochastic calculus are needed.

Theorem A.0.2 (Without proof). *Let X be a squared Bessel process of dimension $\nu \in \mathbb{R}$, i.e., the unique strong solution of the following SDE:*

$$dX_t = \nu dt + 2\sqrt{|X_t|}dW_t \qquad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$ and W is a one-dimensional Brownian motion. Then, if $x_0 \geq 0$:

(i) If $\gamma \geq 0$, then $X \geq 0$.
(ii) If $\gamma \geq 2$, then $X > 0$.

almost surely holds.

Theorem A.0.3. *Let r be a CIR process and $\tau(t) = \frac{\sigma^2}{4k}(e^{kt} - 1)$ a time-transformation. Then, $r(t) = e^{-kt}Y_{\tau(t)}$, where Y is a squared Bessel process of dimension $\nu = \frac{4k\theta}{\sigma^2}$.*

Proof. Let $U_t = e^{kt}r(t)$ and $Y_t := U_{\tau^{-1}(t)}$, where $\tau^{-1}(t) = \frac{1}{k}\ln(1+\frac{4k}{\sigma^2}t)$ is the inverse time transformation to $\tau(t)$. Itô's formula yields:

$$\begin{aligned} dU_t &= kU_t dt + e^{kt}dr(t) = kU_t dt + e^{kt}(k(\theta - r(t))dt + \sigma\sqrt{|r(t)|}dW(t)) \\ &= e^{kt}k\theta dt + \sigma\sqrt{e^{kt}|U_t|}dW(t). \end{aligned}$$

Since $Y_0 = U_0$,

$$Y_t = Y_0 + k\theta\int_0^{\tau^{-1}(t)} e^{ks}ds + \sigma\int_0^{\tau^{-1}(t)}\sqrt{e^{ks}|U_s|}dW(s).$$

The first integral is a standard Riemann integral with value $\theta(e^{k\tau^{-1}(t)}-1) = \nu t$. Using the stochastic substitution rule [2] with the substitution $s = \tau^{-1}(u)$ leads to

$$\begin{aligned} \sigma\int_0^{\tau^{-1}(t)}\sqrt{e^{ks}|U_s|}dW(s) &= \sigma\int_0^t\sqrt{e^{k\tau^{-1}(u)}|Y_u|}\sqrt{(\tau^{-1})'(u)}dW^*(u) \\ &= \sigma\int_0^t\sqrt{u\left(1+\frac{4k}{\sigma^2}\right)|Y_u|}\frac{2}{\sigma}\sqrt{\frac{1}{\left(1+\frac{4k}{\sigma^2}\right)u}}dW^*(u) \\ &= 2\int_0^t\sqrt{|Y_u|}dW^*(u) \end{aligned}$$

Combining both shows that

$$dY_t = \nu dt + 2\sqrt{|Y_t|}dW^*(t),$$

i.e., Y is a Bessel process of dimension $\nu > 0$. Further, $e^{-kt}Y_{\tau(t)} = e^{-kt}U_t = r(t)$, which proves the claim. $\square$

Since $r_0 > 0$ by definition, Theorem A.0.3(i) implies that the CIR process is automatically non-negative. If further $2k\theta > \sigma^2$, then $\nu = \frac{4k\theta}{\sigma^2} > 2$, so that

[2] Let τ be a differentiable deterministic time transformation with positive derivative and Y an integrable stochastic process. Then, there exists a Brownian motion W^* (w.r.t to the time-transformed filtration $\mathcal{F}^*$), such that the following holds:

$$\int_0^{\tau(t)} Y_s dW_s = \int_0^t Y_{\tau(t)}\sqrt{\tau'(u)}dW^*(u),$$

TheoremA.0.3(ii) implies positivity of a CIR process satisfying the Feller condition.

List of Figures

List of Tables

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