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Extremal Problems in Lattices
(Line-free Sets, Moser Sets and Cap Sets)

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Zusammenfassung

In der Kombinatorik treten Gitterprobleme sehr häufig auf. Dabei bezeichnet ein Gitter die Menge der ganzzahligen Punkte eines Hyperwürfels beliebiger Dimension. Im zweidimensionalen entspricht das den ganzzahligen Punkten in einem Quadrat. Wir werden darauf spezielle Klassen von Linien betrachten. Schlussendlich wollen wir dann eine Auswahl der Gitterpunkte finden, so dass diese keine Linien einer Klasse enthält und dabei aber größtmöglich ist. Im ersten Teil der Arbeit werden wir dazu einige allgemeine Resultate, darunter Schranken und optimale Konstruktionen für niedrige Dimensionen geben und danach diese auf höhere Dimensionen verallgemeinern. Im zweiten Teil verwenden wir ganzzahlige, lineare Optimierung um damit dieses Extremwertproblem zu lösen.

Abstract

In combinatorics lattice problems occur rather frequently. Here lattice is the name for a set of integral points of a hypercube in an arbitrary dimension. In the two dimensional case this corresponds to the integral points of a square. We will look at special classes of lines in this lattice. Finally we want a sub set of these lattice points such that there exists no line of one class and that it is as large as possible. In the first part of this thesis we will give some general results such as bounds and constructions for optimal sets in low dimensions. Then we generalize them to higher dimensions. In the second part we will use integer linear programming to solve this extremal problem.

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Chapter 1

Introduction

Some of the readers may know the card game SET® [1]. It is played with a special deck of 81 cards. Each card has four attributes and each attribute can take one of three values. The game is played by shuffling the deck, turning 12 cards face up and finding a *SET* which is a special combination of three cards: For each kind of attribute the three cards must either be all the same or all different in this attribute. After finding a *SET* the corresponding cards are taken away and new cards are dealt until there are 12 cards again. It can sometimes be quite hard to find such *SETs* and it is also possible that there are no *SETs* at all in the 12 cards face up. In this case 3 more cards are dealt. Now the question may arise: Is there a set of 15 cards within the whole deck such that there exists no *SET* in these cards? The answer is yes. In fact it is possible that one can deal up to 20 cards and there is no *SET* in it. But after the 21^{st} card there is guaranteed to be a *SET*. This has a mathematical connection.

We can express the card game by looking at $[n]^d$ with $d = 4$ (the four attributes) and $n = 3$ (the three values which every attribute can take). Here $[n]$ is the integral interval from 1 to n, so $[n] := \{1, 2, 3, ..., n\}$ and $[n]^d := \{(n_1, \ldots, n_d) | n_i \in [n]\}$. If zero is also included then we write $[n]_0$. A *SET* is represented by a so called algebraic line which is a set of three points where for each coordinate the values of the three points are either all the same or all different. This is essentially the definition we saw above. In the first chapter we will see an equivalent definition for arbitrary n and d. Now we are looking for the largest set of points such that there is no algebraic line in it. This problem has two sister problems where we look at combinatorial lines or geometric lines instead of algebraic lines.

For geometric lines the set of three points has to fulfil the following condition: There exists a sorting of the points such that for each coordinate the values are all the same or are all different and in ascending order or are all different and in descending order. Combinatorial lines are slightly more restrictive since for them the last option is not possible.

In this thesis we are looking into these three problems. A lot of work has been done in the case $n = 3$ (See [17] and more recently [8]) but we will mainly involve ourselves with the cases $d = 2$ and $d = 3$ and use some results from them for use in higher dimensions.

For further insight in the card game SET and its mathematical connections one may refer to the paper of Davis and Maclagan [6].

Chapter 2

Problem Definition

At first we define three classes of lines in $[n]^d$ and then look at some examples.

The most crucial definition is the definition of an arithmetic progression.

Definition 1 (Arithmetic Progression)**.** *An arithmetic progression of length n is a sequence of n integral points where the difference between two consecutive members is constant. This means for the sequence $v_1, v_2, \ldots, v_n \in \mathbb{Z}^d$ we have $v_{i+1} - v_i = c$ for $c \in \mathbb{Z}^d$ and for all $i \in [n-1]$.*

An arithmetic progression has essentially the form $\{v_1 + tc | t \in [n-1]_0\}$.

From this we can define three different types of lines. First the geometric lines:

Definition 2 (Geometric Line)**.** *In $[n]^d$ we call an arithmetic progression of length n a geometric line.*

Note that we do not reduce modulo n here.

In another equivalent definition we can take a starting point $v \in [n]^d$ which lies on the boundary ($v_i = 1$ or $v_i = n$ for at least one $i \in [d]$) and a vector $r \in \{-1, 0, 1\}^d$ then the geometric line is the set $\{v + tr | t \in [n-1]_0\}$ if for every $i \in [d]$ it holds that if $r_i = 1$ then $v_i = 1$ and if $r_i = -1$ then $v_i = n$. One may see that the c in the definition for arithmetic progressions and the r in this definition are the same. For example in $[3]^5$ the sequence $\{(1,1,2,3,3), (2,2,2,2,2), (3,3,2,1,1)\}$ is a geometric line. Here the starting point is $v = (1,1,2,3,3)$ and $r = (1,1,0,-1,-1)$.

We define a Moser set $A \subseteq [n]^d$ to be a set without geometric lines. With $g_{n,d}$ we will denote the cardinality of a largest Moser set in $[n]^d$.

If the difference c has only non-negative entries it leads to the following definition.

Definition 3 (Combinatorial Line)**.** *An arithmetic progression of length n in* $[n]^d$ *is called a combinatorial line if the difference between two consecutive elements lies in* $\mathbb{N}^d$*.*

A second definition can be made analogously to the second definition of the geometric lines. For a starting vector $v \in [n]^d$ with $v_i = 1$ for at least one $i \in [d]$ and a vector $r \in \{0,1\}^d$ the combinatorial line is $\{v + tr | t \in [n-1]\}$ if for every $i \in [d]$ it holds that if $r_i = 1$ then $v_i = 1$.

We will call a set without any combinatorial line line-free and the cardinality of a largest line-free set in $[n]^d$ is denoted by $c_{n,d}$.

From the geometric line we can also go the other way and look at a more general case.

Definition 4 (Algebraic Line)**.** *An algebraic line is an arithmetic progression of length n in* $\mathbb{Z}_n^d$ *instead of* $[n]^d$*.*

Throughout this thesis we will always assume that the canonical representatives of the equivalence classes of $\mathbb{Z}_n$ are $[n-1]_0$ but we will frequently use the transformation $((k-1) \bmod n) + 1$ for $k \in \mathbb{Z}$ so that we are in $[n]$. This does not interfere with our observations and it also yields a more natural connection between our problems. We will also use the notation $v \bmod n$ where v is a vector, i.e. that we look at the reduction modulo n on all components of v.

For n composite we will not consider an arithmetic progression as an algebraic line where two elements are the same. Given the arithmetic progression $v_1, \ldots, v_n \in [n]^d$ we forbid $v_i = v_j$ for any pair $(i,j) \in [d]^2$, $i < j$. This means $v_j - v_i = tc \equiv (0) \bmod n$ is not allowed for $t \in [n-1]$, $c \in \mathbb{Z}^d$ and $(0) \in \mathbb{Z}^d$ is the vector with all entries being zero. For all coordinates $k \in [d], tc_k \equiv 0 \bmod n$ and so $\gcd(c_k, n) \neq 1$ is not allowed to hold simultaneously. ($\gcd(a,b)$ is the greatest common divisor of a and b.) Therefor the arithmetic progression has the form $\{((v + tc - (1)) \bmod n) + (1) | t \in [n-1]_0\}$ with $v \in [n]^d$ and $c \in [n]^d \setminus \{(c_1, \ldots, c_d) \in [n]^d | \gcd(c_i, n) \neq 1, \forall i \in [d]\}$ and $(1) \in \mathbb{Z}^d$ is the vector with all entries being one. For example in $[4]^2$ we get $c \in [4]^2 \setminus \{(2,2),(2,4),(4,2),(4,4)\}$.

Sets without any algebraic lines are called cap sets and the cardinality of a largest cap set in $\mathbb{Z}_n^d$ is denoted by $a_{n,d}$.

From the definitions we see that every combinatorial line is a geometric line and every geometric line is an algebraic line but not vice versa. Thus we get that every cap set is a Moser set and every Moser set is a line-free set and therefore the following inequalities hold.

$$a_{n,d} \leq g_{n,d} \leq c_{n,d} \tag{2.1}$$

We can also further generalize the problems if we are not only looking at lines with n points but less than n points.

Definition 5. *For $k \in [n-1]_0$ we will define $c^k_{n,d}, g^k_{n,d}$ and $a^k_{n,d}$ to be the cardinality of a largest set without combinatorial, geometric and algebraic lines of length $n-k$ respectively.*

Note that for $k = 0$ we have our original problems.

Now we can look at some examples of line-free, Moser and cap sets and how to represent them.

In the two-dimensional case we can look at $n \times n$ matrices where each entry represents a point of $[n]^d$. The crosses $\times$ (or alternatively write 1) mark points which are in our considered set and the circles $\circ$ (alternatively 0) mark the ones which are not a part of the set. Since the different lines have nice properties one can easily identify them in this representation. For example geometric lines are either a row , a column or a diagonal of the $n \times n$ matrix. The construction can be expanded to higher dimensions d by looking at an n^d matrix (tensor).

In figure 2.1 there are three sets in the two-dimensional representation.

	1	2	3
1	$\circ$	$\times$	$\times$
2	$\times$	$\times$	$\circ$
3	$\times$	$\circ$	$\times$

	1	2	3
1	$\times$	$\times$	$\circ$
2	$\times$	$\circ$	$\times$
3	$\circ$	$\times$	$\times$

	1	2	3
1	$\times$	$\times$	$\circ$
2	$\times$	$\times$	$\circ$
3	$\circ$	$\circ$	$\circ$

Figure 2.1: Examples in $[3]^2$

The first one of these sets reaches the maximum under all line-free sets through exhaustive search and therefore $c_{3,2} = 6$. But since $\{(1,3),(2,2),\ (3,1)\}$ is a geometric line, it only shows that $g_{3,2} \leq 6$.

The second one looks quite similar to the first one but it has an additional property: It is a Moser set and in fact a largest one, so $g_{3,2} = 6$. $\{(1,2),(2,1),(3,3)\}$ is an algebraic line here and therefore $a_{3,2} \leq 6$.

The last one is a cap set. It is one of the configurations where the maximum is reached and $a_{3,2} = 4$.

For a representation in three dimensions we look at figure 2.2. Here the values in the corner of each 3×3 matrix represent the third dimension. This example shows an optimal configuration for $g_{3,3}$ which is 16.

1	1	2	3
1	○	×	×
2	×	○	×
3	×	×	○

2	1	2	3
1	×	○	×
2	○	○	○
3	×	○	×

3	1	2	3
1	×	×	○
2	×	○	×
3	○	×	×

Figure 2.2: Example in $[3]^3$

Chapter 3

General Observations

We start this chapter by proving an upper and a lower bound for all values of n and d and then move on to the special cases $d = 2$ and $d = 3$. We then use a method from the three-dimensional case in higher dimensions. The last thing in this chapter will be the case $n = 3$.

Theorem 1. *For every* $n, d \in \mathbb{N}$*:* $c_{n,d} \leq n^d - n^{d-1}$.

Proof. The proof is by induction on d.

For $d = 1$ our hypercube is just a line with n points. To get a line-free set we just need to remove one point, so $c_{n,1} = n - 1$.

For the induction step we go from $d - 1$ to d. So suppose $c_{n,d-1} \leq n^{d-1} - n^{d-2}$. $(*)$

We can divide $[n]^d$ into n disjoint slices of dimension $d - 1$. Now we will look at this division on the set where we reach $c_{n,d}$. In every slice we have at most $c_{n,d-1}$ points. If there was one with more points then we would have a combinatorial line in this sub-cube which would lead to a combinatorial line in the original set. So we get:

$$c_{n,d} \leq n \cdot c_{n,d-1} \overset{(*)}{\leq} n \cdot (n^{d-1} - n^{d-2}) = n^d - n^{d-1}.$$

□

From the proof we also get $c_{n,d} \leq n \cdot c_{n,d-1}$ and $c_{n,1} = n - 1$. Since the argument for $d = 1$ also holds for $g_{n,1}$ and $a_{n,1}$ we have $g_{n,1} = a_{n,1} = n - 1$.

Theorem 2. *For every* $n, d \in \mathbb{N}$*:* $a_{n,d} \geq (n - 1)^d$.

Proof. We show the following observation: The set $A = \{(x_1, \ldots, x_d) | x_i \neq n \ \forall i \in [d]\}$ does not contain any algebraic lines. We have already seen that every algebraic line has the form $\mathcal{L} = \{(v + tc - (1)) \bmod n + (1) | t \in [n-1]_0\}$ with $v \in [n]^d$ and $c \in [n]^d \setminus \{(c_1, \ldots, c_d) \in [n]^d | \gcd(c_i, n) \neq 1, \forall i \in [d]\}$. Because of this there exists an index $i \in [d]$ where $c_i \neq n$ and therefore the i-th coordinates of the points in $\mathcal{L}$ go through every value in $[n]$, i.e. $\{l_i | l \in \mathcal{L}\} = [n]$. Thus every line must contain an element where at least one coordinate is n which is excluded in our set A. And hence we get: $|A| = (n-1)^d \leq a_{n,d}$. □

3.1 Problems in two dimensions

In the two-dimensional case the solution for the problem is fairly easy for $c_{n,2}$ and $g_{n,2}$ but needs a little more work for $a_{n,2}$. In all three cases we can only choose $n-1$ points per row and column since otherwise we would have lines. This follows directly from the one-dimensional problem.

3.1.1 Combinatorial Lines

For the case $c_{n,2}$ the main diagonal $\{(k,k) | k \in [n]\}$ is the only additional combinatorial line besides the rows and columns. Now we will construct a set where we reach the upper bound $c_{n,2} \leq n^2 - n$ from Theorem 1. We will look at $[n]^2$ and delete points until there is no more combinatorial line left. At first we will exclude one of the points from the main diagonal. For the remaining rows we observe one at a time and take a column where an element was not already deleted. This always works since we have $n-1$ columns left for $n-1$ rows.

We can also express this in another way: The function $\pi : [n] \to [n]$ with $\pi(x) = y$ for every (x, y) which was not chosen in the desired set, has to be a permutation. Since the first element was taken from the main diagonal we have at least one fixed point.

With this construction which does not contain a combinatorial line we get $c_{n,2} \geq n^2 - n$ and with the upper bound we have:

Theorem 3. *$c_{n,2} = n^2 - n$ holds for all $n \in \mathbb{N} \setminus \{0\}$.*

Figure 3.1 shows an optimal set for $c_{4,2} = 12$. Note how the diagonal is not a combinatorial line by not taking $(1, 1)$.

	1	2	3	4
1	○	×	×	×
2	×	×	×	○
3	×	○	×	×
4	×	×	○	×

Figure 3.1: Optimal configuration in $[4]^2$

3.1.2 Geometric Lines

For $g_{n,2}$ we can proceed similarly but now we also have to be careful about the anti-diagonal $\{(k, n-k+1)|k \in [n]\}$ which is the only other geometric line besides the combinatorial lines.

The optimal set is constructed as above by excluding points from the whole set. We at first choose one element from the main diagonal and then one from the anti-diagonal which is not in the same row or column as the first one. Then for the rest we continue as above. This construction works for every value of n except $n = 2$. Because if we choose an element from the diagonal we cannot choose an element on the anti-diagonal which is not in the same row or column. See also figure 3.2. Therefore $g_{2,2} = 1$.

	1	2
1	○	○
2	○	×

Figure 3.2: Optimal configuration in $[2]^2$

This yields the following theorem.

Theorem 4. *For all $n \in \mathbb{N} \setminus \{0, 2\}$: $g_{n,2} = n^2 - n$.*

As in the combinatorial case we can express this as a permutation $\pi : [n] \to [n]$ with $\pi(x) = y$ for every (x, y) which was not chosen. In addition to the fixed point we need an anti-fixed point.

Definition 6. *Let π be a permutation on n. We call $i \in [n]$ an anti-fixed point of π if $\pi(i) = n - i + 1$.*

The anti-fixed point ensures that the anti-diagonal $\{(k, n-k+1)|k \in [n]\}$ is not a geometric line in the set. Such permutations with exactly one fixed point and one anti-fixed point will play a major role in section 3.3.

3.1.3 Algebraic Lines

As above mentioned for $a_{n,2}$ the problem gets a little more involved. In [13], Jamison proved that if n is a prime, $a_{n,2} = (n-1)^2$ by showing a more general result. We will present a shorter proof by Alon [1]. For all composite values of n nothing is known in general but we can compute some solutions for small n as seen in chapter 4.

We first note that $\mathbb{Z}_q \cong \mathbb{F}_q$ where $\mathbb{F}_q$ is the finite field with q elements and every algebraic line is an affine line in $\mathbb{F}_q$, i.e. we can write them with equations of the form $a_1x_1 + a_2x_2 = b$ with $(a_1, a_2) \neq (0, 0)$ in two dimensions (This can of course also be done in higher dimensions). We also need the following theorem which we will not prove (For a proof see [19]).

Theorem (Chevalley - Warning)**.** *Let $\mathbb{F}_q$ be the finite field with q elements and $f(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, \ldots, x_n]$ be of degree smaller than n with $f(0, \ldots, 0) = 0$. Then there exists a non-trivial zero of the polynomial f, i.e. $\exists(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_q$ not all α_i are zero and $f(\alpha_1, \ldots, \alpha_n) = 0$.*

Theorem 5. *For every prime n: $a_{n,2} = (n-1)^2$.*

Proof. We have already shown the lower bound in Theorem 2. So all we need to prove is $a_{n,2} \leq (n-1)^2$.

Let $A \subseteq \mathbb{Z}_n^2$ be a set such that A contains at least one element of every algebraic line. We can assume that $(0,0) \in A$ by translating the whole set if necessary. Let $B = A \setminus \{(0,0)\}$, then every element in $\mathbb{Z}_n^2 \setminus \{(0,0)\}$ must be on a line intersected by B. Since every algebraic line is an affine line we have that for every $(a_1, a_2) \in \mathbb{Z}_n^2 \setminus \{(0,0)\}$ there exists $(b_1, b_2) \in B$ such that $a_1b_1 + a_2b_2 = 1$.

$$f(x_1, x_2) = \prod_{(b_1,b_2)\in B} (1 - b_1x_1 - b_2x_2)$$

yields $f(a_1, a_2) = 0$ for all $(a_1, a_2) \in \mathbb{Z}_n^2 \setminus \{(0,0)\}$ and $f(0,0) = 1$.

We then look at the following equation with a polynomial in the $2(n-1)$ variables $x_1^1, x_1^2, \ldots, x_1^{n-1}$ and $x_2^1, x_2^2, \ldots, x_2^{n-1}$:

$$\sum_{i=1}^{n-1} f(x_1^i, x_2^i) = n - 1$$

We can now use the theorem of Chevalley and Warning since the only solution of the above equation is $x_1^i = 0$ and $x_2^i = 0$ for all $i \in [n-1]$. Therefore the theorem states that the number of variables is at most as big as the degree of the polynomial on the left hand side which is $|B|$. So we get $|B| \leq 2(n-1) \Rightarrow |A| \leq 2(n-1)+1$ and every $A \subseteq \mathbb{Z}_n^2$ intersecting all algebraic lines must have cardinality at least $2(n-1)+1$. Therefore $a_{n,2} \leq n^2 - 2(n-1) - 1 = (n-1)^2$.

