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Option Pricing
in
Life Insurance

Master's Thesis

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Graz, April 13, 2017

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Abstract

The typical options and guarantees which are usually embedded in life insurance contracts are all intrinsically tied to them. Generally speaking, it is thus not possible to evaluate them on their own but one has to consider the difference resulting from a model incorporating them and one which does not.

The aim of this thesis is to fill gaps between theory and practice by collecting and presenting important aspects of life insurance modeling as well as practical aspects which often do not enter academic works in a most tangible way.

The thesis begins with background information on actuarial practice and an illustration of the present value of future cash flows. It continues with the commonly used simply-stochastic framework before introducing two certainty equivalence models which are contained within the framework and wide spread in practice. Finally a poly-stochastic framework with multi-dimensional time-inhomogeneous continuous affine processes is considered. Models of this framework can allow for dependencies between policy holder behavior and the interest rate while still being rather tractable.

Table of Contents

List of Figures

1. Introduction

Life insurance policies are financial contracts which stipulate payments between the policy holder[1] and the insurer. Characteristically these payments are subject to the state of the insured and they are settled at various dates. The evaluation of initially fixed premiums and benefits is thus already a complex matter because the biometric uncertainty necessitates an expected value concept and the time lags require a present value principle, both of which need to be modeled and calibrated. Insurance policies, however, consist of more than just fixed premiums and benefits. The insurer has costs, is reinsured, and in excess of the appointed benefits, the insurer usually grants the insured certain options and guarantees.

1.1. Objects of Study

Source: [DA13]

Embedded options are thereby certified rights of the policyholder to request certain unilateral contract amendments. Embedded guarantees, on the contrary, are bilateral agreements which fix matters for both sides. Thus, options rather make fixed quantities float while guarantees rather fix floating ones. Because of the lopsidedness, the value of an option is always non-negative for a rational policy holder whereas the value of a guarantee is not necessarily capped at zero.

Prominent examples are a guaranteed life table, a guaranteed interest rate, a guarantee to participate in the insurers profits, the option to surrender at a guaranteed repurchase value, or the possibility to waive future premiums at the cost of ex ante specified benefit reductions.

Even though these rights are de jure part of the contract and the notion of each of these rights indeed alters the value of the contracts, their effects have not been quantified for a long time because they can affect all future contractual payment amounts as well as the valuation basis. Hence, they influence every single aspect of the expected present value calculation and add thus a lot of complexity to the models. In the recent decades, however, not only the urge for accuracy but also the available computational tools as well as the field of actuarial modeling itself have evolved so that the determination has now become obligatory.

[1] For simplicity we assume that the policy holder is also the insured and the beneficiary.

Solvency II, the International Financial Reporting Standards 4 Phase II and the Market Consistent Embedded Value are prominent examples of risk based regimes which use a market consistent approach to derive capital requirements and business ratios. Within all of them, best estimate frameworks cannot stop at basic reserves but have to consider built-in options and guarantees just as well. The details, however, are far from being carved in stone. Solvency II, for example, provides standard procedures but encourages insurers to develop internal models to optimize the accuracy. While the market value of the assets can be obtained fairly easily using standard methods (mark-to-market or mark-to-model) the evaluation of the liabilities turns out to be a more delicate task. In the absence of a liquid market, a mark-to-market approach lacks any basis. On the other hand the presence of multifarious embedded options and guarantees makes insurance contracts complex financial derivatives. Thus mark-to-model procedures have to be elaborate tailor-made solutions.

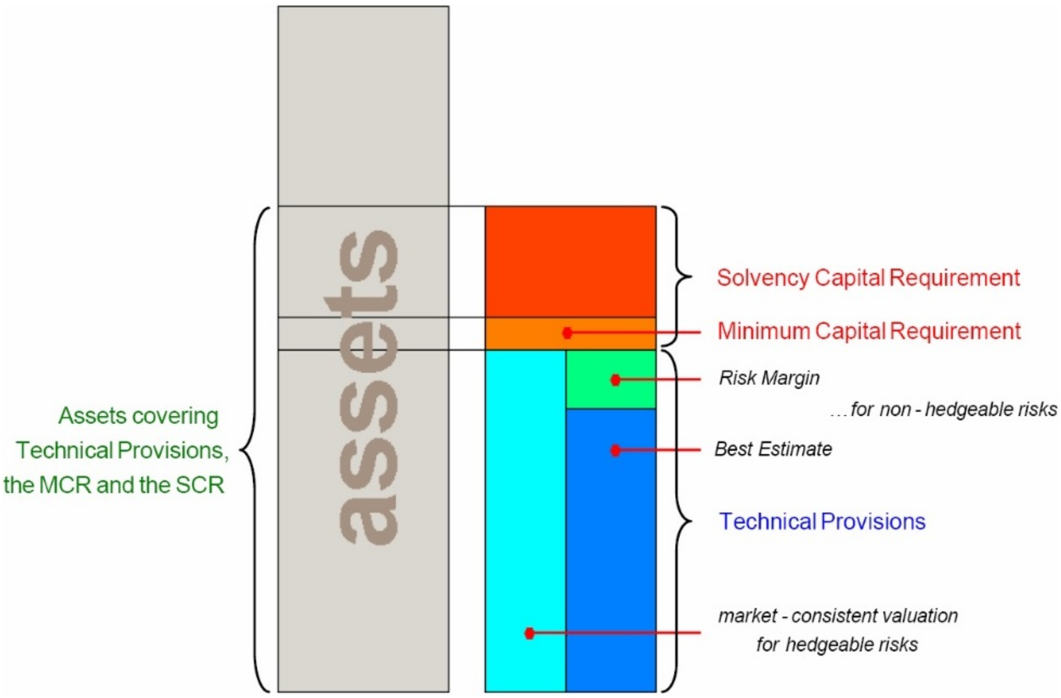


Figure 1.1.: Solvency II Balance Sheet where options and guarantees are part of the Best Estimate. Taken from [MO14].

1.2. Background on Actuarial Practice [Nor02, p. 10ff]

Embedded options and guarantees cannot be considered on their own for they are intrinsically tied to actuarial and accounting practices. Hence, we provide a brief introduction into those practices and we illustrate the evaluation of initially fixed payments before addressing options and guarantees.

1.2.1. Equivalence Principle

Before the first policy is written, the insurance company has to specify most contract terms as a tariff. This includes the risks insured, the respective biometric states, a set of biometric probabilities, admissible payment patterns as well as a discount factor and some cost factors. The factors are known as valuation basis of first order. They are deliberately estimated (i.e. with safety loadings) and then locked (guaranteed).

When a policy is finally written, either the benefit payments or the premium payments are set accordingly to an admissible payment pattern. Subsequently, the other side is calculated by making use of its payment pattern and the equivalence principle which demands that under the valuation basis of first order the expected present value of all benefits has to equal the expected present value of the respective premium components.

1.2.2. Costs

Cost structures vary quite a lot between different companies. For the given purpose,however, it is sufficient to differ expected costs, calculated costs and cost premiums of which the latter two are used for tariffing:

- The *expected costs* are the ones which the insurer actually expects to pay.
- Because the insurer typically gives a guarantee not to raise the cost premium during the term of the contract even though inflation is uncertain; because many costs are hard to itemize on a contract-level; and because not all premiums are paid due to death, surrender or waiver of premiums; the insurer has to add safety loadings. The result is called *calculated costs*.
- To not make the insured pay a different sum each month and to be able to let the insured have a limited premium payment period, the calculated costs and the *cost premiums* can differ as long as they satisfy the equivalence principle and follow an admissible payment pattern.

1.2.3. Reserves

Until the first payment, the equivalence principle thus ensures that the contract has a value of zero under the given valuation basis (probabilities, discount, calculated cost factors) meaning that the expected present value (EPV) of all future cash inflows is the negative of the EPV of all future cash outflows.

As time goes by, this equivalence usually breaks so that one side now owes the other. The first reason for this is that the expected premiums and the expected benefits are not exactly lined up in time which means that one of them is losing volume faster than the other. Endowment benefits are e.g. financed during the whole premium payment period but they are not paid out until the contract matures. Premium payments are often constant over time while mortality is increasing and costs need to be covered even if the premium payment period has already ended.

Therefore the prospective statutory book value reserve is introduced as the difference in the expected present values of future cash inflows and future cash outflows under the valuation basis of first order.

The question *who owes whom and to what extent* can also be answered in a retrospective way by taking a look at past and present payments the way bank accounts are treated. Under the valuation basis of first order the pro- and retrospective reserves give the same results due to the equivalence principle which was used to determine the premiums.

1.2.4. Valuation Basis

The importance of the valuation basis of first order stems from its legal character because the profit participation, surrender values and multiple other conversion rights are deducted from it.

However, the guaranteed discount factors and the estimated probabilities contain biases - volitionally due to the principle of precaution and inadvertently due to errors of estimation. Furthermore certain payment amounts are only steady over time as long as no embedded option or guarantee is exercised. From a best estimate (Solvency II) or fair value (IFRS) perspective, the contracts are thus evaluated under valuation bases of second order. This certainly involves best estimate rates without safety loadings, the prevailing interest rate from the market, and extended state spaces in order to model policy holder behavior such as surrender or waiver of premiums but it also involves stressed bases for variation analysis.

Because each of these changes might alter the expected present value, the resulting best estimate reserves usually differ from the statutory book value reserves. Furthermore, the prospective reserve no longer equals the retrospective reserve because now the retrospective reserve corresponds to the amount which the insured has actually contributed while the prospective reserve indicates the expected amount needed to settle the contract.

Especially for the prospective reserves, the choice of the right valuation basis is still troubling the industry which is why the whole section (3.2) is devoted to that question.

1.2.5. Profit Participations

Sources: [Noro2, p. 14 ff.], [DA13, p. 51].

For the future development of the biometrics and the financial markets are unknown while the valuation basis of first order is guaranteed, the competent supervisory authorities have obliged insurers to use a set of conservative assumptions involving disadvantageous transition rates as well as a low guaranteed interest rate as a valuation basis of first order. As e.g. enacted by the Austrian Financial Market Authority (FMA), this interest rate must not exceed a certain maximum interest rate ordinance, which is a historically averaged value of Austrian bond prices with a haircut of 40% [FMA03].

These acts of caution are expected to result in high profits for the insurer. According to the *Verordnung BGBl. II Nr. 292/2015* every Austrian insurer thus has to return at least 85% of its profits as a bonus after legal reserves were set aside. However, the bonuses do not have to be paid out immediately. Instead, the major part is often set aside to establish provisions which are allocated to the insured's profit accounts two years later. There, they are accumulated alongside the reserves until the contracts end. Hence, every potential (referring to the biometrics) benefit payment in our models is going to trigger a random (referring to the amount) potential profit participation payment.

1.2.6. Instants vs. Periods

Actuaries, just like accountants, work with periods instead of instants and they assume that endowment benefits and premiums happen right after an evaluation date while all other benefits happen right before it. This distinction is important because the reserves represent only the expected present value of **future** payments. This definition causes several minor complications and confusions to addressees who are not familiar with the actuarial notation.

Examples: In the actuarial representation the first premium, which is paid in advance, and a death benefit within the first year, which is payed in arrears, share the same index because they belong to the same period even though they occur a whole year apart from each other. On the other hand a death benefit and an endowment benefit which occur within an infinitesimal short period do not share the same index because the endowment benefit is seen as an in-advance payment of the following period. Thus a death benefit with index i needs to be discounted $i+1$ times while an endowment benefit with index i only needs to be discounted i times. Furthermore the last reserve of a mixed life insurance policy, right before the contract ends, does not contain any death benefits but only endowment benefits even though both are assumed to be paid within an infinitesimally short time interval.

For a portfolio manager who is often interested in the development of the liabilities this is not at all self-explanatory and the answer to the question *How much are we going to need in k years?* should clearly in- or exclude liabilities which will be due in k years.

Because of this and because index shifts are error prone we chose to consider instants instead of periods. In a first step this results in incomplete reserves, which we call volumes, where all current payments are excluded. By adding the respective current payments the volumes become reserves. This means we have substituted the concept of payments in advance and payments in arrears by an in/exclusion concept.

1.3. Illustration of a policy's Best Estimate

Let us now start by considering a simplified mixed life insurance contract.

Component 1: Contract Functions and their Payment Matrix

Within a discrete-time setup each policy can be stripped into single potential (referring to the biometric probability) payments. If the potential payment amounts are deterministic (referring to the amounts, not the biometrics) they can very well be depicted in an indexed way as a two-dimensional payment matrix. We therefore let the first dimension denote time and the second dimension shall denote the index of a predefined exhaustive list of payment purposes. These payment vectors over time are often called contract functions (CoFs). Most of them are subject to the form of contract and are thus known from the beginning. The columns for the profit participation in case of (i.c.o.) death, endowment, or surrender, however, do often depend upon the insurers future prosperity and hence they need to be modeled first. Neglecting costs and reinsurance, the list of CoFs for a mixed life insurance contract might thus, initially, look like this:

RPr	...	risk premium,	SPr	...	savings premium,
DB	...	death benefit,	DP	...	prof. part. i.c.o. death,
EB	...	endowment benefit,	EP	...	prof. part. i.c.o. endowment,
SB	...	surrender benefit,	SP	...	prof. part. i.c.o. surrender.

t	RPr	SPr	DB	EB	SB	DP	EP	SP
0	-17,66	-1.042,34	-	-	-	0,00	10,00	0,00
1	-18,20	-1.041,80	20.000,00	-	1.020,86	?	?	?
2	-21,79	-1.038,21	20.000,00	-	2.072,83	?	?	?
3	-25,09	-1.034,91	20.000,00	-	3.154,44	?	?	?
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	-78,50	- 981,49	20.000,00	-	15.930,78	?	?	?
14	-80,56	- 979,43	20.000,00	-	17.439,61	?	?	?
15	-	-	20.000,00	20.000,00	19.000,00	?	?	?

Table 1.1.: Deterministic CoFs of the second tariff from the results of chapter 3.

Component 2: Biometrics & Cash Flows

In life insurance, all payments are subject to biometric state transitions of the insured. This is almost always directly accounted for in the models. This means that the states and the transitions are modeled explicitly. The component thus needs a set of states, called the biometric state space, and a set of associated transition probabilities[2].

Figure 1.2 shows a standard state space and its possible transitions for the instants zero to n. The structure is very simple because no state can be reentered and all transitions share the same state of departure. This might not be desirable if e.g. invalidity needs to be incorporated but it is the standard state space for mixed life insurance contracts in practice.