□

3.2 Problems in three dimensions

In this section the cases for $c_{n,3}$ and $g_{n,3}$ are dealt with. A method is developed with which optimal sets can be constructed.

3.2.1 Combinatorial Lines

At first we look at $c_{n,3}$. We show that the upper bound in Theorem 1 is tight. First we identify all combinatorial lines in three dimensions. A combinatorial line lying in a plane with some fixed coordinate, is also a combinatorial line in $[n]^3$. Additionally the diagonal $\{(k,k,k)|k \in [n]\}$ is also a combinatorial line.

For the construction we define a mapping from special Latin squares to configurations which have no combinatorial lines and reach the desired optimum. In the last step we then use a result from Latin squares that show that there always exists such configurations. We recall the definition of a Latin square.

Definition 7. *A Latin square of order n is an $n \times n$ array filled with the numbers from 1 to n such that in each column and row every number shows up exactly once.*

We call a Latin square diagonal if additionally the main diagonal contains every number once.

We also need the following result:

Theorem 6. *For every positive integer $n \geq 3$ there exists at least one diagonal Latin square of order n.*

For a proof of this see [14].

Now we can prove:

Theorem 7. *For every integer $n \geq 3$: $c_{n,3} = n^3 - n^2$.*

Proof. We take a diagonal Latin square of order n (due to Theorem 6 there always exists at least one) where w.l.o.g. the elements on the diagonal are in ascending order. Now we define a set of points $S \subseteq [n]^3$: (x, y) has the entry z in the Latin square if and only if $(x, y, z) \in S$. Our desired set is $T := [n]^3 \setminus S$. What needs to be shown is that T is line-free. Since we chose the diagonal in the Latin square in ascending order we have that the diagonal $\{(k, k, k)|k \in [n]\}$ is a subset of S and is therefore not in T. Next for fixed z the permutation π formed by $\pi(x) = y$ with $(x, y, z) \in S$ is a permutation with exactly one fixed point because every number appears exactly once in a column and row in the Latin square. For fixed x we have the same for every $(x, y, z) \in S$ because in every row every number shows up once. And an analogous argument holds for fixed y. So in every two-dimensional sub-slice we have a permutation with exactly one fixed point. So we get $c_{n,3} \geq n^3 - n^2$. And with equation (2.1) we get the statement. □

Figure 3.3 shows a small example how the construction in the proof works.

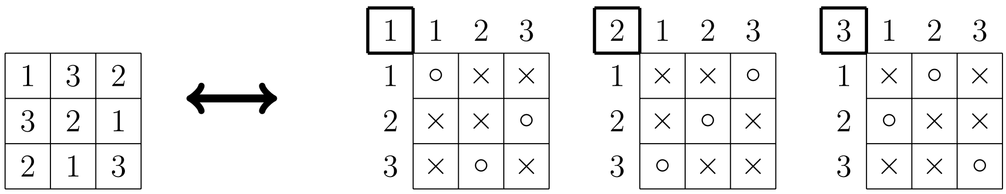

Figure 3.3: Conversion to Latin square in $[3]^3$

3.2.2 Geometric Lines

In this section we want to apply the construction used in the proof of Theorem 7 to get a similar result for the geometric lines. We will show that for n prime we can actually do this.

For the geometric lines in addition to all the combinatorial lines we need to look at all the diagonals in the two-dimensional sub-slices and the three other three-dimensional diagonals $\{(k, k, n-k+1)|k \in [n]\}$, $\{(k, n-k+1, k)|k \in [n]\}$ and $\{(n-k+1, k, k)|k \in [n]\}$. For this reason we use special permutations.

Definition 8. *We call a permutation* $\pi : [n] \to [n]$ *circular fixed, if every circular shift has a fixed point, i.e. for every* $k \in [n]$ *define the* k*-th circular shift permutation to be* $\sigma_k(i) = ((i + k - 1) \bmod n) + 1$ *for all* $i \in [n]$ *and* $\pi \circ \sigma_k$ *has a fixed point.*

Because σ_n is the identity π itself must also have a fixed point. This definition yields that every circular shift must have exactly one fixed point since every element can only become a fixed point once. For example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

is a circular fixed permutation since

$$\pi \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \pi \circ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \pi \circ \sigma_3 = \pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

all have one fixed point: 2, 3 and 1 respectively.

For our purpose these permutations have the property that if we line up the circular shifts in a grid properly (what that means will be explained later) we get a Latin square. If we use the transformation from the combinatorial case we then get a Moser set which reaches the upper bound of Theorem 1. So we want to find such permutations and for all odd primes n we can give some explicitly.

Definition 9. *Let n be an odd prime, $j \in [n]$ and $m \in \{2, \dots, n-2\}$. Define $\pi_{j,m}$ as follows: $\pi_{j,m}(1) = j$ and $\pi_{j,m}(i) = ((\pi_{j,m}(i-1) + m - 1) \bmod n) + 1$ for all $i \in \{2, \dots, n\}$.*

So the $\pi_{j,m}$ are functions which start with value j and the next element is the previous element plus m reduced modulo n.

This construction has very nice properties.

Lemma 1. *Let $\pi_{j,m}$ be a function as above. Then the following holds:*

1. *$\pi_{j,m}(i) = ((j - 1 + (i-1)m) \bmod n) + 1$ for all $i \in [n]$.*

2. *$\pi_{j,m}$ is a circular fixed permutation.*

3. *$\pi_{\pi_{j,m}(n),n-m}$ is the reverse order of the permutation $\pi_{j,m}$ and is therefore also a circular fixed permutation.*

Proof. 1. Follows directly from the definition by inserting $\pi_{j,m}(i-1)$ recursively.

2. We see that $m \in \mathbb{Z}_n^*$ (the set of invertible elements modulo n) because n is prime. In fact $\mathbb{Z}_n^*$ contains all elements of $\mathbb{Z}_n$ except $0 \equiv n$. Also $\mathbb{Z}_n^*$ is cyclic so there exists a generating element g and $\mathbb{Z}_n^* = \{g^k | k \in [n-2]_0\}$. We have $m = g^l$ for some $l \in [n-2]$

and get $m\mathbb{Z}_n^* = g^l\mathbb{Z}_n^* = \mathbb{Z}_n^*$ and especially $m\mathbb{Z}_n = \mathbb{Z}_n$. Because of property 1 and the fact that $j-1$ is just an offset we get that each element in $[n]$ is visited exactly once in $\pi_{j,m}$. Therefore it is a permutation.

Now we have to show that it is circular fixed. We see that the circular shifts of $\pi_{1,m}$ are the $\pi_{j,m}$ with $j \in \{2,\dots,n\}$. So if we show that $\pi_{1,m}$ is circular fixed then the $\pi_{j,m}$ are circular fixed. We observe that $\{(\pi_{1,m}(i) - i) \bmod n | i \in [n]\} = [n]$. This loosely means that the distances between an element i and its image is different for all i. Therefore every circular shift has only one fixed point.

The observation is true because $(\pi_{1,m}(i) - i) \bmod n = ((((i-1)m) \bmod n) + 1 - i) \bmod n = ((i-1)m - (i-1)) \bmod n = (i-1)(m-1) \bmod n$. We have $m \in \{2,\dots,n-2\} \Rightarrow m-1 \in \{1,\dots,n-3\} \subseteq \mathbb{Z}_n^*$ and as before we can argue that all distances appear exactly once. So the $\pi_{j,m}$ are circular fixed.

3. Since $\pi_{j,m}(n)$ is the last element of $\pi_{j,m}$ it must be the first element of the reverse order. Additionally we have $\pi_{j,m}(i) = ((\pi_{j,m}(i-1) + m - 1) \bmod n) + 1$ and so if we add $n-m$ instead of m we get the reverse order.

$\square$

We excluded the values 1 and $n-1$ for m in our definition because for them $\pi_{j,m}$ does not have these properties. From property 3 we get that $\pi_{j,m}$ has exactly one fixed point and one anti-fixed point due to the fact that the fixed points of the reverse order are exactly the anti-fixed points of the original order.

If we look at

$$\left(\pi_{\pi_{k,l}(j),m}(i)\right)_{\substack{i\in[n]\\ j\in[n]}}$$

with fixed $k \in [n]$ and $l, m \in \{2,\dots,n-2\}$ we get a Latin square because for j fixed $\pi_{k,l}(j)$ is also fixed and so $\pi_{\pi_{k,l}(j),m}$ is a circular fixed permutation. The rest follows from the following lemma.

Lemma 2. *The transpose of* $\left(\pi_{\pi_{k,l}(j),m}(i)\right)_{\substack{i\in[n]\\ j\in[n]}}$ *is* $\left(\pi_{\pi_{k,m}(j),l}(i)\right)_{\substack{i\in[n]\\ j\in[n]}}$.

Proof. We have to show that for every $i, j \in [n]$: $\pi_{\pi_{k,l}(i),m}(j) = \pi_{\pi_{k,m}(j),l}(i)$.

$$\begin{aligned}
\pi_{\pi_{k,l}(i),m}(j) &= \pi_{((k-1+(i-1)l) \bmod n)+1,m}(j) \\
&= (((((k-1+(i-1)l) \bmod n)+1-1+(j-1)m)) \bmod n)+1 \\
&= (((k-1+(i-1)l+(j-1)m)) \bmod n)+1 \\
&= (((((k-1+(j-1)m) \bmod n)+1-1+(i-1)l)) \bmod n)+1 \\
&= \pi_{((k-1+(j-1)m) \bmod n)+1,l}(i) = \pi_{\pi_{k,m}(j),l}(i)
\end{aligned}$$

□

If we want to look at the columns of the matrix we can simply transpose it and use the same argument from the rows to prove that the columns are circular fixed permutations. So this matrix is a Latin square and in every row and column there exists exactly one fixed point and one anti-fixed point. The only lines left are the diagonal and anti-diagonal of the Latin square. Both of them need to have exactly one fixed point and one anti-fixed point. This makes sure that the three-dimensional diagonals in our transformed set are not geometric lines. The diagonal has the form $\pi_{\pi_{k,l}(1),m+l}$ and the anti-diagonal has the form $\pi_{\pi_{k,l}(n),m+n-l}$. The following properties have to be met:

1. $(m + l) \bmod n \notin \{0, 1, n-1\}$
2. $(m + n - l) \bmod n \notin \{0, 1, n-1\} \Leftrightarrow m \bmod n \notin \{l-1, l, l+1\}$

This ensures that $\pi_{\pi_{k,l}(1),m+l}$ and $\pi_{\pi_{k,l}(n),m+n-l}$ are circular fixed and therefore have one fixed and one anti-fixed point.

To sum everything up we must find $m, l \in \{2, \dots, n-2\}$ with the above restrictions. For $n \geq 9$ we can find such numbers. For example we can always take $m = 2$ and $l = 4$. In the cases $n = 2, 3, 5, 7$ the restrictions cannot be fulfiled. If we use an arbitrary k then the Latin square

$$\left(a_{i,j}\right)_{\substack{i\in[n] \\ j\in[n]}} = \left(\pi_{\pi_{k,l}(j),m}(i)\right)_{\substack{i\in[n] \\ j\in[n]}}$$

yields an optimal configuration with the transformation described in Theorem 7: $(i, j, q) \notin T$ if and only if $q = a_{i,j}$ where T is our desired set.

For fixed q the Latin square property ensures that each row and column in the two-dimensional sub-slices (i, j, q) with $i, j \in [n]$ has exactly one element which is not chosen. Additionally the diagonals in the Latin square contain every number exactly once and therefore the diagonals in these sub-slices cannot form a line. The rows and the columns of the Latin square are circular-fixed and thus have exactly one fixed and one anti-fixed point. So for fixed i (and equivalently for fixed j) the two-dimensional sub-slices (i, j, q) with $j, q \in [n]$ do not contain a geometric line.

For $n = 7$ we can also explicitly state a Latin square which transforms into an optimal configuration which is depicted in figure 3.4.

1	4	7	3	6	2	5
6	2	5	1	4	7	3
4	7	3	6	2	5	1
2	5	1	4	7	3	6
7	3	6	2	5	1	4
5	1	4	7	3	6	2
3	6	2	5	1	4	7

Figure 3.4: Latin square for $n = 7$

This proves the following theorem.

Theorem 8. *For every prime $n \geq 7$: $g_{n,3} = n^3 - n^2$.*

The diagonals in the Latin square do not have to be circular-fixed permutations necessarily since there are only one of each kind of diagonal. They just have to contain exactly one fixed point and one anti-fixed point. So in our case $m + l \equiv 1 \mod n$. With $k = 1$ this yields the identity permutation which fulfils the requirement if n is odd. We can use this for all n. So we set $k = 1$ and we need to find $m, l \in \{2, \ldots, n-2\}$ such that:

1. $(m + l) \equiv 1 \mod n$

2. $m \not\equiv l \mod n$.

3.3 Problems in higher dimensions

Latin squares can be generalized to higher dimensions. We take a definition based on [3].

Definition 10. *A Latin d-cube of order n is a d-dimensional array of length n filled with the numbers from 1 to n such that if we fix $d - 1$ coordinates the one-dimensional sub-slice contains every number exactly once.*

For simplicity we will use the following notation.

Definition 11. $\pi^{(1)}_{k,l}(i) := \pi_{k,l}(i)$ *and*
$\pi^{(d)}_{k,l_1,\ldots,l_d}(i_1,\ldots,i_d) := \pi_{\pi^{(d-1)}_{k,l_1,\ldots,l_{d-1}}(i_1,\ldots,i_{d-1}),l_d}(i_d)$ *for all* $d \geq 2$*.*

The generalization of the construction for Latin d-cubes is done as in section 3.2.2 for Latin squares. With the above notation this yields:

$$\left(\pi^{(d)}_{k,l_1,\ldots,l_d}(i_1,\ldots,i_d)\right)_{\substack{i_1\in[n]\\ i_2\in[n]\\ \vdots\\ i_d\in[n]}}$$

with $d \in \mathbb{N}$, $k \in [n]$ and $l_1,\ldots,l_d \in \{2,\ldots,n-2\}$.

But there are more restrictions needed than in 3.2.2. We need that every diagonal is circular-fixed. Therefore the sums of the l_i over every set $I \subseteq [d]$, $I \neq \emptyset$ must not lie in $\{0,1,n-1\}$ modulo n, i.e. for every $I \subseteq [d]$

$$\left(\sum_{i\in I} l_i \bmod n\right) \notin \{0,1,n-1\}. \tag{3.1}$$

These restrictions suffice for line-free sets. For Moser sets there are more diagonals. Every disjoint pair of non-empty sets of indices $I,J \subseteq [d]$, $I \neq \emptyset$, $J \neq \emptyset$ and $I \cap J = \emptyset$ must fulfil

$$\left(\left(\sum_{i\in I} l_i - \sum_{j\in J} n - l_j\right) \bmod n\right) \notin \{0,1,n-1\}. \tag{3.2}$$

This ensures that every diagonal is circular fixed and therefore has a fixed point and an anti-fixed point.

For every d the number of restrictions is constant. So n can always be made large enough such that we can find such l_is. k can be chosen arbitrarily. The following two theorems summarize this.

Theorem 9. *For every* $d \in \mathbb{N}$ *there exists a* $N_1 \in \mathbb{N}$ *such that for every* $n \geq N_1$ *with* n *prime:* $c_{n,d} = n^d - n^{d-1}$*.*

Theorem 10. *For every* $d \in \mathbb{N}$ *there exists a* $N_2 \in \mathbb{N}$ *such that for every* $n \geq N_2$ *with* n *prime:* $g_{n,d} = n^d - n^{d-1}$*.*

Proof. Let $T \subseteq [n]^d$ be defined as follows:

$$(i_1,\ldots,i_{d-1},i_d) \notin T \Leftrightarrow i_d = \pi^{(d-1)}_{k,l_1,\ldots,l_{d-1}}(i_1,\ldots,i_{d-1})$$

with $k \in [n]$ and $l_1, \dots, l_{d-1} \in \{2, \dots, n-2\}$ where $l_1, \dots, l_{d-1}$ satisfy the set of equations in 3.1 for Theorem 9 and they satisfy the set of equations in 3.1 and 3.2 for Theorem 10.

For Theorem 9 if we can find $l_1, \dots, l_{d-1}$ such that

$$\sum_{i=1}^{d-1} l_i \leq n-2$$

then every subset $I \subseteq [d-1]$ has the property

$$\sum_{i \in I} l_i \leq n-2$$

. This satisfies the restrictions in 3.1. Since $l_i \geq 2, \forall i \in [d-1]$ we can use $l_i = 2$ and get:

$$\sum_{i=1}^{d-1} 2 \leq n-2 \Leftrightarrow 2(d-1) \leq n-2 \Leftrightarrow 2d \leq n.$$

This shows that if we take $N_1 = 2d$ Theorem 9 holds.

For Theorem 10 we choose $N_2 = 2^d$ and $l_i = 2^i, \forall i \in [d-1]$, the reason being the following: For the equations in 3.1 we have

$$\sum_{i=1}^{d-1} l_i \leq n-2$$

which is true because

$$\sum_{i=1}^{d-1} 2^i = 2^d - 2.$$

For the restrictions in 3.2 since the l_is and n are both even the resulting values of the reduction modulo n can never take the odd values 1 or $2^d - 1$. So we only have to show that we never get 0 which results in the restriction being

$$\sum_{i \in I} 2^i \neq \sum_{j \in J} 2^j$$

where we can omit the reduction modulo 2^d because 2^d is bigger than any sum of 2^i with $i \in [d-1]$. And since any natural number has a unique binary representation the above inequality holds for every pair $I, J \subseteq [d-1]$, $I \neq \emptyset$, $J \neq \emptyset$ and $I \cap J = \emptyset$. □

3.4 Problems for $n = 3$

A lot of research has been done for the case $n = 3$. Especially for the cases of combinatorial and geometric lines many results are known ([17]). But recently also a lot of development has been done in the direction of algebraic lines and cap sets.

In Theorem 6 it was proven that $a_{3,d} \geq 2^d$ which is a trivial lower bound. In [7] the best known lower bound so far was proven with $a_{3,d} = \Omega(2.2174^d)$ by Edel, Ferret, Landjev and Storme. The best known upper bound until May 2016 was $a_{3,d} = O(\frac{3^n}{n^{1+\epsilon}})$ by Bateman and Hawk Katz (See [4]). In May 2016 Ellenberg and Gijswijt independently conducted and then jointly released (See [8]) a proof for a new upper bound $a_{3,d} = o(2.756^d)$. This is quite an improvement over the previous upper bound. For their approach they used the polynomial method and in particular the one developed by Croot, Lev, and Pach in [5]. We will not go into the details of this thesis but for further interest in this topic see the corresponding papers.

3.5 Related Problems

In this section we discuss some related problems. First we will look at the zero sum problem and then we will discuss the Hales-Jewett theorem.