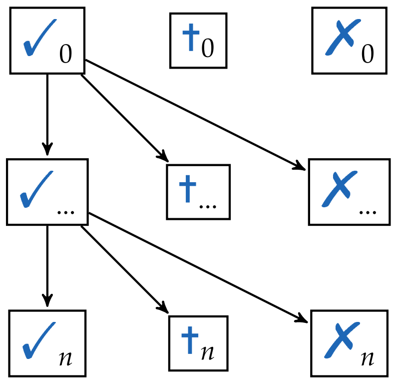


Figure 1.2.: States & transitions.

Henceforth, the contract functions need to be allocated to the transitions. Relating to the mixed life insurance with CoFs from above and the states active ✓, dead †, and surrendered ✗, we have the special situation where all transitions share the same state of departure so that we can identify each transition with its destination. Thus the allocation should look like this

$$\{✓, †, ✗\} \triangleq \{\,\{\text{RPr, SPre, EB, EP}\}, \{\text{DB, DP}\}, \{\text{SB, SP}\}\,\}.$$

Subsequently the respective payments of each state at each time are weighted by the probabilities by which the states are reached to become (expected) cash flows.

Remark: If the state space is too small (like in the Statutory Book Value Model introduced in section 3.1 which does not consider surrender) the respective payments can simply not be incorporated in this way.

[2]Surviving one year is also seen as a transition, namely from alive at t to alive at $t+1$.

Component 3: Interest, Discount & Present Values

For the value of money changes over time, payments which happen at different times cannot be compared or aggregated without scaling them first. This is called discounting if future values are considered, and compounding if past payments are under consideration. The weighted results are called present values.

For the time being, we make due with a deterministic interest rate curve which we take as given. If the interest rate is always positive, the cumulative discount factors become smaller the further the payment occurs in the future. For a typical mixed life insurance policy where the expected premiums happen earlier than the respective expected benefits, this means that the sum of all nominal benefit amounts can exceed those of the premiums while their expected present values are the same.

1.3.1. Expected Present Value

Assuming independence between the components, the valuation of each payment can now be described as the determination of a cuboid's volume. To see this, denote the expected **amount** which is paid if a certain biometric event occurs at a certain time in the first dimension. Now let the second dimension refer to the expected **likelihood** of the respective biometric triggering event and use the third dimension to describe the expected present value of a payment of one Euro payed at the respective future instant (**discount** factor). The expected present value of each payment can thus be seen as a cuboid's volume and the value of a contract as the sum of all payment volumes. This is illustrated in figure (1.3).

1.3.2. Problem Statement

The main task now lies in the determination of the probability- and discount factors (the rates) as well as the determination of the bonus amounts stemming from accumulated profit participations. Furthermore one is often interested in confidence intervals and dependencies between the components.

The value of future options and guarantees is finally obtained as the difference between the result of a model incorporating them less the result of a model which does not.

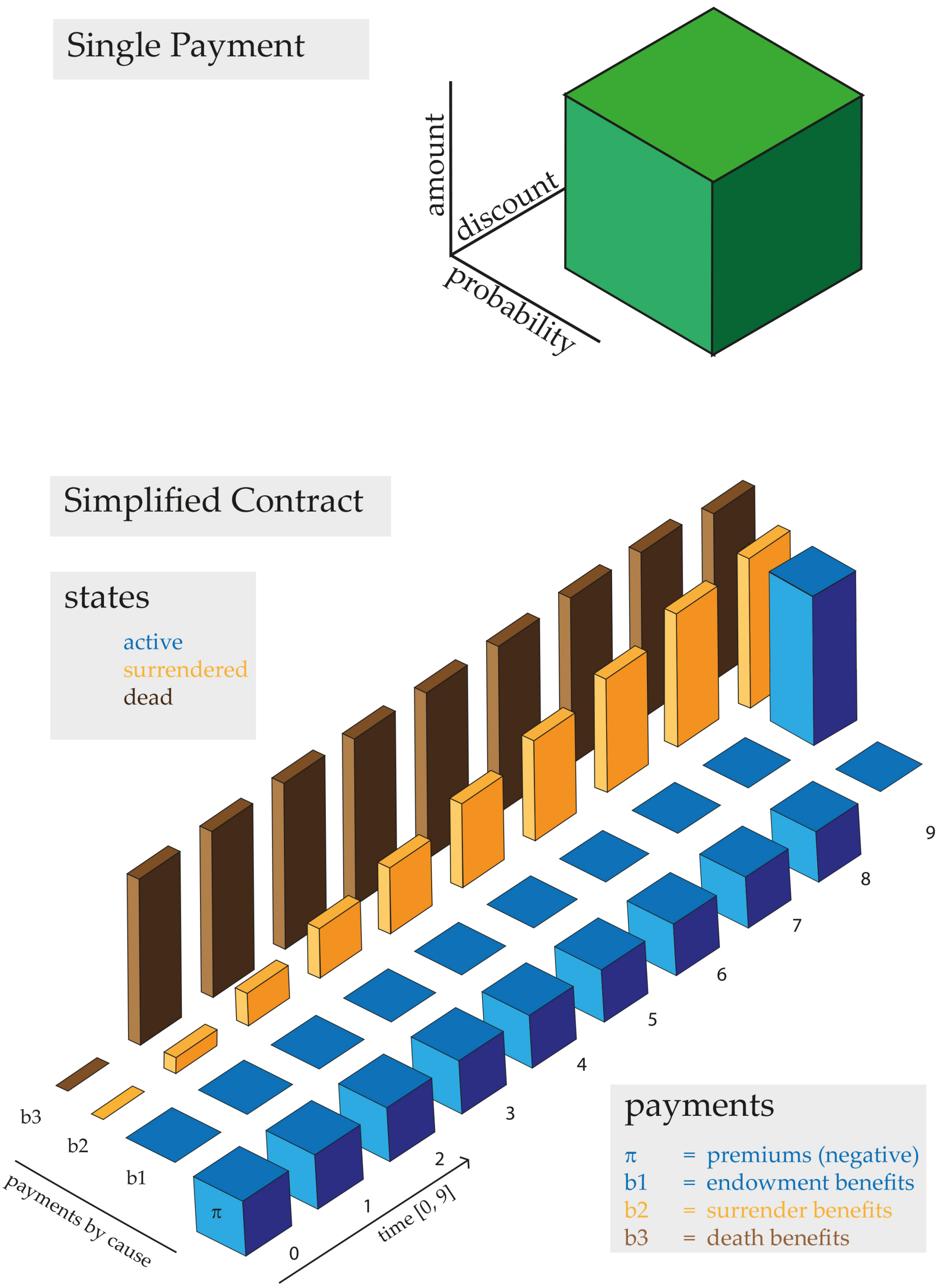


Figure 1.3.: Volumes of a simplified mixed life insurance policy on an annual grid. For simplicity the discount and the probabilities are shown time-invariant and without a scale.

1.4. Ambitions & Motivation

Models are always simplifications of the real world. Their assumptions are intended to filter out all the inaccessible, irrelevant, or unprocessible pieces of information in order to create an idealized world in which the necessary conclusions can be drawn most reliably. In addition a model should always be as transparent as possible so that everyone working with it or its results can be aware of its limits. This is of utmost importance because models are often needed the most during or right after structural breaks which, unfortunately, is exactly when time is a scarce good and model assumptions are likely to be violated.