3.5.1 Zero Sums

Definition 12 (Zero Sum)**.** *Let $x_1, \ldots, x_n \in \mathbb{Z}_n^d$ be a sequence of n vectors. They are called a zero sum if $\sum_{i=1}^n x_i \equiv (0) \bmod n$.*

Now the question arises what the minimal number $f(n, d)$ is such that every set of $f(n, d)$ not necessarily distinct vectors $x_i \in \mathbb{Z}_n^d$ contains a subset of cardinality n which is a zero sum. This problem was first proposed by Harborth [11] and was based on the idea of generalizing a theorem by Erdös, Ginzburg and Ziv [9] which states that for any given sequence of $2n - 1$ integers one may choose n out of them such that there sum equals 0 modulo n. Harborth proved

$$(n-1)2^d + 1 \leq f(n, d) \leq (n-1)n^d + 1 \ .$$

Now the reference to our problem is that every algebraic line l is a zero sum if n is an odd prime, i.e. $\sum_{v \in l} v \equiv 0 \bmod n$. This follows from the fact that every algebraic line has the form $\{v + tr | t \in [n-1]_0\}$ with $v \in [n]^d$ and $r \in [n]^d \setminus (n, \ldots, n)$ for odd primes. If we take the sum of this set we get

$$\sum_{t=0}^{n-1} v + tr = nv + r\sum_{t=0}^{n-1} t = nv + rn\frac{(n-1)}{2} \equiv 0 \bmod n.$$

Since n is odd we get that $\frac{n-1}{2} \in \mathbb{N}$ and so the last equivalence holds.

The reverse that every zero sum of length n is an algebraic line is true for $\mathbb{Z}_3^d$. In $\mathbb{Z}_3$ the only options to get a zero sum with 3 values are $1+1+1$, $2+2+2$, $3+3+3$ and $1+2+3$. So if we take three distinct vectors in $\mathbb{Z}_3^d$ which are a zero sum, in every coordinate one of the above combinations (or any permutation of the last one) must occur. Since all of these combinations are possible in an algebraic line we are finished. For larger n the above statement does not hold anymore. For example in $\mathbb{Z}_5$ the zero sum $1+1+1+3+4 = 10 \equiv 0 \bmod n$ is not an algebraic line.

3.5.2 Hales-Jewett theorem

This section is based on [16].

Our problems always dealt with the question for fixed n and d to find a largest possible set such that there is no line in it. Now we want to ask for fixed n how large we have to make d such that there always exists at least one line.

The Hales-Jewett theorem gives an upper bound for the dimension $d = d(r, n)$ for which there exists a monochromatic combinatorial line in $[n]^d$ for any given colouring with r colours. We will provide a proof by Sharon Shelah which was proven in 1988. The original proof by Hales and Jewett also gives an upper bound but a much worse one than the one from Shelah. With the Hales-Jewett theorem one can prove many other results such as the Van der Warden theorem. There also exists a density version which we state at the end of this section.

We will now look at a different approach to combinatorial lines to make things easier for us to prove.

Let $[n]$ be an alphabet of n symbols and $* \notin [n]$ be a new symbol. Words are strings (or equivalently vectors) over $[n]$ without $*$ and roots are strings with at least one $*$. For a root $t \in ([n] \cup \{*\})^d$ and a symbol $a \in [n]$ we define $t(a) \in [n]$ as the word obtained by substituting $*$ with a.

Definition 13 (Combinatorial Line, second definition)**.** *For t a root we call the set of n*

words $\{t(i)|i \in [n]\}$ *a combinatorial line rooted at* t*.*

The root $(1, *, 2, *, 3) \in ([3] \cup \{*\})^5$ becomes the combinatorial line $\{(1,1,2,1,3),$ $(1,2,2,2,3),$ $(1,3,2,3,3)\}$ for example.

The next result shows how many combinatorial lines there are.

Theorem 11. *The number of combinatorial lines in* $[n]^d$ *is* $(n+1)^d - n^d$*.*

Proof.

$$\begin{aligned}|\{\text{combinatorial lines in } [n]^d\}| &= |\{\text{roots in } ([n] \cup \{*\})^d\}| \\ &= |\{\text{strings in } ([n] \cup \{*\})^d\}| \\ &\quad - |\{\text{strings in } ([n] \cup \{*\})^d \text{ without } *\}| \\ &= (n+1)^d - n^d\end{aligned}$$

□

So now we look at the colouring Hales-Jewett theorem.

Definition 14 (r-colouring)**.** *Let* $r \in \mathbb{N}$ *be a number of colours and* $n \in \mathbb{N}$*. An* r*-colouring of* $[n]^d$ *is a mapping* $\chi : [n]^d \to [r]$*.*

Theorem 12 (Hales-Jewett theorem)**.** *Let* $r \in \mathbb{N}$ *be a number of colours and* $n \in \mathbb{N}$*. There exists a dimension* $d := d(r, n)$ *such that every* r*-colouring of* $[n]^d$ *has a monochromatic combinatorial line.*

Proof. We perform induction on n. r is fixed but arbitrary. For $n = 1$ we get $d(r, 1) = 1$ since $[1]^1$ consists of only one point which whatever colour it has yields a monochromatic combinatorial line of length 1. Now suppose the theorem holds for $n-1$, i.e. $d := d(r, n-1)$ exists. Let $D_1 < \ldots < D_d$ be a sequence of d integers with

$$D_1 = r^{n^d},$$

$$D_i = r^{n^{d+\sum_{j=1}^{i-1} D_j}}, \; i \in \{2, \ldots, d\}$$

and set $D := \sum_{i=1}^{d} D_i$.

We want to show that for any r-colouring χ of $[n]^D$ there exists a monochromatic combinatorial line.

For this we need the following definitions and auxiliary theorem which we will prove later.

Definition 15 (Neighbours)**.** *Two words $a, b \in [n]^d$ are neighbours if they differ in exactly one coordinate with* 1 *being the value in a and n being the value in b.*

Two neighbours which differ in the i-th coordinate are $a = a_1 \dots a_{i-1} 1 a_{i+1} \dots a_d$ and $b = a_1 \dots a_{i-1} n a_{i+1} \dots a_d$ for example.

Definition 16 (Concatenation)**.** *Suppose $t_1, \dots, t_m$ are m roots over $[n]$ which need not be of the same length. Let $t := t_1 \dots t_m$ denote the concatenation of these roots. Let $a := a_1 \dots a_m \in [n]^m$ be a word. $t(a) := t_1(a_1) \dots t_m(a_m)$ is the word obtained by inserting a_i instead of $*$ in t_i for all $i \in [m]$.*

Theorem 13 (auxiliary)**.** *Let χ be an r-colouring of $[n]^D$. Then there exists a root $t := t_1 \dots t_d$ with $|t_i| = D_i$ for all $i \in [d]$ and $\chi(t(a)) = \chi(t(b))$ for all neighbours $a, b \in [n]^d$.*

So we take a root t as above and define a new colouring χ' for $[n-1]^d$ as $\chi'(a) := \chi(t(a))$. Since we now have $n-1$ symbols we get by induction assumption that there exists a root $s = s_1 \dots s_d \in ([n-1] \cup \{*\})^d$ such that the combinatorial line rooted at s is monochromatic with respect to χ'. $t(s)$ yields yet another root and we show that the combinatorial line rooted at $t(s)$ is monochromatic with respect to χ. By the definition of χ' we have $\chi'(s(1)) = \dots = \chi'(s(n-1))$ and therefore $\chi(t(s(1)) = \dots = \chi(t(s(n-1)))$. Now we only need to show that $\chi(t(s(n)))$ is the same colour.

If $t(s)$ has exactly one $*$ then $t(s(n))$ is a neighbour of $t(s(1))$ and therefore has the same colour.

If $t(s)$ has more than one $*$ then $t(s(n))$ still has the same colour because we can pass through a chain of neighbours. For example assume we have three $*$ in the i-th, j-th and k-th position than we get:

$$\begin{aligned}\chi(t(s(n))) &= \chi(\dots \overset{\overset{i}{\downarrow}}{n} \dots \overset{\overset{j}{\downarrow}}{n} \dots \overset{\overset{k}{\downarrow}}{n} \dots) \\ &= \chi(\dots n \dots n \dots 1 \dots) \\ &= \chi(\dots n \dots 1 \dots 1 \dots) \\ &= \chi(\dots 1 \dots 1 \dots 1 \dots) = \chi(t(s(1)))\end{aligned}$$

□

So now the only thing left to do is prove the auxiliary Theorem 13.

Proof. We will provide a backwards induction on i to prove that there exist roots t_i with the desired property. The root t_d can be trivially constructed through the choice of χ. Suppose we now have roots $t_{i+1}, \dots, t_d$. Now we want to construct t_i. We will set $C_{i-1} :=$

$\sum_{j=1}^{i-1} D_j$. (This is the length of the roots $t_1, \ldots, t_{i-1}$ which have to be constructed.) For $k \in \{0, \ldots, D_i\}$ we define

$$W_k := \underbrace{1 \ldots 1}_{k} \underbrace{n \ldots n}_{D_i - k} \ .$$

Let χ_k be an r-colouring of $[n]^{C_{i-1}+d-i}$ as follows:

$$\chi_k(x_1 \ldots x_{C_{i-1}} y_{i+1} \ldots y_d) := \chi(x_1 \ldots x_{C_{i-1}} W_k t_{i+1}(y_{i+1}) \ldots t_d(y_d)) \ .$$

The x_j have length 1 for $j \in [C_{i-1}]$ whereas the y_l have length D_l for $l \in \{i+1, \ldots, d\}$. We now have $D_i + 1$ colourings. The total number of r-colourings for the words in $[n]^{C_{i-1}+d-i}$ is $r^{n^{C_{i-1}+d-i}} \leq r^{n^{C_{i-1}+d}} = D_i$. Therefore by the pigeonhole principle two colourings must be the same. Suppose $\chi_p = \chi_q$ and $p < q$. We can now define our desired root:

$$t_i := \underbrace{1 \ldots 1}_{p} \underbrace{* \ldots *}_{q-p} \underbrace{n \ldots n}_{D_i - q} \ .$$

$t = t_1 \ldots t_d$ satisfies the length condition so we only have to show the condition for the neighbours.

For this first observe that $t_i(1) = W_q$ and $t_i(n) = W_p$. Now suppose a and b are two neighbours that differ in the i-th coordinate:

$$a = a_1 \ldots a_{i-1} 1 a_{i+1} \ldots a_d$$

$$b = a_1 \ldots a_{i-1} n a_{i+1} \ldots a_d$$

then

$$t(a) = t_1(a_1) \ldots t_{i-1}(a_{i-1}) t_i(1) t_{i+1}(a_{i+1}) \ldots t_d(a_d)$$

$$t(b) = t_1(a_1) \ldots t_{i-1}(a_{i-1}) t_i(n) t_{i+1}(a_{i+1}) \ldots t_d(a_d) \ .$$

With the definition of our colourings we get:

$$\begin{aligned}\chi(t(a)) &= \chi(t_1(a_1)\dots t_{i-1}(a_{i-1})t_i(1)t_{i+1}(a_{i+1})\dots t_d(a_d))\\ &= \chi(t_1(a_1)\dots t_{i-1}(a_{i-1})W_q t_{i+1}(a_{i+1})\dots t_d(a_d))\\ &= \chi_q(t_1(a_1)\dots t_{i-1}(a_{i-1})a_{i+1}\dots a_d)\\ &= \chi_p(t_1(a_1)\dots t_{i-1}(a_{i-1})a_{i+1}\dots a_d)\\ &= \chi(t_1(a_1)\dots t_{i-1}(a_{i-1})W_p t_{i+1}(a_{i+1})\dots t_d(a_d))\\ &= \chi(t_1(a_1)\dots t_{i-1}(a_{i-1})t_i(n)t_{i+1}(a_{i+1})\dots t_d(a_d))\\ &= \chi(t(b))\end{aligned}$$

□

There also exists the stronger density version of the Hales-Jewett theorem which we will not prove here. It was first proven by Furstenberg and Katznelson ([10]) in 1989 and in 2009 the Polymath project revealed a simpler and more elementary proof ([18]).

Definition 17 (Density)**.** *Let $A \subseteq [n]^d$. The density δ of A is defined as $\delta := \frac{|A|}{n^d}$.*

Theorem 14 (Density Hales-Jewett theorem)**.** *For every $n \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$ there exists a dimension $d := d(n, \delta)$ such that every subset of $[n]^d$ with density at least δ contains a combinatorial line.*

Chapter 4

Another Approach: Integer Programming

As we have seen in the previous chapters there are many ways to express the problems we are looking at. In this chapter we will look at the approach through integer linear programming following [15].

4.1 Preliminaries

At first we look at integer programming (IP) in general.

A standard integer linear programme has the form:

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leq b, \\ & x \in \mathbb{N}^n, \end{aligned}$$

where $c \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. The relaxation of the problem is:

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leq b, \\ & x \in \mathbb{R}^n, \\ & x \geq 0. \end{aligned}$$

Linear programmes without the integer restriction for x are solvable in polynomial time through interior-point methods. In practice the simplex method of Dantzig is often used which is fast in general but can take exponential time for special cases.

Integer linear programming on the other side is NP-hard. This means that the corresponding decision problem is NP-complete which is the question if for a fixed value $k \in \mathbb{Z}$ the maximum is smaller than k. So integer linear programming is rather time consuming but there are some techniques which systematically can solve this problem.

4.2 Optimization concepts

In this section we will explain two large concepts which are often used in integer linear programming.

4.2.1 Branch-and-Bound

The first and most frequently used concept in integer linear programming is the idea of Branch-and-Bound.

The idea which can also be used for other problems is the following: If we look at an optimization problem $\max_{x \in D} f(x)$ where D is the space of feasible solutions then it is often too time consuming to try every $x \in D$. Therefore we divide D gradually into several subsets (Branches). With suitable bounds we should identify suboptimal solutions early and eliminate them to keep the solution space small. In the worst case we have to enumerate all solutions anyway.

Given a standard integer linear programme:

$$\begin{array}{lrr} \text{maximize} & f = & cx \\ \text{subject to} & & Ax \leq b, \\ & & x \in \mathbb{N}^n \end{array}$$

By omitting the integer restriction we get the continuous relaxation which can be solved with the simplex method. (Because of the required integrity we cannot use the simplex method on our original problem.) We will now use Branch-and-Bound. At first we will set our best objective value so far $\hat{f}$ to be a large negative value (For example we could use the smallest obtainable objective value with regards only to the value restrictions for the variables.) and look at the relaxation. In general the so obtained solution is not integral. This means not all of $x_1, x_2, \ldots, x_n$ are integral. Without loss of generality we assume that this is true for x_1. Now we try to find an integral solution for x_1. Let s_1 be the largest integer smaller than x_1. We can formulate two new optimization problems such that the previous solution is excluded:

$$\begin{aligned} \text{maximize} \quad f = \quad & cx \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \\ & x_1 \leq s_1. \end{aligned} \tag{4.1}$$

$$\begin{aligned} \text{maximize} \quad f = \quad & cx \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \\ & x_1 \geq s_1 + 1. \end{aligned} \tag{4.2}$$

This is called the branching step.

Now we can again solve these problems with the simplex method and the following three cases can occur:

1. The space of feasible solutions becomes empty.
2. An integral optimal solution f is found.
3. x_1 becomes integral but another x_i is not (it does not matter if it was integral in the beginning.)

In case 1 the subproblem is not relevant anymore. This applies also to the other cases if $f \leq \hat{f}$. Otherwise in case 2 we use the computed objective value as our new best objective value so far and substitute $\hat{f}$ by f. And in case 3 we have to further split our problem.

Thus the whole solution space is scanned and an optimal solution is found if there exists one. It is possible there are no suitable solution then the original problem has no feasible one.

4.2.2 Cutting Planes

The concept of cutting planes is also frequently used.

The idea is to look at the continuous relaxation of our integer programme and by adding further inequalities to get an integral solution. At first we look at the relaxation. In general the solution obtained is not integral but it is an upper bound for our problem because every optimal solution of our original problem is a feasible solution for the relaxation. This can be used to measure the quality of a solution of the integral problem. This bound will be tightened by subsequently adding so called cutting planes. A cutting plane (in general a hyper plane) is an additional inequality which every solution of the integral programme fulfils but not the current solution of the relaxation. So if we add this inequality to the relaxation it will lead to another optimal solution. This procedure is continued until an integral solution is obtained (which is then an optimal solution for our problem) or no further inequality can be added.

4.2.3 Branch-and-Cut

The two concepts above can also be combined.

Before we use the Branch-and-Bound method we can add cutting planes and get a solution for the relaxation faster in most cases. In addition further cutting planes can be calculated during the branching process which would not have been found without the restrictions in the sub-problems. These planes can be valid globally, so they are feasible for the original problem, or just locally, here the inequality is only feasible for the current sub-tree. Furthermore we can use heuristics in the sub-problems to obtain feasible solutions faster and then eventually cut off additional sub-trees.

4.3 Integer programming for our problems

We model our problems by taking a d-dimensional tensor x of length n which represents every point in $[n]^d$. (This corresponds directly to how we represent our sets.) All entries of x take values in $\{0,1\}$ with a zero in entry x_v with $v \in [n]^d$ meaning that we do not take the vector v and a one that we take it. (We are abusing notation by writing $x_{(a_1,a_2,\ldots,a_n)} := x_{a_1,a_2,\ldots,a_n}$) So what else do we need to specify the problems? We need special restrictions to handle every line, may it be combinatorial, geometric or algebraic. So the programme looks like this.

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in [n]^d} x_v \\ \text{subject to} \quad & Ax \leq b, \\ & x_v \in \{0,1\}, \quad v \in [n]^d . \end{aligned}$$

Here A and b are unspecified. Since x is an n^d tensor Ax is a tensor product with $A \in \{0,1\}^{m \times n^d}$ where $m \in \mathbb{N}$ is the number of possible lines in our set. So in the m-th sub-tensor we have a set of n vectors $S \subseteq [n]^d$ where $A_{m,v} = 1$ for all $v \in S$ (Here $A_{m,v}$ means that we concatenate m and v into one vector, i.e $A_{m,(a_1,a_2,\dots,a_n)} := A_{m,a_1,a_2,\dots,a_n}$). And b is an $m \times 1$ vector where all entries are $n-1$. So we will have the following equations in the m-th line:

$$\sum_{v \in [n]^d} A_{m,v} x_v \leq n-1.$$

For further insight we look at some examples in $[2]^2$:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{v \in [2]^2} x_v = x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \\ \text{subject to} & Ax \leq b, \\ & x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \in \{0,1\}. \end{array}$$

For line-free sets we get:

$$\begin{array}{lccccccclcl} \text{maximize} & x_{1,1} & + & x_{1,2} & + & x_{2,1} & + & x_{2,2} & = & c_{2,2} \\ \text{subject to} & x_{1,1} & + & x_{1,2} & & & & & \leq & 1 \\ & & & & & x_{2,1} & + & x_{2,2} & \leq & 1 \\ & x_{1,1} & + & & & x_{2,1} & & & \leq & 1 \\ & & & x_{1,2} & + & & & x_{2,2} & \leq & 1 \\ & x_{1,1} & + & & & & & x_{2,2} & \leq & 1 \\ & x_{1,1} & , & x_{1,2} & , & x_{2,1} & , & x_{2,2} & \in & \{0,1\}. \end{array}$$

For Moser sets and in this case also cap sets there is an additional restriction:

$$\begin{array}{lccccccclcl} \text{maximize} & x_{1,1} & + & x_{1,2} & + & x_{2,1} & + & x_{2,2} & = & g_{2,2} = a_{2,2} \\ \text{subject to} & x_{1,1} & + & x_{1,2} & & & & & \leq & 1 \\ & & & & & x_{2,1} & + & x_{2,2} & \leq & 1 \\ & x_{1,1} & + & & & x_{2,1} & & & \leq & 1 \\ & & & x_{1,2} & + & & & x_{2,2} & \leq & 1 \\ & x_{1,1} & + & & & & & x_{2,2} & \leq & 1 \\ & & & x_{1,2} & + & x_{2,1} & & & \leq & 1 \\ & x_{1,1} & , & x_{1,2} & , & x_{2,1} & , & x_{2,2} & \in & \{0,1\}. \end{array}$$

Chapter 5

Computational Values

In this chapter we present the computational values for our problems where we use the software IBM® ILOG® CPLEX® Optimization Studio [12] for optimizing and the modelling language AMPL [2]. All of these values have been computed on a PC with 4 Intel® Core™ i5-6500 @3.20 Ghz processors and 8 GB RAM. At the beginning of each section we will also show the optimization programme for the corresponding problems. The tables show the computed values, the time taken and the maximum size of the decision tree used by the Branch-and-Bound method which is a good indicator for how much space was needed. In appendix A all configurations are listed which were computed this way. Additionally we give the statistics for them which provides us with additional information about the structure.

Definition 18. *A g-dimensional statistic in direction(s) $D \subseteq [d]$ with $|D| = d - g$ for a set $A \subseteq [n]^d$ is a $n^{|D|}$-tuple $(i_j)_{j \in [n]^{|D|}}$ with $i_j = |A \cap \{(k_1, \ldots, k_d) \in [n]^d | (k_l)_{l \in D} = j\}|$.*

For cap sets we also allow affine transformations before making the statistic. We will mainly list $d-1$-dimensional statistics for $d \geq 3$. Another statistic for cap sets with $n = 4$ is to look at the number of points in sub-hypercubes of length 2.

5.1 Computational values for $d = 2$

5.1.1 Algebraic lines

In the two-dimensional case for cap sets we have to specify every algebraic line in our restrictions. At first we take the lines which represent each row and column (one-dimensional slices) of the variable matrix. This is the case when either k or m in the restriction below is equal to n. Then we need to specify the remaining lines. For n prime this is fairly easy because every line is a permutation. For n composite we already said that we only look at lines where no coordinate is repeated amongst the points. So we have similar restrictions.

$$\begin{aligned}
\text{maximize} \quad & \sum_{i,j\in[n]} x_{i,j} \\
\text{subject to} \quad & \sum_{i\in[n]} x_{((j+(i-1)k-1) \bmod n)+1,((l+(i-1)m-1) \bmod n)+1} \leq n-1 \\
& \qquad \forall j,l \in [n] \text{ and } (k,m) \in (\{i|\, \gcd(i,n)=1\} \cup \{n\})^2 \setminus \{(n,n)\}, \\
& x_{j,k} \in \{0,1\} \qquad \forall j,k \in [n].
\end{aligned}$$

Note how for n prime Table 5.1 shows the values we have proven in Theorem 5.

n	$a_{n,2}$	CPU time	max. size of decision tree
3	4	< 0.01 sec	< 0.01 MB
4	10	< 0.01 sec	< 0.01 MB
5	16	0.1875 sec	< 0.01 MB
6	28	0.09375 sec	< 0.01 MB
7	36	93.9062 sec	9.27 MB
8	52	4.46875 sec	< 0.01 MB
9	66	673.781 sec	12.72 MB
10	86	27.2344 sec	0.46 MB
12	130	12.9531 sec	0.4 MB

Table 5.1: Table of computational values for $a_{n,2}$

For $n = 11$ the program aborted after two hours of computation with an out of memory error.

The following sets in figure 5.1 are the two solutions computed by our programme for $n = 4$ and $n = 5$:

$a_{n,4}$: $\{(1,1), (1,2), (1,3), (2,1), (2,3), (2,4), (3,2), (4,1), (4,3), (4,4)\}$

$a_{n,5}$: $\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4)\}$

Whilst the solution for the case $n = 5$ (and especially for every other prime computed) looks like the solution proven by Jamison [13] and Alon [1], for $n = 4$ the point set does not exhibit any particular structure.

	1	2	3	4
1	×	×	○	×
2	×	○	×	○
3	×	×	○	×
4	○	×	○	×

	1	2	3	4	5
1	×	×	×	×	○
2	×	×	×	×	○
3	×	×	×	×	○
4	×	×	×	×	○
5	○	○	○	○	○

Figure 5.1: Configurations for $a_{4,2}$ and $a_{5,2}$

5.2 Computational values for $d = 3$

5.2.1 Combinatorial lines

We have already proven these values but for completeness we have computed them anyway. To specify the lines we need all one-dimensional slices which are the top three restriction sets in the second column below. Then we need the main diagonal in every two-dimensional slice (top three in the first column) and the overall main diagonal (fourth line below). As we have shown in the proof of Theorem 7 in section 2 we can assume that all elements on the main diagonal are zero, which speeds up the computation time.

$$\begin{array}{lllll}
\text{maximize} & \sum_{i,j,k\in[n]} x_{i,j,k} & & & \\
\text{subject to} & \sum_{i\in[n]} x_{i,i,j} \leq n-1 & \forall j\in[n], & \sum_{i\in[n]} x_{i,j,k} \leq n-1 & \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{i,j,i} \leq n-1 & \forall j\in[n], & \sum_{i\in[n]} x_{j,i,k} \leq n-1 & \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{j,i,i} \leq n-1 & \forall j\in[n], & \sum_{i\in[n]} x_{j,k,i} \leq n-1 & \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{i,i,i} \leq n-1 & & x_{i,i,i} = 0 & \forall i\in[n], \\
& x_{j,k,l} \in \{0,1\} & \forall j,k,l\in[n]. & &
\end{array}$$

Table 5.2 shows exactly the values provided by Theorem 7.

5.2.2 Geometric lines

For the geometric lines we need all the combinatorial lines which are listed in the first four rows below. The other three lines represent the diagonals in the two-dimensional slices

n	$c_{n,3}$	CPU time	max. size of decision tree
3	18	< 0.01 sec	< 0.01 MB
4	48	0.03125 sec	< 0.01 MB
5	100	0.046875 sec	< 0.01 MB
6	180	0.15625 sec	< 0.01 MB
7	294	0.65625 sec	< 0.01 MB
8	448	0.34375 sec	< 0.01 MB
9	648	7.10938 sec	1.70 MB
10	900	5.10938 sec	0.17 MB
11	1210	5.20312 sec	0.14 MB
12	1584	1.96875 sec	< 0.01 MB
13	2028	21.5781 sec	1.06 MB
14	2548	8.46875 sec	< 0.01 MB
15	3150	5.89062 sec	< 0.01 MB
16	3840	8.59375 sec	< 0.01 MB
17	4624	260.188 sec	17.26 MB
18	5508	14.3906 sec	< 0.01 MB
19	6498	1978.83 sec	76.09 MB
20	7600	761.375 sec	30.10 MB

Table 5.2: Table of computational values for $c_{n,3}$

(second column) and the other diagonals in the third dimension (first column).

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i,j,k\in[n]} x_{i,j,k} \\
\text{subject to} \quad & \sum_{i\in[n]} x_{i,i,i} \le n-1, \\
& \sum_{i\in[n]} x_{i,i,j} \le n-1 \quad \forall j\in[n], \qquad \sum_{i\in[n]} x_{i,j,k} \le n-1 \quad \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{i,j,i} \le n-1 \quad \forall j\in[n], \qquad \sum_{i\in[n]} x_{j,i,k} \le n-1 \quad \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{j,i,i} \le n-1 \quad \forall j\in[n], \qquad \sum_{i\in[n]} x_{j,k,i} \le n-1 \quad \forall j,k\in[n], \\
& \sum_{i\in[n]} x_{i,i,n-i+1} \le n-1, \qquad \sum_{i\in[n]} x_{i,n-i+1,j} \le n-1 \quad \forall j\in[n], \\
& \sum_{i\in[n]} x_{i,n-i+1,i} \le n-1, \qquad \sum_{i\in[n]} x_{n-i+1,j,i} \le n-1 \quad \forall j\in[n], \\
& \sum_{i\in[n]} x_{n-i+1,i,i} \le n-1, \qquad \sum_{i\in[n]} x_{j,i,n-i+1} \le n-1 \quad \forall j\in[n], \\
& x_{j,k,l} \in \{0,1\} \quad \forall j,k,l\in[n].
\end{aligned}
$$

As we can see in Table 5.3 for the prime values larger than 6 we get the values we have proven in Theorem 8. Also note that the composite values have the form $n^3 - n^2$.

5.2.3 Algebraic lines

The three-dimensional case for algebraic lines is very similar to the two-dimensional case.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i,j,k\in[n]} x_{i,j,k} \\
\text{subject to} \quad & \sum_{i\in[n]} x_{((j+(i-1)k-1) \bmod n)+1,((l+(i-1)m-1) \bmod n)+1,((p+(i-1)q-1) \bmod n)+1} \le n-1 \\
& \qquad \forall j,l,p\in[n] \text{ and } (k,m,q)\in[n]^3 \setminus \{i\in[n] \,|\, \gcd(i,n)\neq 1\}^3, \\
& x_{j,k,l} \in \{0,1\} \quad \forall j,k,l\in[n].
\end{aligned}
$$

Table 5.4 shows the values $a_{3,3}$, $a_{4,3}$ and $a_{5,3}$.

For $n = 5$ the program already took more than two hours before it stopped with an out of memory error therefore we unfortunately have not many values here.

n	$g_{n,3}$	CPU time	max. size of decision tree
3	16	< 0.01 sec	< 0.01 MB
4	45	0.09375 sec	< 0.01 MB
5	97	1.89062 sec	< 0.01 MB
6	177	35.0625 sec	1.73 MB
7	294	0.21875 sec	< 0.01 MB
8	448	246.906 sec	83.4 MB
9	648	293.625 sec	54.1 MB
10	900	1164.19 sec	214.28 MB
11	1210	285.594 sec	84.45 MB
12	1584	486.766 sec	57.73 MB
13	2028	795.875 sec	44.3 MB
14	2548	478.5 sec	36.57 MB
15	3150	2953.58 sec	97.26 MB
16	3840	2606.14 sec	33.06 MB
17	4624	2254.72 sec	31.14 MB
18	5508	890.328 sec	18.86 MB
19	6498	1475.55 sec	18.46 MB
20	7600	1529.52 sec	19.52 MB

Table 5.3: Table of computational values for $g_{n,3}$

n	$a_{n,3}$	CPU time	max. size of decision tree
3	9	0.078125 sec	< 0.01 MB
4	36	0.40625 sec	< 0.01 MB

Table 5.4: Table of computational values for $a_{n,3}$

5.3 Computational values for $d = 4$

5.4 Combinatorial lines

The four-dimensional case for combinatorial lines is also similar to the three-dimensional case. We have to take care of all one-dimensional slices and all diagonals in two-dimensional and three-dimensional slices.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i,j,k,l\in[n]} x_{i,j,k,l} \\
\text{subject to} \quad & \sum_{i\in[n]} x_{i,i,i,j} \leq n-1 \quad \forall j \in [n], && \sum_{i\in[n]} x_{i,j,k,l} \leq n-1 \quad \forall j,k,l \in [n], \\
& \sum_{i\in[n]} x_{i,i,j,i} \leq n-1 \quad \forall j \in [n], && \sum_{i\in[n]} x_{j,i,k,l} \leq n-1 \quad \forall j,k,l \in [n], \\
& \sum_{i\in[n]} x_{i,j,i,i} \leq n-1 \quad \forall j \in [n], && \sum_{i\in[n]} x_{j,k,i,l} \leq n-1 \quad \forall j,k,l \in [n], \\
& \sum_{i\in[n]} x_{j,i,i,i} \leq n-1 \quad \forall j \in [n], && \sum_{i\in[n]} x_{j,k,l,i} \leq n-1 \quad \forall j,k,l \in [n], \\
& \sum_{i\in[n]} x_{i,i,j,k} \leq n-1 \quad \forall j,k \in [n], && \sum_{i\in[n]} x_{i,j,i,k} \leq n-1 \quad \forall j,k \in [n], \\
& \sum_{i\in[n]} x_{i,j,k,i} \leq n-1 \quad \forall j,k \in [n], && \sum_{i\in[n]} x_{j,k,i,i} \leq n-1 \quad \forall j,k \in [n], \\
& \sum_{i\in[n]} x_{j,i,k,i} \leq n-1 \quad \forall j,k \in [n], && \sum_{i\in[n]} x_{j,i,i,k} \leq n-1 \quad \forall j,k \in [n], \\
& \sum_{i\in[n]} x_{i,i,i,i} \leq n-1, \\
& \quad x_{j,k,l,m} \in \{0,1\} \quad \forall j,k,l,m \in [n].
\end{aligned}
$$

In Table 5.5 we have some values for the 4-dimensional case of combinatorial lines.

n	$c_{n,4}$	CPU time	max. size of decision tree
3	52	0.046875	$<$ 0.01 MB
4	183	761.656 sec	102.21 MB
5	500	0.984375 sec	$<$ 0.01 MB

Table 5.5: Table of computational values for $c_{n,4}$

For $n = 6$ and $n = 7$ there occurred an out of memory error.

5.5 Computational values for $d = 3$ and $n - 1$ progressions

5.5.1 Combinatorial lines

Now we remember the definition of $c^1_{n,3}$: It is the largest set without a combinatorial line of length $n - 1$ in $[n]^3$. Therefore we have to satisfy that in every combinatorial line of length $n-1$ at most $n-2$ points are chosen. For $n=3$ we have additional restrictions since for example $(1,1,1)$ and $(3,3,3)$ is also a progression of length 2 in $[3]^3$.

$$
\begin{array}{lll}
\text{maximize} & \sum_{i,j,k\in[n]} x_{i,j,k} & \\
\text{subject to} & \sum_{i\in[n-1]} x_{i+j,i+k,i+l} \le n-2 & \forall j,k,l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+k,i+l,j} \le n-2 & \forall j\in[n] \text{ and } k,l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+k,j,i+l} \le n-2 & \forall j\in[n] \text{ and } k,l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,i+k,i+l} \le n-2 & \forall j\in[n] \text{ and } k,l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+l,j,k} \le n-2 & \forall j,k\in[n] \text{ and } l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,i+l,k} \le n-2 & \forall j,k\in[n] \text{ and } l\in\{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,k,i+l} \le n-2 & \forall j,k\in[n] \text{ and } l\in\{0,1\}, \\
& x_{j,k,l}\in\{0,1\} & \forall j,k,l\in[n].
\end{array}
$$

Table 5.6 shows computational values for $n \le 8$.

n	$c^1_{n,3}$	CPU time	max. size of decision tree
3	7	0.015625 sec	< 0.01 MB
4	39	0.15625 sec	< 0.01 MB
5	90	0.96875 sec	< 0.01 MB
6	173	0.109375 sec	< 0.01 MB
7	283	242.344 sec	38.63 MB
8	439	1.70312 sec	< 0.01 MB

Table 5.6: Table of computational values for $c^1_{n,3}$

5.5.2 Geometric lines

We need all geometric lines of length $n-1$. Therefore we can add all restrictions from the combinatorial lines of length $n-1$ and some other which are similar to the restrictions in the case of geometric lines of length n.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i,j,k\in[n]} x_{i,j,k} \\
\text{subject to} \quad & \sum_{i\in[n-1]} x_{i+j,i+k,i+l} \le n-2 && \forall j,k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+k,i+l,j} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+k,j,i+l} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,i+k,i+l} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+l,j,k} \le n-2 && \forall j,k \in [n] \text{ and } l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,i+l,k} \le n-2 && \forall j,k \in [n] \text{ and } l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,k,i+l} \le n-2 && \forall j,k \in [n] \text{ and } l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{n-i+1-j,i+k,i+l} \le n-2 && \forall j,k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+j,n-i+1-k,i+l} \le n-2 && \forall j,k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+j,i+k,n-i+1-l} \le n-2 && \forall j,k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{i+k,n-i+1-l,j} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{n-i+1-l,j,i+k} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& \sum_{i\in[n-1]} x_{j,i+k,n-i+1-l} \le n-2 && \forall j \in [n] \text{ and } k,l \in \{0,1\}, \\
& x_{j,k,l} \in \{0,1\} && \forall j,k,l \in [n].
\end{aligned}
$$

Table 5.7 shows computational values for $n \le 6$.

n	$g^1_{n,3}$	CPU time	max. size of decision tree
3	4	< 0.01 sec	< 0.01 MB
4	32	0.125 sec	< 0.01 MB
5	82	257.531 sec	0.88 MB

Table 5.7: Table of computational values for $g^1_{n,3}$

5.6 Computational values for $d = 3$ and $n - 2$ progressions

5.6.1 Combinatorial lines

Here we need all combinatorial lines of length $n - 2$. The restrictions are nearly identical to the ones we had in the $(n - 1)$-length case. We have to be careful for $n = 4$ and $n = 5$ because for both we need additional restrictions. For $n = 4$ $(1, 1, 1)$ and $(4, 4, 4)$ is a two-progression and for $n = 5$ $(1, 1, 1)$, $(3, 3, 3)$ and $(5, 5, 5)$ is a three-progression not considered by the restrictions below.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i,j,k \in [n]} x_{i,j,k} & & \\
\text{subject to} \quad & \sum_{i \in [n-2]} x_{i+j,i+k,i+l} \le n - 3 & & \forall j, k, l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{i+k,i+l,j} \le n - 3 & & \forall j \in [n] \text{ and } k, l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{i+k,j,i+l} \le n - 3 & & \forall j \in [n] \text{ and } k, l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{j,i+k,i+l} \le n - 3 & & \forall j \in [n] \text{ and } k, l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{i+l,j,k} \le n - 3 & & \forall j, k \in [n] \text{ and } l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{j,i+l,k} \le n - 3 & & \forall j, k \in [n] \text{ and } l \in \{0, 1, 2\}, \\
& \sum_{i \in [n-2]} x_{j,k,i+l} \le n - 3 & & \forall j, k \in [n] \text{ and } l \in \{0, 1, 2\}, \\
& x_{j,k,l} \in \{0, 1\} & & \forall j, k, l \in [n].
\end{aligned}
$$

Table 5.8 shows computational values for $n \le 9$.

n	$c^2_{n,3}$	CPU time	max. size of decision tree
4	13	0.046875 sec	< 0.01 MB
5	75	0.40625 sec	< 0.01 MB
6	152	143.109 sec	43.41 MB
7	276	0.046875 sec	< 0.01 MB
8	423	312.562 sec	41.75 MB
9	626	1.73438 sec	< 0.01 MB

Table 5.8: Table of computational values for $c^2_{n,3}$

The programme could not complete with $n = 10$, due to memory constraints.

5.6.2 Geometric lines

As in the case of length $n-1$ for length $n-2$ we have similar restrictions and need to be careful as in the combinatorial case.