Once the assumptions are made, the objects and their relations within the idealized world are fully specified. The algorithms that are applicable within this world are thus not part of the model. They are seen as algorithmically defined implications drawn from the model. As long as the algorithms are correct, the results do hardly/not depend on the algorithms whereas they do depend on the assumptions. The distinction is therefore crucial to differ modeling risk from implementation risk.

We thus formulated three ambitions of which the first two ambitions **comprehensibility** and **practicability** are deemed to be indispensable hygiene factors while **sufficiency** is considered the target.

1.4.1. Comprehensibility

There is quite a set of addressees (members of the board, financial market supervision, actuaries, accountants, risk managers, investors, policy holders,...) who might all be interested in the results. Due to their different backgrounds, however, they are used to different models and representations. The reserves we found were determined by using

- a time-discrete or a time-continuous model,
- working in a pro- or retrospective way,
- within a full or arbitrarily simplified state space,
- which processed probabilities or intensities
- for which different calibration procedures were used.
- Some in/excluded costs, profit participations, taxes or reinsurance,
- some incorporated/neglected policyholder behavior, etc.

Building a bridge of understanding between them by presenting a modular, easy to understand framework thus became the first ambition. For this we split up the reserve models into the three components (payments, biometrics, and discount - as already illustrated in section 1.3) and show how each component can be modeled within the same Markovian framework as a

1. deterministic,
2. certainty-equivalence,
3. simply-stochastic,
4. or poly-stochastic

model component. The first three cases are treated in discrete- and the latter is treated in continuous-time.

Simply-stochastic components neglect input uncertainties. The only deviation from their expected values are due to idiosyncratic risks[3]. A tangible example for this class of models is the model of a fair dice where the probabilities are known but the result is stochastic unless the dice is thrown infinitely many times in which case the model deteriorates to its **certainty equivalence** core where all sides appear simultaneously with a weight of 1/6. Even though this is a deterministic result, this is not the result of a **deterministic** component which would result in a single value. Referring to the dice model this could be any side of the dice. In terms of complexity, the certainty equivalence case is thus in between the deterministic and the simply stochastic approach. Within **poly-stochastic** components some input parameters are already random variables themselves.

1.4.2. Practicability

Bearing in mind that many insurance companies have millions of contracts, the second ambition concerned the practicability. Working on a single-contract-level, analytic expressions would be of great help but - as will be discussed in subsection (3.2.1) - analytic expressions for the profit participation are nowhere in sight. Thus, a sophisticated simulation sounds like a nice solution but that is not feasible on a contract level. Therefore either the number of contracts, the number of paths, or both need to be reduced.

[3]Assuming that the parameters are 100% correct and that one runs a given number of simulations, the only risk that is left is whether the received set of realizations is representative. This risk is also known as stochastic or binomial risk.

The number of contracts is usually reduced by data clustering algorithms[4]. They result in model points which are a set of (artificial) sample contracts that should represent the portfolio as well as possible. Unfortunately, it is very hard to tell whether the resulting fit is suitable for all aspects under concern without another model. Hence, model points are deemed to be a makeshift solution and will not be discussed here.

The alternative is to reduce the number of paths. This can be done in a gentle way by using variance reduction methods like the antithetic variates method, but it can also be done in a radical way by only considering the certainty equivalence path.

- The first option works like a catalyst, speeding up the simulation with very little side effects[5]. Nevertheless, thousands of paths for several different base scenarios for millions of contracts where each path can be up to ninety years long stays a mammoth task even without nested simulations which become more and more popular[6].
- Considering the certainty equivalence core means two things. Firstly, that one uses deterministic model parameters and therefore neglects the uncertainty of the model's parameters as well as their dependencies and secondly, that one assumes to have an infinitely big portfolio. If some components are furthermore modeled as deterministic components, then even the respective time values are no longer determinable.

Due to its simplicity the second approach has been the standard for decades now and it will thus be the starting point of this thesis.

[4]See e.g. the book of Frees, Derrig, and Meyers [Fre+14]
[5]See e.g. the book of Albrecher, Runggaldier, and Schachermayer [HRS00].
[6]See e.g. the working paper of Bauer, Bergmann, and Reuss [BBR10].

1.4.3. Sufficiency

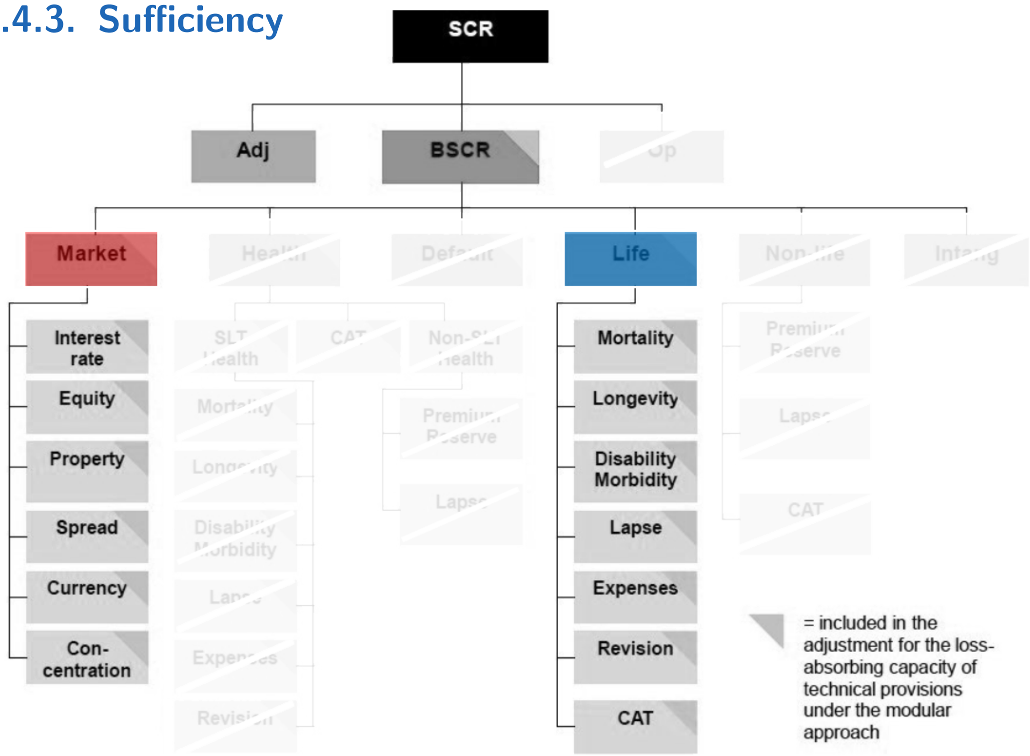


Figure 1.4.: Structure of the EIOPA Standard Formula for life insurance from [EIO14].

The third ambition was to provide a framework that can be used for all the risk based regimes mentioned in section (1.1). This means that the concept must be capable of evaluating life insurance policies taking into account

- market risk,
- policy holder behaviour,
- biometric risk,
- and risk of inflation

in order to provide the necessary tool to treat embedded derivatives like

- a guaranteed interest rate,
- a surrender option,
- a guaranteed life table,
- a right to participate in profits,
- a waiver of premium feature,
- or a guaranteed cost premium

separately as well as (partially) conjointly.

2. Simply-Stochastic Framework

In this chapter we are interested in a class of models where the stochastic present value of all future cash flows is of the following form

$$ {}_iV[t] := \sum_{\substack{u\in T\\u>t}} \sum_{j,k\in S} \overbrace{v[t,u]}^{\substack{\text{stochastic}\\\text{discount}}} \underbrace{\mathbb{1}_{\left\{\begin{smallmatrix} X_{u-1}=j \\ X_u \quad =k\end{smallmatrix}\middle| X_t=i\right\}}}_{\substack{\text{biometric}\\\text{indicator}}} \overbrace{a_{jk}[u]}^{\substack{\text{stochastic}\\\text{payment}}} \tag{2.1} $$

where $T = \{0,\ldots,n\}$ denotes the remaining term of the contract, $t \in T$ the observation date, $i, j, k \in S$ biometric states, and each of the three independent model components is driven by one or more[1] simply-stochastic[2] processes.

For the discount and the biometrics we are going to restrict all processes to Markov chains because the general setup does not provide enough structure for practical use. For the payment amounts, however, a distinction of cases is required. While most of them (e.g. fixed premiums and benefits) are deterministic anyway and some (e.g. costs under stochastic inflation) can conveniently be modeled by Markov chains, others (e.g. the profit participations) are accumulated over time which makes them path dependent. As soon as the discount is no longer deterministic, these payment amounts thus need to be derived through a Monte Carlo simulation which is not within the scope of this thesis.

For it is not obvious from the formula (2.1), we want to emphasize that within a pure Markovian setting all three model components can be modeled in the same way as a one-dimensional random walk over a discretized plane. The following section on Markov chains is thus relevant for all three model components.

[1]Each model component might consist of several independent components itself. Death, surrender, and waiver could for example form three separate biometric components but they could also be modeled as a single component.

[2]This means that each path is stochastic (unless the reachable part of the state space is a one-element set) but the transition probabilities are still deterministic.

2.1. Markov Chains

Source: **[SDS09]**.

Definition 2.1 (*Stochastic Process*)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a Kolmogorov probability space consisting of a sample space Ω, a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$. Let furthermore S denote a non-void finite state space with n elements, $T = [s, \ldots, u]$ an interval on $\mathbb{N}_0$, and let $\{X_t\}_{t \in T}$ be a collection of S-valued random variables. Then $\{X_t\}_{t \in T}$ is called a time-discrete stochastic process with finite state space S.

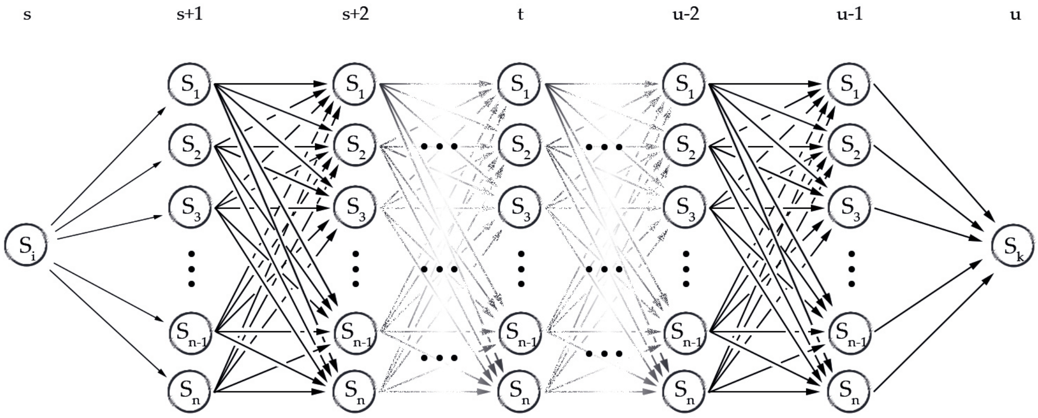


Figure 2.1.: Potential paths of a time-discrete stochastic process with finite state space.

Graphic (2.1) illustrates the potential set of paths such a process might be able to use to get to state $k \in S$ at time $u \in T$ given that it started in state $i \in S$ at time $s \in T$. The main problem here is to quantify the probability by which this happens. Usually one assumes to know the transition probabilities between two time-adjacent states because they are the easiest quantities to estimate. In this general setting, however, each of those transition probabilities might be path-dependent meaning that this information might not exist as a scalar value. This is because the probability is simply different if the path history is not the same. We thus restrict ourselves to Markov chains where the history is irrelevant.

Definition 2.2 (*Markov Chain*)

A time-discrete stochastic process $\{X_t\}_{t\in T}$ mapping from Ω to S is called a Markov chain if for all $n\in\mathbb{N}$, $t_0 < t_2 < \cdots < t_n \in T$, and $i_0, \dots, i_n \in S$ with

$$\mathbb{P}[X_{t_0} = i_0, \dots, X_{t_{n-1}} = i_{n-1}] > 0$$

it holds that

$$\mathbb{P}[X_{t_n} = i_n | X_{t_0} = i_0, \dots, X_{t_{n-1}} = i_{n-1}] = \mathbb{P}[X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}]. \quad (2.2)$$

Definition 2.3 (*Conditional Transition Probabilities*)

For a Markov chain $\{X_t\}_{t\in T}$, two instants $s \le t \in T$, and two states $i, j \in S$ of which the first must not be impossible to reach i.e. $\mathbb{P}[X_s = i] > 0$ the conditional transition probability from state i to state j within the time interval $(s, t]$ is denoted by

$$p_{ij}[s,t] := \mathbb{P}[X_t = j | X_s = i]. \quad (2.3)$$

For $\mathbb{P}[X_s = i] = 0$ we define $p_{ij}[s,s] := p_{ij}[s,s+1] := \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

Definition 2.4 (*Family of Transition Matrices*)

For a Markov chain $\{X_t\}_{t\in T}$ with finite state space S the set of all conditional transition probabilities between two instants $s \le t \in T$ can be written in matrix form as

$$P[s,t] = \{p_{ij}[s,t]\}_{i,j\in S} \quad (2.4)$$

forming the family of transition matrices.

Lemma 2.1 (*Chapman-Kolmogorov*)

Let $\{X_t\}_{t\in T}$ denote a Markov chain and $\{P[s,t]\}_{s\leq t\in T}$ its family of transition matrices. Then for all $s \leq t \leq u \in T$ and for all $i, j, k \in S$ for which $\mathbb{P}[X_s = i] > 0$, the Chapman-Kolmogorov equation states

$$
\begin{aligned}
P[s,u] &= P[s,t]P[t,u], \\
p_{ik}[s,u] &= \sum_{j\in S} p_{ij}[s,t]p_{jk}[t,u].
\end{aligned}
\tag{2.5}
$$

Proof 2.1

Let $S^* = \{j \in S \mid \mathbb{P}[X_t = j | X_s = i] > 0\}$ denote the states which can be reached at time t if starting from state i at time s. Using the law of total probability, we can write

$$
\begin{aligned}
p_{ik}[s,u] &= \mathbb{P}[X_u = k | X_s = i] \\
&= \sum_{j\in S^*} \mathbb{P}[X_u = k, | X_s = i, X_t = j]\mathbb{P}[X_t = j | X_s = i] \\
&= \sum_{j\in S} p_{ij}[s,t]p_{jk}[t,u].
\end{aligned}
\tag{2.6}
$$

Remark: The Chapman-Kolmogorv equation shows how to construct path probabilities from time-step probabilities in a recursive way.