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{i,j,k\in[n]} x_{i,j,k} & \\
\text{subject to} & \displaystyle\sum_{i\in[n-2]} x_{i+j,i+k,i+l} \leq n-3 & \forall j,k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+k,i+l,j} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+k,j,i+l} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{j,i+k,i+l} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+l,j,k} \leq n-3 & \forall j,k \in [n] \text{ and } l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{j,i+l,k} \leq n-3 & \forall j,k \in [n] \text{ and } l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{j,k,i+l} \leq n-3 & \forall j,k \in [n] \text{ and } l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{n-i+1-j,i+k,i+l} \leq n-3 & \forall j,k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+j,n-i+1-k,i+l} \leq n-3 & \forall j,k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+j,i+k,n-i+1-l} \leq n-3 & \forall j,k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{i+k,n-i+1-l,j} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{n-i+1-l,j,i+k} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& \displaystyle\sum_{i\in[n-2]} x_{j,i+k,n-i+1-l} \leq n-3 & \forall j \in [n] \text{ and } k,l \in \{0,1,2\}, \\
& x_{j,k,l} \in \{0,1\} & \forall j,k,l \in [n].
\end{array}
$$

Table 5.9 shows computational values for $n = 4$ and $n = 5$.
Here for $n = 6$ and $n = 7$ an out of memory error occurred.

n	$g^2_{n,3}$	CPU time	max. size of decision tree
4	8	< 0.01 sec	< 0.1 MB
5	64	12.4531 sec	< 0.1 MB

Table 5.9: Table of computational values for $g^2_{n,3}$

5.7 Computational values for $n = 4$ and 3 progressions

5.7.1 Algebraic lines

In this section we computed $a^1_{4,d}$ for different d. We show how the programme looks for $d = 3$. For the other values the programmes look similar.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j,k,l\in[n]} x_{j,k,l} \\
\text{subject to} \quad & \sum_{i\in\{1,2,3\}} x_{((j+(i-1)*k-1) \bmod 4)+1,((l+(i-1)*m-1) \bmod 4)+1,((n+(i-1)*o-1) \bmod 4)+1} <= 2, \\
& \sum_{i\in\{2,3,4\}} x_{((j+(i-1)*k-1) \bmod 4)+1,((l+(i-1)*m-1) \bmod 4)+1,((n+(i-1)*o-1) \bmod 4)+1} <= 2, \\
& \sum_{i\in\{1,3,4\}} x_{((j+(i-1)*k-1) \bmod 4)+1,((l+(i-1)*m-1) \bmod 4)+1,((n+(i-1)*o-1) \bmod 4)+1} <= 2, \\
& \sum_{i\in\{1,2,4\}} x_{((j+(i-1)*k-1) \bmod 4)+1,((l+(i-1)*m-1) \bmod 4)+1,((n+(i-1)*o-1) \bmod 4)+1} <= 2, \\
& \qquad \forall j,l,n \in [4] \text{ and } (k,m,o) \in [4]^3 \setminus \{2,4\}^3, \\
& x_{j,k,l} \in \{0,1\} \qquad \forall j,k,l \in [n].
\end{aligned}
$$

Table 5.10 shows values for $d \leq 4$.

d	$a^1_{4,d}$	CPU time	max. size of decision tree
2	6	0.0625 sec	< 0.01 MB
3	16	0.546875 sec	< 0.01 MB
4	42	1057.81 sec	7.49 MB

Table 5.10: Table of computational values for $a^1_{4,d}$

Chapter 6

Conclusion

In chapter 3 we at first explored the case $d = 2$. It resulted in the fact that we could reach the trivial upper bound for almost every value of n in the cases of combinatorial and geometric lines. For algebraic lines we showed that the trivial lower bound is the actual value in the case of n prime.

We then moved on to the case $d = 3$ and developed a construction for combinatorial lines with diagonal Latin squares to show that we can also reach the upper bound for almost every n. We then used a similar approach for geometric lines to show the same for n prime but invested some more time in developing permutations to fit into a Latin square.

We could then use these permutations also in higher dimensions and show that for every d there exists a threshold such that for every prime n over this threshold we reach the upper bound. This can be shown in both combinatorial and geometric cases.

After this we used the opportunity to look at similar problems and proved a weaker version of the Hales-Jewett theorem which is very crucial in Ramsey theory.

For chapter 4 we showed that our problems could also be explained in integer linear programmes and used this to compute several values which are shown in chapter 5.

Chapter 7

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Appendices

Appendix A

Configurations and Statistics

Note that all configurations and statistics here are just the ones computed by our programme. There are possibly other optimal configurations where the statistics differ.

A.1 Configurations for $d = 2$

A.1.1 Algebraic lines

A.1.1.1 $n = 3$

	1	2	3
1	×	×	○
2	×	×	○
3	○	○	○

Figure A.1: Configuration for $a_{3,2} = 4$

A.1.1.2 $n = 4$

	1	2	3	4
1	×	×	○	×
2	×	○	×	○
3	×	×	○	×
4	○	×	○	×

Figure A.2: Configuration for $a_{4,2} = 10$

A.1.1.3 $n = 5$

	1	2	3	4	5
1	×	×	×	×	○
2	×	×	×	×	○
3	×	×	×	×	○
4	×	×	×	×	○
5	○	○	○	○	○

Figure A.3: Configuration for $a_{5,2} = 16$

A.1.1.4 $n = 6$

	1	2	3	4	5	6
1	×	×	○	×	×	×
2	×	×	○	×	×	×
3	×	×	×	×	○	○
4	×	○	×	×	×	×
5	×	×	×	○	×	×
6	○	×	×	×	×	○

Figure A.4: Configuration for $a_{6,2} = 28$

A.1.1.5 $n = 7$

	1	2	3	4	5	6	7
1	×	×	×	×	×	×	○
2	×	×	×	×	×	×	○
3	×	×	×	×	×	×	○
4	×	×	×	×	×	×	○
5	×	×	×	×	×	×	○
6	×	×	×	×	×	×	○
7	○	○	○	○	○	○	○

Figure A.5: Configuration for $a_{7,2} = 36$

A.1.1.6 $n = 8$

	1	2	3	4	5	6	7	8
1	×	×	○	×	×	×	○	×
2	×	○	×	×	×	×	×	×
3	×	○	×	×	×	○	×	×
4	×	×	○	×	×	×	×	×
5	○	×	×	×	○	×	×	×
6	×	×	×	×	×	×	×	○
7	×	×	×	○	×	×	×	○
8	×	×	×	×	○	×	×	×

Figure A.6: Configuration for $a_{8,2} = 52$

A.1.1.7 $n = 9$

	1	2	3	4	5	6	7	8	9
1	×	×	○	×	×	○	○	○	○
2	○	×	×	×	×	×	×	×	×
3	×	×	×	×	×	×	○	×	×
4	×	×	×	×	×	×	○	○	×
5	×	×	×	×	○	×	×	×	×
6	×	×	×	×	×	○	×	×	×
7	×	×	×	×	×	×	○	○	×
8	×	○	×	×	×	×	×	×	×
9	×	×	×	○	×	×	×	×	×

Figure A.7: Configuration for $a_{9,2} = 66$

A.1.1.8 $n = 10$

	1	2	3	4	5	6	7	8	9	10
1	×	×	×	×	×	×	×	×	○	×
2	×	×	×	×	○	×	×	×	×	×
3	×	○	×	×	×	×	×	×	○	×
4	×	×	×	○	×	×	○	×	×	×
5	×	×	×	×	×	×	×	○	×	×
6	×	×	○	×	×	×	×	×	×	×
7	×	○	×	×	×	×	×	×	○	×
8	○	×	×	×	×	×	×	×	×	×
9	×	×	×	×	×	×	×	×	×	○
10	×	×	×	○	×	○	×	×	×	×

Figure A.8: Configuration for $a_{10,2} = 86$

A.1.1.9 $n = 12$

	1	2	3	4	5	6	7	8	9	10	11	12
1	×	×	×	×	×	×	×	○	×	×	×	×
2	×	×	×	○	×	×	×	×	×	×	×	×
3	×	×	×	×	×	×	×	×	×	×	×	○
4	×	×	○	×	×	×	×	×	×	×	×	×
5	×	×	×	×	×	○	×	×	○	×	×	×
6	×	○	×	×	×	×	×	×	×	×	×	×
7	×	×	×	×	×	×	×	×	×	○	×	×
8	○	×	×	×	○	×	×	×	×	×	×	×
9	×	×	×	×	×	×	○	×	×	×	×	×
10	×	×	×	○	×	×	×	×	×	×	×	×
11	×	×	×	×	×	×	×	×	×	×	○	×
12	×	×	×	×	×	×	×	×	×	×	×	○

Figure A.9: Configuration for $a_{12,2} = 130$

A.2 Configurations for $d = 3$

A.2.1 Combinatorial lines

We will only list the configurations until $n = 9$ because it would take too much space to list all larger values. The statistics in this section are all the same in every direction because of Theorem 7. The optimal values have the form $n^3 - n^2$ and so in every two-dimensional sub-slice the number of points must be $n^2 - n$.

A.2.1.1 $n = 3$

1	1	2	3
1	○	×	×
2	×	×	○
3	×	○	×

2	1	2	3
1	×	×	○
2	×	○	×
3	○	×	×

3	1	2	3
1	×	○	×
2	○	×	×
3	×	×	○

Figure A.10: Configuration for $c_{3,3} = 18$

direction	statistic
x (1)	(6,6,6)
y (2)	(6,6,6)
z (3)	(6,6,6)

Table A.1: 2-dimensional statistics for $c_{3,3}$

A.2.1.2 $n = 4$

1	1	2	3	4
1	○	×	×	×
2	×	×	×	○
3	×	○	×	×
4	×	×	○	×

2	1	2	3	4
1	×	×	○	×
2	×	○	×	×
3	×	×	×	○
4	○	×	×	×

3	1	2	3	4
1	×	×	×	○
2	○	×	×	×
3	×	×	○	×
4	×	○	×	×

4	1	2	3	4
1	×	○	×	×
2	×	×	○	×
3	○	×	×	×
4	×	×	×	○

Figure A.11: Configuration for $c_{4,3} = 48$

direction	statistic
x (1)	(12,12,12,12)
y (2)	(12,12,12,12)
z (3)	(12,12,12,12)

Table A.2: 2-dimensional statistics for $c_{4,3}$

A.2.1.3 $n = 5$

1	1	2	3	4	5
1	×	×	○	×	×
2	×	○	×	×	×
3	○	×	×	×	×
4	×	×	×	×	○
5	×	×	×	○	×

2	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	×	×	○
4	×	×	×	○	×
5	×	×	○	×	×

3	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

4	1	2	3	4	5
1	×	×	×	×	○
2	×	×	×	○	×
3	×	×	○	×	×
4	×	○	×	×	×
5	○	×	×	×	×

5	1	2	3	4	5
1	×	×	×	○	×
2	×	×	○	×	×
3	×	○	×	×	×
4	○	×	×	×	×
5	×	×	×	×	○

Figure A.12: Configuration for $c_{5,3} = 100$

direction	statistic
x (1)	(20,20,20,20,20)
y (2)	(20,20,20,20,20)
z (3)	(20,20,20,20,20)

Table A.3: 2-dimensional statistics for $c_{5,3}$

A.2.1.4 $n = 6$

1	1	2	3	4	5	6
1	○	×	×	×	×	×
2	×	×	×	×	○	×
3	×	×	×	○	×	×
4	×	○	×	×	×	×
5	×	×	×	×	×	○
6	×	×	○	×	×	×

2	1	2	3	4	5	6
1	×	×	×	×	×	○
2	×	×	○	×	×	×
3	×	○	×	×	×	×
4	×	×	×	○	×	×
5	○	×	×	×	×	×
6	×	×	×	×	○	×

3	1	2	3	4	5	6
1	×	○	×	×	×	×
2	×	×	×	○	×	×
3	×	×	×	×	○	×
4	○	×	×	×	×	×
5	×	×	○	×	×	×
6	×	×	×	×	×	○

4	1	2	3	4	5	6
1	×	×	×	×	○	×
2	○	×	×	×	×	×
3	×	×	○	×	×	×
4	×	×	×	×	×	○
5	×	○	×	×	×	×
6	×	×	×	○	×	×

5	1	2	3	4	5	6
1	×	×	○	×	×	×
2	×	○	×	×	×	×
3	×	×	×	×	×	○
4	×	×	×	×	○	×
5	×	×	×	○	×	×
6	○	×	×	×	×	×

6	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	×	×	×	×	○
3	○	×	×	×	×	×
4	×	×	○	×	×	×
5	×	×	×	×	○	×
6	×	○	×	×	×	×

Figure A.13: Configuration for $c_{6,3} = 180$

direction	statistic
x (1)	(30,30,30,30,30,30)
y (2)	(30,30,30,30,30,30)
z (3)	(30,30,30,30,30,30)

Table A.4: 2-dimensional statistics for $c_{6,3}$

A.2.1.5 $n = 7$

1	1	2	3	4	5	6	7
1	○	×	×	×	×	×	×
2	×	×	×	×	○	×	×
3	×	×	×	○	×	×	×
4	×	×	×	×	×	×	○
5	×	×	×	×	×	○	×
6	×	×	○	×	×	×	×
7	×	○	×	×	×	×	×

2	1	2	3	4	5	6	7
1	×	×	×	×	×	×	○
2	×	×	×	×	×	○	×
3	×	×	×	×	○	×	×
4	×	×	×	○	×	×	×
5	×	×	○	×	×	×	×
6	×	○	×	×	×	×	×
7	○	×	×	×	×	×	×

3	1	2	3	4	5	6	7
1	×	×	×	○	×	×	×
2	○	×	×	×	×	×	×
3	×	×	○	×	×	×	×
4	×	×	×	×	○	×	×
5	×	○	×	×	×	×	×
6	×	×	×	×	×	×	○
7	×	×	×	×	×	○	×

4	1	2	3	4	5	6	7
1	×	×	×	×	○	×	×
2	×	×	×	○	×	×	×
3	×	×	×	×	×	×	○
4	×	○	×	×	×	×	×
5	○	×	×	×	×	×	×
6	×	×	×	×	×	○	×
7	×	×	○	×	×	×	×

5	1	2	3	4	5	6	7
1	×	○	×	×	×	×	×
2	×	×	×	×	×	×	○
3	×	×	×	×	×	○	×
4	×	×	○	×	×	×	×
5	×	×	×	×	○	×	×
6	○	×	×	×	×	×	×
7	×	×	×	○	×	×	×

6	1	2	3	4	5	6	7
1	×	×	○	×	×	×	×
2	×	○	×	×	×	×	×
3	○	×	×	×	×	×	×
4	×	×	×	×	×	○	×
5	×	×	×	×	×	×	○
6	×	×	×	○	×	×	×
7	×	×	×	×	○	×	×

7	1	2	3	4	5	6	7
1	×	×	×	×	×	○	×
2	×	×	○	×	×	×	×
3	×	○	×	×	×	×	×
4	○	×	×	×	×	×	×
5	×	×	×	○	×	×	×
6	×	×	×	×	○	×	×
7	×	×	×	×	×	×	○

Figure A.14: Configuration for $c_{7,3} = 294$

direction	statistic
x (1)	(42,42,42,42,42,42,42)
y (2)	(42,42,42,42,42,42,42)
z (3)	(42,42,42,42,42,42,42)

Table A.5: 2-dimensional statistics for $c_{7,3}$

A.2.1.6 $n = 8$

1	1	2	3	4	5	6	7	8
1	○	×	×	×	×	×	×	×
2	×	×	×	×	×	×	○	×
3	×	×	×	×	×	×	×	○
4	×	×	×	×	×	○	×	×
5	×	○	×	×	×	×	×	×
6	×	×	×	○	×	×	×	×
7	×	×	×	×	○	×	×	×
8	×	×	○	×	×	×	×	×

2	1	2	3	4	5	6	7	8
1	×	×	×	×	○	×	×	×
2	×	○	×	×	×	×	×	×
3	×	×	×	×	×	○	×	×
4	×	×	×	×	×	×	×	○
5	×	×	×	×	×	×	○	×
6	×	×	○	×	×	×	×	×
7	×	×	×	○	×	×	×	×
8	○	×	×	×	×	×	×	×

3	1	2	3	4	5	6	7	8
1	×	×	×	×	×	×	×	○
2	×	×	×	×	×	○	×	×
3	×	×	○	×	×	×	×	×
4	×	○	×	×	×	×	×	×
5	×	×	×	○	×	×	×	×
6	×	×	×	×	×	×	○	×
7	○	×	×	×	×	×	×	×
8	×	×	×	×	○	×	×	×

4	1	2	3	4	5	6	7	8
1	×	×	×	×	×	×	○	×
2	×	×	○	×	×	×	×	×
3	○	×	×	×	×	×	×	×
4	×	×	×	○	×	×	×	×
5	×	×	×	×	×	×	×	○
6	×	×	×	×	○	×	×	×
7	×	×	×	×	×	○	×	×
8	×	○	×	×	×	×	×	×

5	1	2	3	4	5	6	7	8
1	×	○	×	×	×	×	×	×
2	○	×	×	×	×	×	×	×
3	×	×	×	○	×	×	×	×
4	×	×	×	×	×	×	○	×
5	×	×	×	×	○	×	×	×
6	×	×	×	×	×	×	×	○
7	×	×	○	×	×	×	×	×
8	×	×	×	×	×	○	×	×

6	1	2	3	4	5	6	7	8
1	×	×	×	○	×	×	×	×
2	×	×	×	×	○	×	×	×
3	×	○	×	×	×	×	×	×
4	○	×	×	×	×	×	×	×
5	×	×	○	×	×	×	×	×
6	×	×	×	×	×	○	×	×
7	×	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×

7	1	2	3	4	5	6	7	8
1	×	×	×	×	×	○	×	×
2	×	×	×	×	×	×	×	○
3	×	×	×	×	○	×	×	×
4	×	×	○	×	×	×	×	×
5	○	×	×	×	×	×	×	×
6	×	○	×	×	×	×	×	×
7	×	×	×	×	×	×	○	×
8	×	×	×	○	×	×	×	×

8	1	2	3	4	5	6	7	8
1	×	×	○	×	×	×	×	×
2	×	×	×	○	×	×	×	×
3	×	×	×	×	×	×	○	×
4	×	×	×	×	○	×	×	×
5	×	×	×	×	×	○	×	×
6	○	×	×	×	×	×	×	×
7	×	○	×	×	×	×	×	×
8	×	×	×	×	×	×	×	○

Figure A.15: Configuration for $c_{8,3} = 448$

direction	statistic
x (1)	(56,56,56,56,56,56,56,56)
y (2)	(56,56,56,56,56,56,56,56)
z (3)	(56,56,56,56,56,56,56,56)

Table A.6: 2-dimensional statistics for $c_{8,3}$

A.2.1.7 $n = 9$

1	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	×	×	○
2	×	×	×	○	×	×	×	×	×
3	×	×	×	×	×	○	×	×	×
4	×	×	×	×	○	×	×	×	×
5	○	×	×	×	×	×	×	×	×
6	×	×	○	×	×	×	×	×	×
7	×	○	×	×	×	×	×	×	×
8	×	×	×	×	×	×	×	○	×
9	×	×	×	×	×	×	○	×	×

2	1	2	3	4	5	6	7	8	9
1	×	○	×	×	×	×	×	×	×
2	×	×	×	×	○	×	×	×	×
3	×	×	×	○	×	×	×	×	×
4	○	×	×	×	×	×	×	×	×
5	×	×	×	×	×	○	×	×	×
6	×	×	×	×	×	×	×	×	○
7	×	×	×	×	×	×	○	×	×
8	×	×	○	×	×	×	×	×	×
9	×	×	×	×	×	×	×	○	×

3	1	2	3	4	5	6	7	8	9
1	×	×	×	×	○	×	×	×	×
2	×	×	×	×	×	×	×	○	×
3	×	○	×	×	×	×	×	×	×
4	×	×	×	×	×	×	○	×	×
5	×	×	×	×	×	×	×	×	○
6	×	×	×	×	×	○	×	×	×
7	×	×	○	×	×	×	×	×	×
8	○	×	×	×	×	×	×	×	×
9	×	×	×	○	×	×	×	×	×