Lemma 2.2 (*Summary on Transition Matrices*)

The transition matrices of a Markov chain satisfy the following conditions for all $s < t < u \in T$

- $p_{ij}[s,t] \geq 0 \quad \forall i, j \in S$,
- $\sum_{j\in S} p_{ij}[s,t] = 1 \quad \forall i \in S$,
- *$P[s,s]$ is the identity matrix,*
- $P[s,u] = P[s,t]P[t,u]$.

Proof 2.2

The first three conditions follow from the definition of the conditional probabilities while the last one is equation 2.5.

Lemma 2.3 (*Markov Chains & Transitions Matrices*)

A stochastic process $\{X_t\}_{t\in T}$ is a Markov chain if and only if $\forall n \in \mathbb{N}$, states $i_0, \dots, i_n \in S$ and instants $t_0 < \dots < t_n \in T$ the transition matrices $\{P[s,t]\}_{s \le t\in T}$ satisfy

$$\mathbb{P}[X_{t_0} = i_0, \dots, X_{t_n} = i_n] = \mathbb{P}[X_{t_0} = i_0] \prod_{k=1}^{n} p_{i_{k-1} i_k}[t_{k-1}, t_k]. \tag{2.7}$$

Proof 2.3

The implication of the Markov property follows from the definition of the conditional probabilities because the possibility to separate the probability of a path into the product of single step probabilities shows the Markovian no-memory property. The second implication is a repeated application of the Chapman-Kolmogorov equation.

In order to be able to use this powerful concept for the biometric development as well as for the development of the interest rate, inflation, and other independent influences simultaneously, the following lemma is a very important result.

Lemma 2.4 (*Product of Independent Markov Chains*)

Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ denote two independent Markov chains. Then $\{(X_t, Y_t)\}_{t\in T}$ is a Markov chain with state space $S = S_X \times S_Y$ and transition matrix

$$\begin{aligned} P[s,t] &= \left\{ p_{(i,j)(k,l)}[s,t] \right\}_{(i,j)(k,l)\in S} \\ &= \left\{ p^X_{ik}[s,t] p^Y_{jl}[s,t] \right\}_{(i,j)(k,l)\in S}, \qquad s \le t \in T. \end{aligned} \tag{2.8}$$

Proof 2.4

The proof follows directly from the independence of the probabilities:

$$\begin{aligned} &\mathbb{P}\left[X_{t_n} = i_n, Y_{t_n} = j_n \middle| X_{t_{n-1}}, \dots, X_{t_0}, Y_{t_{n-1}}, \dots, Y_{t_0}\right] \\ =&\mathbb{P}\left[X_{t_n} = i_n | \dots\right] \mathbb{P}\left[Y_{t_n} = j_n | \dots\right] \\ =&\mathbb{P}\left[X_{t_n} = i_n \middle| X_{t_{n-1}}, Y_{t_{n-1}}\right] \mathbb{P}\left[Y_{t_n} = j_n \middle| X_{t_{n-1}}, Y_{t_{n-1}}\right] \\ =&\mathbb{P}\left[X_{t_n} = i_n, Y_{t_n} = j_n \middle| X_{t_{n-1}}, Y_{t_{n-1}}\right] \end{aligned} \tag{2.9}$$

2.2. Markovian Components

In this section we are interested in the independent Markovian components from formula (2.1), their common structure, and how they interact.

2.2.1. Component 1: Payments

Definition 2.5 (*Markovian Contract Functions - CoFs*)

> *Let $u \in T = \{0, \ldots, n\}$ denote an instant during the remaining term of the contract under consideration and let B denote a finite set of payment purposes*[3]*. For each payment purpose $b \in B$ let S_{Z^b} denote the finite state space of a Markov chain $(Z_u^b)_{u \in T}$ driving the potential (according to the biometrics) random (according to Z_u^b) payment amount $a^b[u, Z_u^b]$. Then for each $b \in B$*
>
> $$\left(a^b[u, Z_u^b]\right)_{u \in T}$$
>
> *is called a Markovian contract function.*

Remark: Assuming that the state space S_{Z^b} contains only a single state, the surface collapses and becomes a path. As a consequence the stochastics vanish and the payments become deterministic.

Remark: For there are many ways the connection between the values of Z^b and the Markovian CoF can be modeled, we cannot list them all but we want to illustrate a driving random walk nether the less: The Markov chain could e.g. be used to simply distort a given deterministic course by adding noise. In this case each state is a value around the neutral element which is added or multiplied onto the value from the deterministic course. The disretized plane on which the random walk is performed could then be formed by the possible distortion factors over time.

Remark: Within a stochastic setting, the terms 'payoffs' and 'contingent claims' are very common as well. In all cases, the inherent uncertainty exclusively refers to the amount that is payed but not to the biometric probability by which it is payed.

[3] E.g. premium components, benefits, costs, etc.

2.2.2. Component 2: Biometrics

Definition 2.6 (*Grouped CoFs*)

Let S_X denote the finite state space of the biometric Markov chain X and $j,k \in S_X$ two states thereof. Then for each transition $\xi = (j \mapsto k)$ the sum of all CoFs belonging to ξ i.e.

$$\left(a_{jk}[u, Z_u]\right)_{u\in T} := \sum_{b \cong \xi} \left(a^b[u, Z_u^b]\right)_{u\in T}$$

where Z_u denotes the vector $(Z_u^b)_{b\in B}$, is called a grouped CoF .

Remark: The aggregation of the CoFs by the biometric transitions does mainly serve a less loaded notation because in practice the reduction in computational effort does not justify the loss in granularity.

Definition 2.7 (*Stochastic Cash Flows*)

Let $(X_u)_{u\in T}$ denote the Markov chain modeling the biometric state of the insured and let $i,j,k \in S_X$ denote three of its biometric states. Let furthermore $0 \le t_X, t_Z \le u-1 < u \in T$ denote some points in time. Then

$$_{i}a_{jk}\left[t_X, t_Z, u\right] := \mathbb{E}\left[\mathbb{1}_{\{X_{u-1}=j,\, X_u=k\}}\, a_{jk}[u, Z_u] \,\middle|\, X_{t_X}=i,\, Z_{t_Z}\right] \qquad (2.10)$$

denotes the random payments that are due if the insured switches from state j to state k during $(u-1, u]$ given that she is in state i at time t_X and the economic situation at time t_Z is known.

Remark: Here the discretized plane on which the biometric random walk is performed is formed by the grouped CoFs.

Remark: There are two common choices for t_X. For $t_X=0$ the stochastic cash flow is called *unconditional cash flow* while for $t_X=u-1$ the cash flow is called *conditional cash flow*. The conditional perspective is commonly used when single contracts are considered because a single contract cannot die fractionally. Portfolios, however, are rather displayed unconditionally because this perspective accounts for the dilution of the portfolio due to surrender and death. An exception to this is the Statutory Book Value model which also displays portfolios conditionally.

2.2.3. Component 3: Interest / Discount

Definition 2.8 (*Markovian Interest Rates*)

Let $\{Y_t\}_{t\in T}$ denote a Markov chain with state space S^Y and let $\{r_i[t]\}_{i\in S^Y,\, t\in T}$ denote a deterministic two-dimensional function. Then we model the stochastic short rate r_t as a random walk on this surface i.e.

$$r_t := r_{Y_t}[t] = \sum_{j\in S^Y} r_j[t]\mathbb{1}_{Y_t=j}. \tag{2.11}$$

Definition 2.9 (*Stochastic Discount*)

Let $P(t,u)$ denote the price at time t of a risk-free bond paying one unit at time u. Then the stochastic discount is defined as

$$v[t_Y,t,u] := \mathbb{E}\left[P(t,u)|Y_{t_Y}\right] = \mathbb{E}\left[\prod_{u=s+1}^{t} \frac{1}{1+r_u}\middle|Y_{t_Y}\right], \qquad \forall t \le u \in T. \tag{2.12}$$

Remark: The discount is closely related to the survival probability. Both are usually driven indirectly because the discount is driven by the short rate while the survival probability is usually driven by the mortality. The main difference is the link function which is one over $(1+r)$ for the discount but $(1-q)$ for the survival. In both cases the multi-annual values are the product of the single-annual values.

Definition 2.10 (*Stochastic Present Value*)

With the notation introduced above, the random present value of all cash flows between $t+1$ and n shall now be defined as

$${}_iV\left[t_X,t_Y,t_Z,t\right] = \sum_{\substack{u\in T\\ u>t}} \sum_{j,k\in S_X} v[t_Y,t,u]\ {}_ia_{jk}[t_X,t_Z,u]. \tag{2.13}$$

2.2.4. Expected Values

Definition 2.11 (*Prospective Contract Volume*)

The prospective contract volume is defined as the expected present value of all future cash flows

$$ {}_i\mathbb{V}\left[t_X, t_Y, t_Z, t\right] := \mathbb{E}\left[\, {}_iV\left[t_X, t_Y, t_Z, t\right]\right]. \tag{2.14}$$

In order to have a less loaded notation we drop t_X, t_Y, t_Z from now on.

Lemma 2.5 (*Prospective Contract Volume*)

Assuming independence between our three components and using the Chapman-Kolmogorov equation from lemma (2.1) as well as the product property from lemma (2.4), it holds that the prospective contract volume is

$$ {}_i\mathbb{V}[t] = \sum_{\substack{u\in T\\ t<u}} \mathbb{E}\left[v[t,u]\right] \left(\sum_{j,k\in S} p_{ij}[t,u-1]\, p_{jk}[u-1,u]\, \mathbb{E}\left[a_{jk}[u,Z_u]\right] \right). \tag{2.15}$$

Definition 2.12 (*Prospective Reserve*)

The prospective reserve is finally obtained by adding the respective current current cash flows

$$ {}_iRes[t] = {}_i\mathbb{V}\left[t\right] + \sum_{i,j\in S} \mathbb{1}_{incl.}\, a_{ij}[t]. \tag{2.16}$$

Lemma 2.6 (*Thiele's Difference Equation for the Prospective Contract Volume*)

Using the Chapman-Kolmogorov equation from lemma (2.1) to split up the discount and the probabilities into their annual factors, the prospective contract volume can be developed recursively:

$$ {}_i\mathbb{V}[t] = \mathbb{E}\left[v[t,t+1]\right] \sum_{j\in S} p_{ij}[t,t+1] \left({}_j\mathbb{V}[s,t+1] + \mathbb{E}\left[a_{ij}[t+1,Z_{t+1}]\right]\right) \tag{2.17}$$

Eq. (2.17) is often called the *fundament of modern life insurance mathematics*. Having it at hand, the evolution of the reserve is calculated very efficiently.

3. Certainty Equivalence Models

In this chapter we introduce two simple models which are contained within the simply stochastic framework. Both have deterministic payment- and discounting components as well as certainty-equivalence biometrics. The first model is the Statutory Book Value (BV) model, according to which the contract's legal parameters are defined. Subsequently several aspects of market consistent actuarial valuation are discussed before presenting and extending the Market Value (MV) model of Haas and Ladreiter in section (3.3) which is still a very simple valuation model.

3.1. The Statutory Book Value Model

Sources: [Bow+86; Sch06]. The statutory Book Value model (BV) is a very old and simple model with a highly efficient algorithm which does not work with cash flows, but calculates the expected present values directly via a commutation table. The model has been introduced when computing power was more or less equal to mental power and so the algebra had to be designed in a very resource friendly way - especially for calculating the reserve for the whole portfolio of policies. Unfortunately the resulting formulary is so rigid that even a change within a single benefit vector can yield to changes within the formulary. Therefore we only define the model and exemplify the algorithm on the basis of two simple contracts.

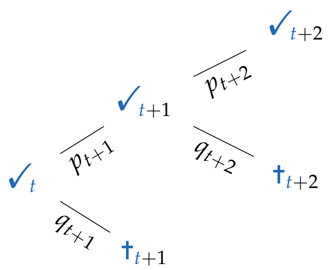


Figure 3.1.: States & transitions.

The Model

Only the biometric state space is not degenerated and consists of the set $\{\text{alive } ✓, \text{ dead } †\}$. The transitions are fully specified by a mortality table $(\bar{q}_x)_{x=0,\ldots,\omega}$ which provides the annual conditional mortality probabilities up to an ultimate age of $\omega \approx 130$. The discounting is done using a constant guaranteed interest rate $\bar{r}$. Future profit participations are not modeled so that all payments under consideration are deterministic.

Algorithm Part I: Commutation Tables

Using this information, the first step is to calculate so called commutation tables for each combination of guaranteed interest rate and mortality table within the portfolio. Because it is unusual to use this model with instants instead of periods, the resulting deviations are highlighted in red.

q_x	$:= \bar{q}_{x-1}, \quad q_0 = 0$	first order mortality table
v	$:= \dfrac{1}{1+\bar{r}}$	annual discount factor
l_0	$:= 100.000$	population of newborns
l_x	$:= (1 - q_{x-1+1}) l_{x-1}$	expected $x \in [1, \omega]$year olds
D_x	$:= l_x \, v^x$	discounted number of people alive
N_x	$:= D_x + D_{x+1} + \ldots + D_\omega$	reverse cumulative sum of D_x
d_x	$:= q_x \, l_{x-1}, \quad d_0 = 0$	expected deaths within $(x-1, x]$
C_x	$:= d_x \, v^{x+1-1}$	discounted number of deaths
M_x	$:= C_{x+1} + C_{x+1+1} + \ldots + C_{\omega+1}$	reverse cumulative sum of C_{x+1}
R_x	$:= M_x + M_{x+1} + \ldots + M_\omega$	reverse cumulative sum of M_x

Algorithm Part II: Expected Present Values

In a second step the expected present value functions of each tariff are expressed in terms of the commutation tables and the age x of the insured.

We therefore consider two widely used mixed life insurance tariffs. Both tariffs shall pay an endowment benefit of 1 if the insured survives until the contract ends after $n \leq \omega - x \in \mathbb{N}$ years. If the insured dies during the term of contract the first tariff shall pay a benefit of 1 at the end of the same year while the death benefit of the second tariff shall be a stepwise function with $k \leq n$ steps of equal height which occur within the first k years of the contract.