4	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	×	○	×
2	×	×	×	×	×	○	×	×	×
3	×	×	×	×	×	×	○	×	×
4	×	×	×	○	×	×	×	×	×
5	×	○	×	×	×	×	×	×	×
6	×	×	×	×	○	×	×	×	×
7	○	×	×	×	×	×	×	×	×
8	×	×	×	×	×	×	×	×	○
9	×	×	○	×	×	×	×	×	×

5	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	○	×	×	×
2	×	×	×	×	×	×	○	×	×
3	○	×	×	×	×	×	×	×	×
4	×	○	×	×	×	×	×	×	×
5	×	×	○	×	×	×	×	×	×
6	×	×	×	○	×	×	×	×	×
7	×	×	×	×	×	×	×	○	×
8	×	×	×	×	○	×	×	×	×
9	×	×	×	×	×	×	×	×	○

6	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	○	×	×
2	○	×	×	×	×	×	×	×	×
3	×	×	×	×	×	×	×	×	○
4	×	×	○	×	×	×	×	×	×
5	×	×	×	×	○	×	×	×	×
6	×	×	×	×	×	×	×	○	×
7	×	×	×	○	×	×	×	×	×
8	×	○	×	×	×	×	×	×	×
9	×	×	×	×	×	○	×	×	×

7	1	2	3	4	5	6	7	8	9
1	○	×	×	×	×	×	×	×	×
2	×	×	○	×	×	×	×	×	×
3	×	×	×	×	○	×	×	×	×
4	×	×	×	×	×	×	×	×	○
5	×	×	×	×	×	×	×	○	×
6	×	×	×	×	×	×	○	×	×
7	×	×	×	×	×	○	×	×	×
8	×	×	×	○	×	×	×	×	×
9	×	○	×	×	×	×	×	×	×

8	1	2	3	4	5	6	7	8	9
1	×	×	○	×	×	×	×	×	×
2	×	○	×	×	×	×	×	×	×
3	×	×	×	×	×	×	×	○	×
4	×	×	×	×	×	○	×	×	×
5	×	×	×	○	×	×	×	×	×
6	○	×	×	×	×	×	×	×	×
7	×	×	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×	×
9	×	×	×	×	○	×	×	×	×

9	1	2	3	4	5	6	7	8	9
1	×	×	×	○	×	×	×	×	×
2	×	×	×	×	×	×	×	×	○
3	×	×	○	×	×	×	×	×	×
4	×	×	×	×	×	×	×	○	×
5	×	×	×	×	×	×	○	×	×
6	×	○	×	×	×	×	×	×	×
7	×	×	×	×	○	×	×	×	×
8	×	×	×	×	×	○	×	×	×
9	○	×	×	×	×	×	×	×	×

Figure A.16: Configuration for $c_{9,3} = 648$

direction	statistic
x (1)	(72,72,72,72,72,72,72,72,72)
y (2)	(72,72,72,72,72,72,72,72,72)
z (3)	(72,72,72,72,72,72,72,72,72)

Table A.7: 2-dimensional statistics for $c_{9,3}$

A.2.2 Geometric lines

We will only list the configurations until $n = 9$ because it would take too much space to list all larger values. The statistics in this section for $n \geq 7$ are all the same in every direction because the optimal values have the form $n^3 - n^2$ and so in every two-dimensional sub-slice the number of points must be $n^2 - n$.

A.2.2.1 $n = 3$

1	1	2	3
1	○	×	×
2	×	○	×
3	×	×	○

2	1	2	3
1	×	○	×
2	○	○	○
3	×	○	×

3	1	2	3
1	×	×	○
2	×	○	×
3	○	×	×

Figure A.17: Configuration for $g_{3,3} = 16$

direction	statistic
x (1)	(6,4,6)
y (2)	(6,4,6)
z (3)	(6,4,6)

Table A.8: 2-dimensional statistics for $g_{3,3}$

A.2.2.2 $n = 4$

1	1	2	3	4
1	×	○	×	×
2	×	×	×	○
3	×	×	○	×
4	○	×	×	×

2	1	2	3	4
1	×	×	○	×
2	○	○	×	×
3	×	○	○	×
4	×	×	×	○

3	1	2	3	4
1	○	×	×	×
2	×	×	○	×
3	×	×	×	○
4	×	○	×	×

4	1	2	3	4
1	×	×	×	○
2	×	○	×	×
3	○	×	×	×
4	○	×	○	×

Figure A.18: Configuration for $g_{4,3} = 45$

direction	statistic
x (1)	(12,11,11,11)
y (2)	(11,11,11,12)
z (3)	(12,10,12,11)

Table A.9: 2-dimensional statistics for $g_{4,3}$

A.2.2.3 $n = 5$

1	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	○	×
3	×	○	×	×	×
4	×	×	○	×	○
5	×	×	×	○	×

2	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	○	×	×	×
5	×	×	○	×	×

3	1	2	3	4	5
1	×	○	×	×	×
2	×	×	×	○	×
3	×	×	○	×	×
4	×	×	×	×	○
5	○	×	×	×	×

4	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	○
3	○	×	×	×	×
4	×	×	×	○	×
5	×	○	×	×	×

5	1	2	3	4	5
1	×	×	○	×	×
2	○	×	×	×	×
3	×	×	×	○	×
4	×	○	×	×	×
5	×	×	×	×	○

Figure A.19: Configuration for $g_{5,3} = 97$

direction	statistic
x (1)	(20,19,20,18,20)
y (2)	(20,19,20,19,19)
z (3)	(19,19,20,19,20)

Table A.10: 2-dimensional statistics for $g_{5,3}$

A.2.2.4 $n = 6$

1	1	2	3	4	5	6
1	×	×	×	×	○	×
2	○	×	×	×	×	×
3	×	×	○	×	×	×
4	×	×	×	×	×	○
5	×	○	×	×	×	×
6	×	×	×	○	×	×

2	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	○	○	×	×	×
3	×	×	×	×	×	○
4	○	×	×	×	×	×
5	×	×	×	×	○	×
6	○	○	×	×	×	×

3	1	2	3	4	5	6
1	○	×	×	×	×	×
2	×	×	×	×	×	○
3	×	×	×	○	×	×
4	×	○	×	×	×	×
5	×	×	○	×	×	×
6	×	×	×	×	○	×

4	1	2	3	4	5	6
1	×	○	×	×	×	×
2	×	×	×	×	○	×
3	○	×	×	×	×	×
4	×	×	×	○	×	×
5	×	×	×	×	×	○
6	×	×	○	×	×	×

5	1	2	3	4	5	6
1	×	○	×	×	×	×
2	×	×	×	○	×	×
3	×	×	×	×	○	×
4	×	×	○	×	×	×
5	○	×	×	×	×	×
6	×	×	×	×	×	○

6	1	2	3	4	5	6
1	×	×	○	×	×	○
2	○	×	×	×	×	×
3	×	○	×	×	×	×
4	×	×	×	×	○	×
5	×	×	×	○	×	×
6	×	×	×	×	×	○

Figure A.20: Configuration for $g_{6,3} = 177$

direction	statistic
x (1)	(29,29,30,30,30,29)
y (2)	(29,29,30,30,30,29)
z (3)	(30,28,30,30,30,29)

Table A.11: 2-dimensional statistics for $g_{6,3}$

A.2.2.5 $n = 7$

1	1	2	3	4	5	6	7
1	×	×	×	×	×	∘	×
2	×	×	×	∘	×	×	×
3	×	×	∘	×	×	×	×
4	×	×	×	×	∘	×	×
5	×	∘	×	×	×	×	×
6	×	×	×	×	×	×	∘
7	∘	×	×	×	×	×	×

2	1	2	3	4	5	6	7
1	×	×	×	×	∘	×	×
2	×	×	∘	×	×	×	×
3	×	×	×	×	×	∘	×
4	∘	×	×	×	×	×	×
5	×	×	×	∘	×	×	×
6	×	∘	×	×	×	×	×
7	×	×	×	×	×	×	∘

3	1	2	3	4	5	6	7
1	∘	×	×	×	×	×	×
2	×	×	×	×	∘	×	×
3	×	∘	×	×	×	×	×
4	×	×	×	×	×	×	∘
5	×	×	∘	×	×	×	×
6	×	×	×	∘	×	×	×
7	×	×	×	×	×	∘	×

4	1	2	3	4	5	6	7
1	×	×	∘	×	×	×	×
2	×	×	×	×	×	×	∘
3	∘	×	×	×	×	×	×
4	×	×	×	∘	×	×	×
5	×	×	×	×	×	∘	×
6	×	×	×	×	∘	×	×
7	×	∘	×	×	×	×	×

5	1	2	3	4	5	6	7
1	×	×	×	∘	×	×	×
2	∘	×	×	×	×	×	×
3	×	×	×	×	∘	×	×
4	×	∘	×	×	×	×	×
5	×	×	×	×	×	×	∘
6	×	×	×	×	×	∘	×
7	×	×	∘	×	×	×	×

6	1	2	3	4	5	6	7
1	×	∘	×	×	×	×	×
2	×	×	×	×	×	∘	×
3	×	×	×	×	×	×	∘
4	×	×	∘	×	×	×	×
5	×	×	×	×	∘	×	×
6	∘	×	×	×	×	×	×
7	×	×	×	∘	×	×	×

7	1	2	3	4	5	6	7
1	×	×	×	×	×	×	∘
2	×	∘	×	×	×	×	×
3	×	×	×	∘	×	×	×
4	×	×	×	×	×	∘	×
5	∘	×	×	×	×	×	×
6	×	×	∘	×	×	×	×
7	×	×	×	×	∘	×	×

Figure A.21: Configuration for $g_{7,3} = 294$

direction	statistic
x (1)	(42,42,42,42,42,42,42)
y (2)	(42,42,42,42,42,42,42)
z (3)	(42,42,42,42,42,42,42)

Table A.12: 2-dimensional statistics for $g_{7,3}$

A.2.2.6 $n = 8$

1	1	2	3	4	5	6	7	8
1	×	○	×	×	×	×	×	×
2	×	×	×	×	×	×	×	○
3	×	×	○	×	×	×	×	×
4	×	×	×	×	○	×	×	×
5	○	×	×	×	×	×	×	×
6	×	×	×	×	×	×	○	×
7	×	×	×	○	×	×	×	×
8	×	×	×	×	×	○	×	×

2	1	2	3	4	5	6	7	8
1	×	×	×	×	×	×	×	○
2	×	×	○	×	×	×	×	×
3	×	×	×	×	○	×	×	×
4	×	○	×	×	×	×	×	×
5	×	×	×	×	×	○	×	×
6	○	×	×	×	×	×	×	×
7	×	×	×	×	×	×	○	×
8	×	×	×	○	×	×	×	×

3	1	2	3	4	5	6	7	8
1	×	×	×	×	×	○	×	×
2	○	×	×	×	×	×	×	×
3	×	×	×	×	×	×	○	×
4	×	×	×	○	×	×	×	×
5	×	×	×	×	×	×	×	○
6	×	×	○	×	×	×	×	×
7	×	×	×	×	○	×	×	×
8	×	○	×	×	×	×	×	×

4	1	2	3	4	5	6	7	8
1	○	×	×	×	×	×	×	×
2	×	×	×	×	×	×	○	×
3	×	×	×	○	×	×	×	×
4	×	×	×	×	×	○	×	×
5	×	○	×	×	×	×	×	×
6	×	×	×	×	×	×	×	○
7	×	×	○	×	×	×	×	×
8	×	×	×	×	○	×	×	×

5	1	2	3	4	5	6	7	8
1	×	×	○	×	×	×	×	×
2	×	×	×	×	○	×	×	×
3	×	○	×	×	×	×	×	×
4	○	×	×	×	×	×	×	×
5	×	×	×	○	×	×	×	×
6	×	×	×	×	×	○	×	×
7	×	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×

6	1	2	3	4	5	6	7	8
1	×	×	×	○	×	×	×	×
2	×	○	×	×	×	×	×	×
3	×	×	×	×	×	×	×	○
4	×	×	○	×	×	×	×	×
5	×	×	×	×	×	×	○	×
6	×	×	×	×	○	×	×	×
7	×	×	×	×	×	○	×	×
8	○	×	×	×	×	×	×	×

7	1	2	3	4	5	6	7	8
1	×	×	×	×	×	×	○	×
2	×	×	×	○	×	×	×	×
3	×	×	×	×	×	○	×	×
4	×	×	×	×	×	×	×	○
5	×	×	×	×	○	×	×	×
6	×	○	×	×	×	×	×	×
7	○	×	×	×	×	×	×	×
8	×	×	○	×	×	×	×	×

8	1	2	3	4	5	6	7	8
1	×	×	×	×	○	×	×	×
2	×	×	×	×	×	○	×	×
3	○	×	×	×	×	×	×	×
4	×	×	×	×	×	×	○	×
5	×	×	○	×	×	×	×	×
6	×	×	×	○	×	×	×	×
7	×	○	×	×	×	×	×	×
8	×	×	×	×	×	×	×	○

Figure A.22: Configuration for $g_{8,3} = 448$

direction	statistic
x (1)	(56,56,56,56,56,56,56,56)
y (2)	(56,56,56,56,56,56,56,56)
z (3)	(56,56,56,56,56,56,56,56)

Table A.13: 2-dimensional statistics for $g_{8,3}$

A.2.2.7 $n = 9$

1	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	○	×	×
2	×	×	×	×	×	○	×	×	×
3	×	×	×	×	○	×	×	×	×
4	×	×	×	×	×	×	×	×	○
5	○	×	×	×	×	×	×	×	×
6	×	×	×	○	×	×	×	×	×
7	×	○	×	×	×	×	×	×	×
8	×	×	×	×	×	×	×	○	×
9	×	×	○	×	×	×	×	×	×

2	1	2	3	4	5	6	7	8	9
1	○	×	×	×	×	×	×	×	×
2	×	×	×	×	×	×	×	×	○
3	×	×	×	×	×	×	×	○	×
4	×	×	×	×	×	×	○	×	×
5	×	×	×	×	×	○	×	×	×
6	×	×	○	×	×	×	×	×	×
7	×	×	×	×	○	×	×	×	×
8	×	○	×	×	×	×	×	×	×
9	×	×	×	○	×	×	×	×	×

3	1	2	3	4	5	6	7	8	9
1	×	×	×	○	×	×	×	×	×
2	×	×	○	×	×	×	×	×	×
3	×	○	×	×	×	×	×	×	×
4	×	×	×	×	×	○	×	×	×
5	×	×	×	×	×	×	×	×	○
6	×	×	×	×	×	×	×	○	×
7	×	×	×	×	×	×	○	×	×
8	○	×	×	×	×	×	×	×	×
9	×	×	×	×	○	×	×	×	×

4	1	2	3	4	5	6	7	8	9
1	×	○	×	×	×	×	×	×	×
2	×	×	×	×	×	×	○	×	×
3	○	×	×	×	×	×	×	×	×
4	×	×	○	×	×	×	×	×	×
5	×	×	×	×	○	×	×	×	×
6	×	×	×	×	×	×	×	×	○
7	×	×	×	○	×	×	×	×	×
8	×	×	×	×	×	○	×	×	×
9	×	×	×	×	×	×	×	○	×

5	1	2	3	4	5	6	7	8	9
1	×	×	×	×	○	×	×	×	×
2	×	×	×	×	×	×	×	○	×
3	×	×	○	×	×	×	×	×	×
4	×	○	×	×	×	×	×	×	×
5	×	×	×	○	×	×	×	×	×
6	○	×	×	×	×	×	×	×	×
7	×	×	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×	×
9	×	×	×	×	×	○	×	×	×

6	1	2	3	4	5	6	7	8	9
1	×	×	○	×	×	×	×	×	×
2	×	×	×	○	×	×	×	×	×
3	×	×	×	×	×	×	○	×	×
4	×	×	×	×	○	×	×	×	×
5	×	×	×	×	×	×	×	○	×
6	×	×	×	×	×	○	×	×	×
7	○	×	×	×	×	×	×	×	×
8	×	×	×	×	×	×	×	×	○
9	×	○	×	×	×	×	×	×	×

7	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	×	×	○
2	×	○	×	×	×	×	×	×	×
3	×	×	×	×	×	○	×	×	×
4	○	×	×	×	×	×	×	×	×
5	×	×	○	×	×	×	×	×	×
6	×	×	×	×	○	×	×	×	×
7	×	×	×	×	×	×	×	○	×
8	×	×	×	○	×	×	×	×	×
9	×	×	×	×	×	×	○	×	×

8	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	○	×	×	×
2	○	×	×	×	×	×	×	×	×
3	×	×	×	○	×	×	×	×	×
4	×	×	×	×	×	×	×	○	×
5	×	×	×	×	×	×	○	×	×
6	×	○	×	×	×	×	×	×	×
7	×	×	○	×	×	×	×	×	×
8	×	×	×	×	○	×	×	×	×
9	×	×	×	×	×	×	×	×	○

9	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	×	○	×
2	×	×	×	×	○	×	×	×	×
3	×	×	×	×	×	×	×	×	○
4	×	×	×	○	×	×	×	×	×
5	×	○	×	×	×	×	×	×	×
6	×	×	×	×	×	×	○	×	×
7	×	×	×	×	×	○	×	×	×
8	×	×	○	×	×	×	×	×	×
9	○	×	×	×	×	×	×	×	×

Figure A.23: Configuration for $g_{9,3} = 648$

direction	statistic
x (1)	(72,72,72,72,72,72,72,72,72)
y (2)	(72,72,72,72,72,72,72,72,72)
z (3)	(72,72,72,72,72,72,72,72,72)

Table A.14: 2-dimensional statistics for $g_{9,3}$

A.2.3 Algebraic lines

A.2.3.1 $n = 3$

1	1	2	3
1	○	○	○
2	○	○	○
3	○	×	○

2	1	2	3
1	○	×	○
2	×	○	○
3	×	×	○

3	1	2	3
1	×	×	○
2	×	○	×
3	○	○	○

Figure A.24: Configuration for $a_{3,3} = 9$

direction	statistic
x (1)	(3,3,3)
y (2)	(4,4,1)
z (3)	(1,4,4)

Table A.15: 2-dimensional statistics for $a_{3,3}$

A.2.3.2 $n = 4$

1	1	2	3	4
1	×	×	×	○
2	×	○	×	×
3	○	×	○	×
4	○	×	○	×

2	1	2	3	4
1	○	○	○	○
2	×	○	×	×
3	○	×	○	×
4	×	○	×	×

3	1	2	3	4
1	×	×	×	○
2	×	×	×	○
3	×	○	×	×
4	○	×	○	○

4	1	2	3	4
1	×	○	×	×
2	○	○	○	×
3	×	×	×	○
4	○	○	○	×

Figure A.25: Configuration for $a_{4,3} = 36$

direction	statistic
x (1)	(9,10,10,7)
y (2)	(9,8,9,10)
z (3)	(10,8,10,8)

Table A.16: 2-dimensional statistics for $a_{4,3}$

A.3 Configurations for $d = 4$

A.3.1 Combinatorial lines

A.3.1.1 $n = 3$

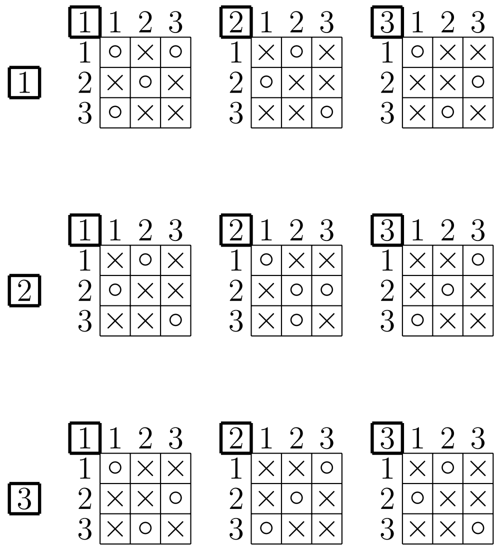

Figure A.26: Configuration for $c_{3,4} = 52$

direction	statistic
1	(17,17,18)
2	(17,17,18)
3	(17,17,18)
4	(17,17,18)

Table A.17: 3-dimensional statistics for $c_{3,4}$

A.3.1.2 $n = 4$

1

1	1	2	3	4
1	○	×	×	○
2	○	×	×	×
3	×	○	×	×
4	○	×	○	×

2	1	2	3	4
1	×	×	○	×
2	×	○	×	×
3	○	×	×	×
4	×	×	×	○

3	1	2	3	4
1	×	○	×	×
2	×	×	×	○
3	×	×	○	×
4	○	×	×	×

4	1	2	3	4
1	○	×	×	×
2	×	×	○	×
3	×	×	×	○
4	×	○	×	×

2

1	1	2	3	4
1	×	×	○	×
2	×	○	×	×
3	○	×	×	×
4	×	×	×	○

2	1	2	3	4
1	×	○	×	×
2	○	○	○	○
3	×	○	×	×
4	×	○	×	×

3	1	2	3	4
1	○	×	×	×
2	×	○	×	×
3	×	×	×	○
4	×	×	○	×

4	1	2	3	4
1	×	×	×	○
2	×	○	×	×
3	×	×	○	×
4	○	×	×	×

3

1	1	2	3	4
1	×	○	×	×
2	×	×	×	○
3	×	×	○	×
4	○	×	×	×

2	1	2	3	4
1	○	×	×	×
2	×	○	×	×
3	×	×	×	○
4	×	×	○	×

3	1	2	3	4
1	×	×	○	○
2	×	×	○	×
3	○	×	○	×
4	×	○	×	×

4	1	2	3	4
1	×	×	○	×
2	○	×	×	×
3	×	○	×	×
4	×	×	×	○

4

1	1	2	3	4
1	○	×	×	×
2	×	×	○	×
3	×	×	×	○
4	×	○	×	×

2	1	2	3	4
1	×	×	×	○
2	×	○	×	×
3	×	×	○	×
4	○	×	×	×

3	1	2	3	4
1	×	×	○	×
2	○	×	×	×
3	×	○	×	×
4	×	×	×	○

4	1	2	3	4
1	×	○	×	×
2	×	×	×	○
3	○	×	×	○
4	×	×	○	○

Figure A.27: Configuration for $c_{4,4} = 183$

direction	statistic
1	(46,45,46,46)
2	(46,45,46,46)
3	(46,45,46,46)
4	(46,45,46,46)

Table A.18: 3-dimensional statistics for $c_{4,4}$

A.3.1.3 $n = 5$

1

1	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	○	×	×
4	×	×	×	×	○
5	×	×	×	○	×

2	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

3	1	2	3	4	5
1	×	×	○	×	×
2	×	×	×	○	×
3	○	×	×	×	×
4	×	○	×	×	×
5	×	×	×	×	○

4	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	×

5	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	×	×	×	×
5	×	×	○	×	×

2

1	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

2	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	×

3	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	×	×	×	×
5	×	×	○	×	×

4	1	2	3	4	5
1	×	×	○	×	×
2	×	×	×	○	×
3	○	×	×	×	×
4	×	○	×	×	×
5	×	×	×	×	○

5	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	○	×	×
4	×	×	×	×	○
5	×	×	×	○	×

3

1	1	2	3	4	5
1	×	×	○	×	×
2	×	×	×	○	×
3	○	×	×	×	×
4	×	○	×	×	×
5	×	×	×	×	○

2	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	×	×	×	×
5	×	×	○	×	×

3	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

4	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	○	×	×
4	×	×	×	×	○
5	×	×	×	○	×

5	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	×

4

1	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	×

2	1	2	3	4	5
1	×	×	○	×	×
2	×	×	×	○	×
3	○	×	×	×	×
4	×	○	×	×	×
5	×	×	×	×	○

3	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	○	×	×
4	×	×	×	×	○
5	×	×	×	○	×

4	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	×	×	×	×
5	×	×	○	×	×

5	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

5

1	1	2	3	4	5
1	×	×	×	○	×
2	×	○	×	×	×
3	×	×	×	×	○
4	○	×	×	×	×
5	×	×	○	×	×

2	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	×	×
3	×	×	○	×	×
4	×	×	×	×	○
5	×	×	×	○	×

3	1	2	3	4	5
1	×	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	×

4	1	2	3	4	5
1	○	×	×	×	×
2	×	×	×	×	○
3	×	×	×	○	×
4	×	×	○	×	×
5	×	○	×	×	×

5	1	2	3	4	5
1	×	×	○	×	×
2	×	×	×	○	×
3	○	×	×	×	×
4	×	○	×	×	×
5	×	×	×	×	○

Figure A.28: Configuration for $c_{5,4} = 500$

direction	statistic
1	(100,100,100,100,100)
2	(100,100,100,100,100)
3	(100,100,100,100,100)
4	(100,100,100,100,100)

Table A.19: 3-dimensional statistics for $c_{5,4}$

The statistics are all the same in every direction because the optimal value has the form $n^4 - n^3$ and so in every two-dimensional sub-slice the number of points must be $n^3 - n^2$.

A.4 Configurations for $d = 3$ and $n - 1$ progressions

A.4.1 Combinatorial lines

A.4.1.1 $n = 3$

1	1	2	3
1	×	○	○
2	○	○	×
3	○	×	○

2	1	2	3
1	○	○	×
2	○	○	○
3	×	○	○

3	1	2	3
1	○	×	○
2	×	○	○
3	○	○	×

Figure A.29: Configuration for $c^1_{3,3} = 7$

direction	statistic
x (1)	(3,2,3)
y (2)	(3,2,3)
z (3)	(3,2,3)

Table A.20: 2-dimensional statistics for $c^1_{3,3}$

A.4.1.2 $n = 4$

1	1	2	3	4
1	×	○	×	×
2	○	○	×	○
3	×	×	○	×
4	×	○	×	×

2	1	2	3	4
1	○	×	×	○
2	×	×	○	×
3	×	○	×	×
4	○	×	×	○

3	1	2	3	4
1	×	×	○	×
2	×	○	○	×
3	○	○	○	○
4	×	×	○	×

4	1	2	3	4
1	×	○	×	×
2	○	×	×	○
3	×	×	○	×
4	×	○	×	×

Figure A.30: Configuration for $c^1_{4,3} = 39$

direction	statistic
x (1)	(11,8,9,11)
y (2)	(11,8,9,11)
z (3)	(10,10,8,11)

Table A.21: 2-dimensional statistics for $c^1_{4,3}$

A.4.1.3 $n = 5$

1	1	2	3	4	5
1	○	×	×	×	○
2	×	○	×	×	×
3	×	×	×	○	×
4	×	×	○	×	×
5	○	×	×	×	○

2	1	2	3	4	5
1	×	○	×	×	×
2	○	×	×	○	×
3	×	×	○	×	○
4	×	○	×	×	×
5	×	×	○	×	×

3	1	2	3	4	5
1	×	×	×	○	×
2	×	×	○	×	○
3	×	○	×	×	×
4	○	×	×	○	×
5	×	○	×	×	×

4	1	2	3	4	5
1	×	×	○	×	×
2	×	○	×	×	×
3	○	×	×	○	×
4	×	×	○	×	○
5	×	×	×	○	×

5	1	2	3	4	5
1	○	×	×	×	○
2	×	×	○	×	×
3	×	○	×	×	×
4	×	×	×	○	×
5	○	×	×	×	○

Figure A.31: Configuration for $c^1_{5,3} = 90$

direction	statistic
x (1)	(18,18,18,18,18)
y (2)	(18,18,18,18,18)
z (3)	(18,18,18,18,18)

Table A.22: 2-dimensional statistics for $c^1_{5,3}$

A.4.1.4 $n = 6$

1	1	2	3	4	5	6
1	×	×	×	×	○	×
2	×	×	×	○	×	×
3	×	×	○	×	×	×
4	×	○	×	×	×	×
5	○	×	×	×	×	○
6	×	×	×	×	○	×

2	1	2	3	4	5	6
1	×	×	○	×	×	×
2	×	○	×	×	×	×
3	○	×	×	×	×	○
4	×	×	×	×	○	×
5	×	×	×	○	×	×
6	×	×	○	×	×	×

3	1	2	3	4	5	6
1	○	×	×	×	×	○
2	×	×	×	×	○	×
3	×	×	×	○	×	×
4	×	×	○	×	×	×
5	×	○	×	×	×	×
6	○	×	×	×	×	○

4	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	×	○	×	×	×
3	×	○	×	×	×	×
4	○	×	×	×	×	○
5	×	×	×	×	○	×
6	×	×	×	○	×	×

5	1	2	3	4	5	6
1	×	○	×	×	×	×
2	○	×	×	×	×	○
3	×	×	×	×	○	×
4	×	×	×	○	×	×
5	×	×	○	×	×	×
6	×	○	×	×	×	×

6	1	2	3	4	5	6
1	×	×	×	×	○	×
2	×	×	×	○	×	×
3	×	×	○	×	×	×
4	×	○	×	×	×	×
5	○	×	×	×	×	○
6	×	×	×	×	○	×

Figure A.32: Configuration for $c^1_{6,3} = 173$

direction	statistic
x (1)	(29,29,29,29,28,29)
y (2)	(29,29,29,29,28,29)
z (3)	(29,29,28,29,29,29)

Table A.23: 2-dimensional statistics for $c^1_{6,3}$

A.4.1.5 $n = 7$

1	1	2	3	4	5	6	7
1	×	×	×	×	×	○	×
2	×	×	×	×	○	×	×
3	×	×	○	×	×	×	×
4	×	×	×	○	×	×	×
5	×	○	×	×	×	×	×
6	○	×	×	×	×	×	○
7	×	×	×	×	×	○	×

2	1	2	3	4	5	6	7
1	×	○	×	×	×	×	×
2	×	×	○	×	×	×	×
3	×	×	×	×	×	○	×
4	×	×	×	○	×	×	×
5	○	×	×	×	×	×	○
6	×	×	×	×	○	×	×
7	×	○	×	×	×	×	×

3	1	2	3	4	5	6	7
1	×	×	×	×	○	×	×
2	×	○	×	×	×	×	×
3	○	×	×	×	×	×	○
4	×	×	○	×	×	×	×
5	×	×	×	×	×	○	×
6	×	×	×	○	×	×	×
7	×	×	×	×	○	×	×

4	1	2	3	4	5	6	7
1	×	×	×	○	×	×	×
2	×	×	×	○	×	×	×
3	×	×	×	○	○	×	×
4	○	○	×	×	×	×	○
5	×	×	×	○	×	×	×
6	×	×	○	×	×	○	×
7	×	×	×	○	×	×	×

5	1	2	3	4	5	6	7
1	×	×	○	×	×	×	×
2	○	×	×	×	×	×	○
3	×	×	×	○	×	×	×
4	×	×	×	×	×	○	×
5	×	×	×	×	○	×	×
6	×	○	×	×	×	×	×
7	×	×	○	×	×	×	×

6	1	2	3	4	5	6	7
1	○	×	×	×	×	×	○
2	×	×	×	×	×	○	×
3	×	○	×	×	×	×	×
4	×	×	×	×	○	×	×
5	×	×	○	×	×	×	×
6	×	×	×	○	×	×	×
7	○	×	×	×	×	×	○

7	1	2	3	4	5	6	7
1	×	×	×	×	×	○	×
2	×	×	×	×	○	×	×
3	×	×	○	×	×	×	×
4	×	×	×	○	×	×	×
5	×	○	×	×	×	×	×
6	○	×	×	×	×	×	○
7	×	×	×	×	×	○	×

Figure A.33: Configuration for $c^1_{7,3} = 283$

direction	statistic
x (1)	(41,41,40,40,41,39,41)
y (2)	(41,41,41,38,41,40,41)
z (3)	(41,41,41,38,41,40,41)

Table A.24: 2-dimensional statistics for $c^1_{7,3}$

A.4.1.6 $n = 8$

1	1	2	3	4	5	6	7	8
1	×	×	×	○	×	×	×	×
2	×	○	×	×	×	×	×	×
3	×	×	×	×	○	×	×	×
4	○	×	×	×	×	×	×	○
5	×	×	○	×	×	×	×	×
6	×	×	×	×	×	×	○	×
7	×	×	×	×	×	○	×	×
8	×	×	×	○	×	×	×	×

2	1	2	3	4	5	6	7	8
1	×	○	×	×	×	×	×	×
2	○	×	×	×	×	×	×	○
3	×	×	○	×	×	×	×	×
4	×	×	×	×	×	○	×	×
5	×	×	×	×	×	×	○	×
6	×	×	×	○	×	×	×	×
7	×	×	×	×	○	×	×	×
8	×	○	×	×	×	×	×	×

3	1	2	3	4	5	6	7	8
1	×	×	×	×	○	×	×	×
2	×	×	○	×	×	×	×	×
3	×	○	×	×	×	×	×	×
4	×	×	×	×	×	×	○	×
5	○	×	×	×	×	×	×	○
6	×	×	×	×	×	○	×	×
7	×	×	×	○	×	×	×	×
8	×	×	×	×	○	×	×	×

4	1	2	3	4	5	6	7	8
1	○	×	×	×	×	×	×	○
2	×	×	×	×	×	○	×	×
3	×	×	×	×	×	×	○	×
4	×	×	×	×	○	×	×	×
5	×	×	×	○	×	×	×	×
6	×	○	×	×	×	×	×	×
7	×	×	○	×	×	×	×	×
8	○	×	×	×	×	×	×	○

5	1	2	3	4	5	6	7	8
1	×	×	○	×	×	×	×	×
2	×	×	×	×	×	×	○	×
3	○	×	×	×	×	×	×	○
4	×	×	×	○	×	×	×	×
5	×	×	×	×	×	○	×	×
6	×	×	×	×	○	×	×	×
7	×	○	×	×	×	×	×	×
8	×	×	○	×	×	×	×	×

6	1	2	3	4	5	6	7	8
1	×	×	×	×	×	×	○	×
2	×	×	×	○	×	×	×	×
3	×	×	×	×	×	○	×	×
4	×	○	×	×	×	×	×	×
5	×	×	×	×	○	×	×	×
6	×	×	○	×	×	×	×	×
7	○	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×

7	1	2	3	4	5	6	7	8
1	×	×	×	×	×	○	×	×
2	×	×	×	×	○	×	×	×
3	×	×	×	○	×	×	×	×
4	×	×	○	×	×	×	×	×
5	×	○	×	×	×	×	×	×
6	○	×	×	×	×	×	×	○
7	×	×	×	×	×	×	○	×
8	×	×	×	×	×	○	×	×

8	1	2	3	4	5	6	7	8
1	×	×	×	○	×	×	×	×
2	×	○	×	×	×	×	×	×
3	×	×	×	×	○	×	×	×
4	○	×	×	×	×	×	×	○
5	×	×	○	×	×	×	×	×
6	×	×	×	×	×	×	○	×
7	×	×	×	×	×	○	×	×
8	×	×	×	○	×	×	×	×

Figure A.34: Configuration for $c^1_{8,3} = 439$

direction	statistic
x (1)	(55,55,55,54,55,55,55,55)
y (2)	(55,55,55,54,55,55,55,55)
z (3)	(55,55,55,54,55,55,55,55)

Table A.25: 2-dimensional statistics for $c^1_{8,3}$

A.4.2 Geometric lines

A.4.2.1 $n = 3$

1	1	2	3
1	×	○	○
2	○	○	×
3	○	○	○

2	1	2	3
1	○	○	○
2	○	○	○
3	×	○	○

3	1	2	3
1	○	×	○
2	○	○	○
3	○	○	○

Figure A.35: Configuration for $g^1_{3,3} = 4$

direction	statistic
x (1)	(2,1,1)
y (2)	(2,1,1)
z (3)	(2,1,1)

Table A.26: 2-dimensional statistics for $g^1_{3,3}$

A.4.2.2 $n = 4$

1	1	2	3	4
1	○	×	○	×
2	×	×	○	○
3	×	○	×	×
4	○	×	×	○

2	1	2	3	4
1	×	×	○	○
2	○	○	○	○
3	×	○	○	×
4	×	×	○	×

3	1	2	3	4
1	×	○	×	×
2	×	○	○	×
3	○	○	○	○
4	○	○	×	×

4	1	2	3	4
1	○	×	×	○
2	×	×	○	×
3	○	○	×	×
4	×	○	×	○

Figure A.36: Configuration for $g^1_{4,3} = 32$

direction	statistic
x (1)	(9,7,7,9)
y (2)	(9,7,7,9)
z (3)	(9,7,7,9)

Table A.27: 2-dimensional statistics for $g^1_{4,3}$

A.4.2.3 $n = 5$

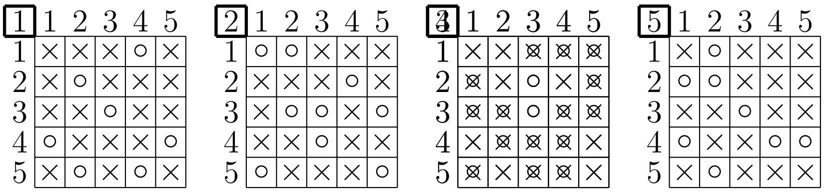

Figure A.37: Configuration for $g^1_{5,3} = 82$

direction	statistic
x (1)	(18,17,14,16,17)
y (2)	(17,16,14,17,18)
z (3)	(18,16,15,16,17)

Table A.28: 2-dimensional statistics for $g^1_{5,3}$

A.5 Configurations for $d = 3$ and $n - 2$ progressions

A.5.1 Combinatorial lines

A.5.1.1 $n = 4$

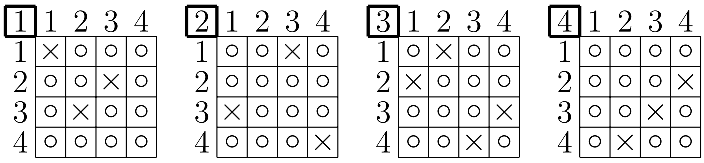

Figure A.38: Configuration for $c^2_{4,3} = 13$

direction	statistic
x (1)	(3,3,4,3)
y (2)	(3,3,4,3)
z (3)	(3,3,4,3)

Table A.29: 2-dimensional statistics for $c^2_{4,3}$

A.5.1.2 $n = 5$

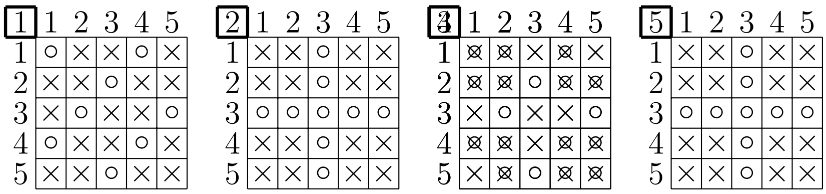

Figure A.39: Configuration for $c^2_{5,3} = 75$

direction	statistic
x (1)	(18,16,11,17,17)
y (2)	(18,16,11,17,17)
z (3)	(17,16,11,17,16)

Table A.30: 2-dimensional statistics for $c^2_{5,3}$

A.5.1.3 $n = 6$

1	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	×	×	○	×	×
3	×	×	○	×	×	○
4	○	○	×	×	○	×
5	×	×	×	○	×	×
6	×	×	○	×	×	×

2	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	×	○	×	×	×
3	×	○	×	×	○	○
4	○	×	×	○	×	×
5	×	×	○	×	×	×
6	×	×	×	○	×	×

3	1	2	3	4	5	6
1	×	×	○	×	×	○
2	×	○	×	×	○	○
3	○	×	×	○	○	×
4	×	×	○	×	×	×
5	×	○	○	×	×	×
6	×	×	○	×	×	○

4	1	2	3	4	5	6
1	○	○	×	×	○	×
2	○	×	×	○	×	×
3	×	×	○	×	×	×
4	×	○	×	○	×	○
5	○	×	×	×	○	○
6	○	○	×	×	○	×

5	1	2	3	4	5	6
1	×	×	×	○	×	×
2	×	×	○	×	×	×
3	×	○	○	×	×	×
4	○	×	×	×	○	○
5	×	×	×	○	×	×
6	×	×	×	○	×	×

6	1	2	3	4	5	6
1	×	×	○	×	×	×
2	×	×	○	×	×	×
3	○	○	×	×	×	○
4	×	×	×	○	○	×
5	×	×	×	○	×	×
6	×	×	○	×	×	×

Figure A.40: Configuration for $c^2_{6,3} = 152$

direction	statistic
x (1)	(27,27,22,22,27,27)
y (2)	(27,27,22,22,27,27)
z (3)	(27,27,23,21,27,27)

Table A.31: 2-dimensional statistics for $c^2_{6,3}$

A.5.1.4 $n = 7$

1	1	2	3	4	5	6	7
1	×	×	×	×	○	×	×
2	×	×	×	○	×	×	×
3	×	×	○	×	×	×	×
4	×	○	×	×	×	×	○
5	○	×	×	×	×	○	×
6	×	×	×	×	○	×	×
7	×	×	×	○	×	×	×

2	1	2	3	4	5	6	7
1	×	×	×	○	×	×	×
2	×	×	○	×	×	×	×
3	×	○	×	×	×	×	○
4	○	×	×	×	×	○	×
5	×	×	×	×	○	×	×
6	×	×	×	○	×	×	×
7	×	×	○	×	×	×	×

3	1	2	3	4	5	6	7
1	×	×	○	×	×	×	×
2	×	○	×	×	×	×	○
3	○	×	×	×	×	○	×
4	×	×	×	×	○	×	×
5	×	×	×	○	×	×	×
6	×	×	○	×	×	×	×
7	×	○	×	×	×	×	○

4	1	2	3	4	5	6	7
1	×	○	×	×	×	×	○
2	○	×	×	×	×	○	×
3	×	×	×	×	○	×	×
4	×	×	×	○	×	×	×
5	×	×	○	×	×	×	×
6	×	○	×	×	×	×	○
7	○	×	×	×	×	○	×

5	1	2	3	4	5	6	7
1	○	×	×	×	×	○	×
2	×	×	×	×	○	×	×
3	×	×	×	○	×	×	×
4	×	×	○	×	×	×	×
5	×	○	×	×	×	×	○
6	○	×	×	×	×	○	×
7	×	×	×	×	○	×	×

6	1	2	3	4	5	6	7
1	×	×	×	×	○	×	×
2	×	×	×	○	×	×	×
3	×	×	○	×	×	×	×
4	×	○	×	×	×	×	○
5	○	×	×	×	×	○	×
6	×	×	×	×	○	×	×
7	×	×	×	○	×	×	×

7	1	2	3	4	5	6	7
1	×	×	×	○	×	×	×
2	×	×	○	×	×	×	×
3	×	○	×	×	×	×	○
4	○	×	×	×	×	○	×
5	×	×	×	×	○	×	×
6	×	×	×	○	×	×	×
7	×	×	○	×	×	×	×

Figure A.41: Configuration for $c^2_{7,3} = 276$

direction	statistic
x (1)	(40,40,39,38,39,40,40)
y (2)	(40,40,39,38,39,40,40)
z (3)	(40,40,39,38,39,40,40)

Table A.32: 2-dimensional statistics for $c^2_{7,3}$

A.5.1.5 $n = 8$

1	1	2	3	4	5	6	7	8
1	○	×	×	×	×	×	○	×
2	×	×	×	×	×	○	×	×
3	×	×	×	○	×	×	×	×
4	×	×	○	×	×	×	×	×
5	×	×	×	×	○	×	×	×
6	×	○	×	×	×	×	×	○
7	○	×	×	×	×	×	○	×
8	×	×	×	×	×	○	×	×

2	1	2	3	4	5	6	7	8
1	×	×	×	×	×	○	×	×
2	×	×	×	○	×	×	×	×
3	×	×	○	×	×	×	×	×
4	×	○	×	×	×	×	×	○
5	×	×	×	×	○	×	×	×
6	○	×	×	×	×	×	○	×
7	×	×	×	×	×	○	×	×
8	×	×	×	○	×	×	×	×

3	1	2	3	4	5	6	7	8
1	×	×	×	○	×	×	×	×
2	×	×	○	×	×	×	×	×
3	×	○	×	×	○	×	×	○
4	○	×	×	×	×	×	○	×
5	×	×	○	×	×	×	×	×
6	×	×	×	×	×	○	×	×
7	×	×	×	○	×	×	×	×
8	×	×	○	×	×	×	×	×

4	1	2	3	4	5	6	7	8
1	×	×	○	×	×	×	×	×
2	×	○	×	×	×	×	×	○
3	○	×	×	×	×	×	○	×
4	×	×	×	×	○	×	×	×
5	×	×	×	○	×	×	×	×
6	×	×	×	×	×	○	×	×
7	×	×	○	×	×	×	×	×
8	×	○	×	×	×	×	×	○

5	1	2	3	4	5	6	7	8
1	×	×	×	×	○	×	×	×
2	×	×	×	×	○	×	×	×
3	×	×	○	×	×	×	×	×
4	×	×	×	○	×	×	×	×
5	○	○	×	×	×	○	○	○
6	×	×	×	×	○	×	×	×
7	×	×	×	×	○	×	×	×
8	×	×	×	×	○	×	×	×

6	1	2	3	4	5	6	7	8
1	×	○	×	×	×	×	×	○
2	○	×	×	×	×	×	○	×
3	×	×	×	×	×	○	×	×
4	×	×	×	×	×	○	×	×
5	×	×	×	×	○	×	×	×
6	×	×	○	○	×	×	×	×
7	×	○	×	×	×	×	×	○
8	○	×	×	×	×	×	○	×

7	1	2	3	4	5	6	7	8
1	○	×	×	×	×	×	○	×
2	×	×	×	×	×	○	×	×
3	×	×	×	○	×	×	×	×
4	×	×	○	×	×	×	×	×
5	×	×	×	×	○	×	×	×
6	×	○	×	×	×	×	×	○
7	○	×	×	×	×	×	○	×
8	×	×	×	×	×	○	×	×

8	1	2	3	4	5	6	7	8
1	×	×	×	×	×	○	×	×
2	×	×	×	○	×	×	×	×
3	×	×	○	×	×	×	×	×
4	×	○	×	×	×	×	×	○
5	×	×	×	×	○	×	×	×
6	○	×	×	×	×	×	○	×
7	×	×	×	×	×	○	×	×
8	×	×	×	○	×	×	×	×

Figure A.42: Configuration for $c^2_{8,3} = 423$

direction	statistic
x (1)	(53,54,53,53,52,51,53,54)
y (2)	(53,54,53,53,52,51,53,54)
z (3)	(53,54,53,53,52,51,53,54)

Table A.33: 2-dimensional statistics for $c^2_{8,3}$

A.5.1.6 $n = 9$

1	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	○	×	×
2	×	×	×	×	○	×	×	×	×
3	×	×	○	×	×	×	×	×	×
4	○	×	×	×	×	×	×	○	×
5	×	×	×	×	×	○	×	×	×
6	×	×	×	○	×	×	×	×	×
7	×	○	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×	×
9	×	×	×	×	○	×	×	×	×

2	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	○	×	×	×
2	×	×	×	○	×	×	×	×	×
3	×	○	×	×	×	×	×	×	○
4	×	×	×	×	×	×	○	×	×
5	×	×	×	×	○	×	×	×	×
6	×	×	○	×	×	×	×	×	×
7	○	×	×	×	×	×	×	○	×
8	×	×	×	×	×	○	×	×	×
9	×	×	×	○	×	×	×	×	×

3	1	2	3	4	5	6	7	8	9
1	×	×	×	×	○	×	×	×	×
2	×	×	○	×	×	×	×	×	×
3	○	×	×	×	×	×	×	○	×
4	×	×	×	×	×	○	×	×	×
5	×	×	×	○	×	×	×	×	×
6	×	○	×	×	×	×	×	×	○
7	×	×	×	×	×	×	○	×	×
8	×	×	×	×	○	×	×	×	×
9	×	×	○	×	×	×	×	×	×

4	1	2	3	4	5	6	7	8	9
1	×	×	×	○	×	×	×	×	×
2	×	○	×	×	×	×	×	×	○
3	×	×	×	×	×	×	○	×	×
4	×	×	×	×	○	×	×	×	×
5	×	×	○	×	×	×	×	×	×
6	○	×	×	×	×	×	×	○	×
7	×	×	×	×	×	○	×	×	×
8	×	×	×	○	×	×	×	×	×
9	×	○	×	×	×	×	×	×	○

5	1	2	3	4	5	6	7	8	9
1	×	×	○	×	×	×	×	×	×
2	○	×	×	×	×	×	×	○	×
3	×	×	×	×	×	○	×	×	×
4	×	×	×	○	×	×	×	×	×
5	×	○	×	×	×	×	×	×	○
6	×	×	×	×	×	×	○	×	×
7	×	×	×	×	○	×	×	×	×
8	×	×	○	×	×	×	×	×	×
9	○	×	×	×	×	×	×	○	×

6	1	2	3	4	5	6	7	8	9
1	×	○	×	×	×	×	×	×	○
2	×	×	×	×	×	×	○	×	×
3	×	×	×	×	○	×	×	×	×
4	×	×	○	×	×	×	×	×	×
5	○	×	×	×	×	×	×	○	×
6	×	×	×	×	×	○	×	×	×
7	×	×	×	○	×	×	×	×	×
8	×	○	×	×	×	×	×	×	○
9	×	×	×	×	×	×	○	×	×

7	1	2	3	4	5	6	7	8	9
1	○	×	×	×	×	×	×	○	×
2	×	×	×	×	×	○	×	×	×
3	×	×	×	○	×	×	×	×	×
4	×	○	×	×	×	×	×	×	○
5	×	×	×	×	×	×	○	×	×
6	×	×	×	×	○	×	×	×	×
7	×	×	○	×	×	×	×	×	×
8	○	×	×	×	×	×	×	○	×
9	×	×	×	×	×	○	×	×	×

8	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	×	○	×	×
2	×	×	×	×	○	×	×	×	×
3	×	×	○	×	×	×	×	×	×
4	○	×	×	×	×	×	×	○	×
5	×	×	×	×	×	○	×	×	×
6	×	×	×	○	×	×	×	×	×
7	×	○	×	×	×	×	×	×	○
8	×	×	×	×	×	×	○	×	×
9	×	×	×	×	○	×	×	×	×

9	1	2	3	4	5	6	7	8	9
1	×	×	×	×	×	○	×	×	×
2	×	×	×	○	×	×	×	×	×
3	×	○	×	×	×	×	×	×	○
4	×	×	×	×	×	×	○	×	×
5	×	×	×	×	○	×	×	×	×
6	×	×	○	×	×	×	×	×	×
7	○	×	×	×	×	×	×	○	×
8	×	×	×	×	×	○	×	×	×
9	×	×	×	○	×	×	×	×	×

Figure A.43: Configuration for $c^2_{9,3} = 626$

direction	statistic
x (1)	(70,70,69,69,70,70,68,70,70)
y (2)	(70,70,70,69,69,69,69,70,70)
z (3)	(70,70,70,69,69,69,69,70,70)

Table A.34: 2-dimensional statistics for $c^2_{9,3}$

A.5.2 Geometric lines

A.5.2.1 $n = 4$

1	1	2	3	4
1	×	○	○	○
2	○	○	×	○
3	○	○	○	○
4	○	×	○	○

2	1	2	3	4
1	○	○	○	○
2	○	○	○	○
3	○	○	○	○
4	○	○	○	×

3	1	2	3	4
1	○	×	○	○
2	○	○	○	○
3	○	○	○	×
4	×	○	○	○

4	1	2	3	4
1	○	○	○	○
2	○	○	○	○
3	○	○	○	○
4	○	○	×	○

Figure A.44: Configuration for $g^2_{4,3} = 8$

direction	statistic
x (1)	(2,1,1,4)
y (2)	(2,2,2,2)
z (3)	(3,1,3,1)

Table A.35: 2-dimensional statistics for $g^2_{4,3}$

A.5.2.2 $n = 5$

1	1	2	3	4	5
1	×	×	○	×	×
2	×	×	○	×	×
3	○	○	○	○	○
4	×	×	○	×	×
5	×	×	○	×	×

2	1	2	3	4	5
1	×	×	○	×	×
2	×	×	○	×	×
3	○	○	○	○	○
4	×	×	○	×	×
5	×	×	○	×	×

3/4	1	2	3	4	5
1	⊗	⊗	○	⊗	⊗
2	⊗	⊗	○	⊗	⊗
3	○	○	○	○	○
4	⊗	⊗	○	⊗	⊗
5	⊗	⊗	○	⊗	⊗

5	1	2	3	4	5
1	×	×	○	×	×
2	×	×	○	×	×
3	○	○	○	○	○
4	×	×	○	×	×
5	×	×	○	×	×

Figure A.45: Configuration for $g^2_{5,3} = 64$

direction	statistic
x (1)	(16,16,0,16,16)
y (2)	(16,16,0,16,16)
z (3)	(16,16,0,16,16)

Table A.36: 2-dimensional statistics for $g^2_{5,3}$

A.6 Configurations for $n = 4$ and 3 progressions

A.6.1 Algebraic lines

A.6.1.1 $d = 2$

	1	2	3	4
1	×	○	○	×
2	○	○	○	○
3	×	×	○	○
4	○	×	○	×

Figure A.46: Configuration for $a^1_{4,2} = 6$

A.6.1.2 $d = 3$

1	1	2	3	4
1	○	○	×	×
2	○	○	○	○
3	○	○	×	×
4	○	○	○	○

2	1	2	3	4
1	○	×	○	×
2	○	○	○	○
3	○	○	○	○
4	○	×	○	×

3	1	2	3	4
1	○	×	×	○
2	○	○	○	○
3	○	×	×	○
4	○	○	○	○

4	1	2	3	4
1	○	○	○	○
2	○	○	○	○
3	○	×	○	×
4	○	×	○	×

Figure A.47: Configuration for $a^1_{4,3} = 16$

direction	statistic
x (1)	(6,0,6,4)
y (2)	(0,6,4,6)
z (3)	(4,4,4,4)

Table A.37: 2-dimensional statistics for $a^1_{4,3}$

For the statistics of sub-cubes we count the points which are contained in every sub-cube of length two. For this reason we count the occurrences of the number of points in all possible pairs of two-dimensional sub-slices in the z direction (pages). This means we fix two z values and look at every sub-cube of length two in there and count the points.

number of points in sub-cube	occurrences
0	2
1	14
2	8
3	8
4	2
5	2

Table A.38: sub-cube statistics for $a^1_{4,3}$ where $z = 1$ and $z = 2$

number of points in sub-cube	occurrences
0	6
1	8
2	10
3	8
4	2
6	2

Table A.39: sub-cube statistics for $a^1_{4,3}$ where $z = 1$ and $z = 3$

number of points in sub-cube	occurrences
0	2
1	14
2	8
3	8
4	2
5	2

Table A.40: sub-cube statistics for $a^1_{4,3}$ where $z = 1$ and $z = 4$

number of points in sub-cube	occurrences
0	2
1	14
2	8
3	8
4	2
5	2

Table A.41: sub-cube statistics for $a^1_{4,3}$ where $z = 2$ and $z = 3$

number of points in sub-cube	occurrences
0	6
1	8
2	10
3	8
4	2
6	2

Table A.42: sub-cube statistics for $a^1_{4,3}$ where $z = 2$ and $z = 4$

number of points in sub-cube	occurrences
0	2
1	14
2	8
3	8
4	2
5	2

Table A.43: sub-cube statistics for $a^1_{4,3}$ where $z = 3$ and $z = 4$

A.6.1.3 $d = 4$

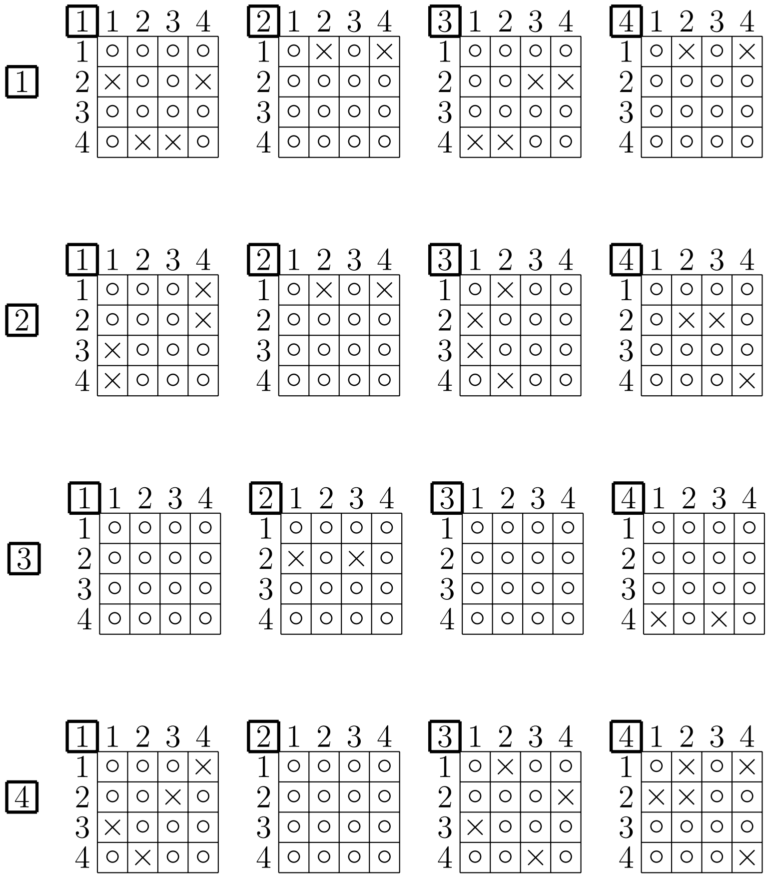

Figure A.48: Configuration for $a^1_{4,4} = 42$

direction	statistic
1	(12,14,4,12)
2	(11,12,7,12)
3	(12,6,12,12)
4	(12,13,4,13)

Table A.44: 3-dimensional statistics for $a^1_{4,4}$