In order to determine the net premium, the expected present value (EPV) of all future benefits needs to be divided by the premium annuity which is the EPV of a premium of 1. Assuming a premium payment period of m years this results in the following formulary for the first tariff.

$$
\begin{aligned}
E_{x+t:\overline{n-t}|} &= \frac{D_{x+n}}{D_{x+t}} && \text{EPV of the endowment benefit} \\
A_{x+t:\overline{n-t}|} &= \frac{M_{x+t} - M_{x+n}}{D_{x+t}} && \text{EPV of the death benefits} \\
AE_{x+t:\overline{n-t}|} &= A_{x+t:\overline{n-t}|} + E_{x+t:\overline{n-t}|} && \text{EPV of all benefits} \\
\ddot{a}_{x+t:\overline{m-t}|} &= \frac{N_{x+t} - N_{x+m}}{D_{x+t}} && \text{premium annuity} \\
P &= \frac{AE_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}} && \text{premium amount}
\end{aligned}
\tag{3.1}
$$

For the second Tariff the definition of $A_{x+t:\overline{n-t}|}$ has to be changed to

$$
A_{x+t:\overline{n-t}|} = \frac{\frac{R_{x+t} - R_{x+max\{t,k\}}}{k} + \frac{min\{t,k\}}{k} M_x - M_{x+n}}{D_x}. \tag{3.2}
$$

Subsequently the statutory book value net reserve is calculated as the present values of future liabilities less future premiums

$$
Res_t = AE_{x+t:\overline{n-t}|} - \ddot{a}_{x+t:\overline{m-t}|} P. \tag{3.3}
$$

Remark: The resulting development of the reserve is thus conditioned on the survival of the insured which will not be the case in all other models.

Costs & Profits

Once the cost factors are set, the costs are treated analogically. The profit account is not modeled into the future.

3.2. Market Consistent Actuarial Valuation

According to Wüthrich, Bühlmann, and Furrer, market-consistent actuarial valuation is the answer to the question of how actuarial methods need to be changed in order to give values for insurance policies as if a market existed for them[1]. Baur gives a consistent, yet more technical, definition. According to her, a market consistent approach rests upon the fundamental theorems of asset pricing and risk-neutral valuation[2].

This definition thus starts off with a complete and arbitrage-free market. In such a market, each tradeable item (and every item that can be constructed from tradeable items by an admissible trading strategy) has a unique price and there is no free lunch with vanishing risk. The two fundamental theorems of asset pricing say, that such a market exists if and only if there exists a unique equivalent martingale measure Q under which the expected present value of every actively traded item equals its current market price. The existence of Q is furthermore linked to the non-existence of arbitrage and the uniqueness of Q to the uniqueness of the market prices[3].

Unfortunately, when pricing insurance contracts, the assumption of a complete market is not valid initially. This is because insurance policies themselves are not traded and some of their risk factors - like the biometric behavior - are not hedgeable as long as one cannot trade in them[4].

Therefore, there have been attempts to introduce derivatives with various biometric underlyings into the financial market. These *securitizations* have worked well for CAT mortality bonds while longevity bonds did not sell well due to their long term character and the resulting unpredictability[5]. Hence, there can still be no replicating trading strategy which determines the price of a typical life insurance policy uniquely.

There is, however, the possibility to turn things around by explicitly declaring one of the equivalent martingale measures the market consistent one.

[1] See e.g. [WBF08].
[2] See e.g. [Bau09].
[3] See e.g. [MB00], [Bjö09] or [Bau09].
[4] See e.g. [LNP16].
[5] See e.g. [CBD08].

Selection criteria therefore might be derived from best estimate assumptions, assumptions leading to risk-minimizing trading strategies, or other suitable utility concepts.

We emphasize that one is not free to choose at this point for it is the market that has to choose. In the absence of a real market, one is, however, free to argue what such a market might look like and the aforementioned concepts are possible implications of such considerations. A certain quantum of arbitrariness is thus in the nature of things and as a consequence, every choice requires a solid and persuasive reasoning[6].

3.2.1. Component 1: Payments

For deterministic payment amounts there is no need to model them for the only uncertainties concern the other two model components. Contingent claims, however, depend upon a specified stochastic event and are thus derivatives. If the random sources behind them are driving tradeable assets, we can thus use the theory of financial derivatives[7] in order to price the claims in terms of these assets. Otherwise the market is not complete and situational solutions are required.

Example: The profit participation presents a very common yet very complex contingent claim. This is because, firstly, it depends upon the insurers profits which themselves depend upon the financial market, the biometrics, cost development, etc. Secondly, this guarantee is strongly asymmetric because losses are not shared. Thirdly, these profits are then subject to local accounting standards and management rules which distort them. Fourthly, these profits are subsequently accumulated and compounded until the contract ends. This interferes with the future profits of the insurer and once again depends upon the mortality and the policy holder behavior. Finally, these cash flows must then be discounted and aggregated to arrive at the desired expected present value. Especially because of the local accounting rules and the management decisions, a closed form expression is nowhere in sight and a sophisticated simulation seems to be indispensable.

[6] See e.g. [Bau09].
[7] See e.g. [Bjö09].

3.2.2. Component 2: Biometrics

Since there is no way to buy **mortality**, a mark-to-market approach lacks any basis. For longevity bonds are not (yet) publicly traded, there is also no way to price mortality in terms of other assets. The common market consistent approach is thus to exchange the initial mortality table by a historically-estimated mortality table plus a risk margin for this is what a rational counterparty would presumably do as well.

Concerning **policy holder behavior** like surrender or forms of partial redemption like the waiver of premiums option, Milbrodt writes in [MS97]

> According to Cantelli's theorem, in a multiple decrement model, the cause of decrement "withdrawal" (cancellation) may be neglected without affecting premiums and reserves, if the withdrawal benefit equals the reserve.

For the withdrawal benefits are usually even lower than the reserve and because the BV model belongs to a system where precaution trumps best estimates, this is the main reason why the BV model can make due with the two biometric states active and dead. This, however, is not market consistent because the statement does only hold true in combination with the valuation basis of first order which, itself, is not market consistent. Once again there is no market where one can buy surrender. There are, however, two well known mark-to-model approaches which give significantly different results.

- Real-World Policyholder: The assumption is that the policyholder's behavior can be historically estimated. The biometric state space is thus extended, the transition probabilities are historically estimated and a risk-margin is added.
- Rational Policyholder: The assumption is that the policyholders trigger the option whenever it is best for them. This results in an optimal stopping problem.

While the second approach seems to be superior, many practitioners prefer the first option because policy holders have proven to not trigger the option then - but when they need the money which is not within the scope of the model. Furthermore it requires nested simulations to determine the optimal stopping time which increases the complexity of the model significantly.

3.2.3. Component 3: Interest & Discount

The statutory book value model uses the valuation basis of first order and thus a constant, guaranteed interest rate for all compounding purposes. The discount in this model is thus completely blind for changes at the financial markets. This is clearly not market consistent.

Another common mistake is to extrapolate the history and add a risk margin arguing that one cannot invest in a risk-free interest rate. This is plain wrong for there are enough traded items one can use to price in terms of them. But because the source of risk-free interest is indeed not that tangible in practice, we want to provide a short comment. There are many ways interest rates can be derived and quoted. While different quotations are convertible into each other (see appendix A), different derivations (e.g. from interest rate swaps, bond prices, etc.) usually lead to different results. This ambiguity is commonly solved by an arbitrage argument, saying that the derived interest rates are composed of a risky and a risk-free component whereby only the risk-free component is the same for all the rates. If the derived bond-rate is thus different from the derived swap-rate then this is because the two assets have different risk margins. In Econometrics these risk margins are called the market price of risk while in mathematics one is talking about the Girsanov kernel between the real world measure and the risk-neutral measure.

Remark: Within a deterministic setting, the discount is a deterministic function of two points in time and the respective compounding factor for the same interval is simply a different quotation. Within a stochastic setting, however, the two concepts diverge because their relation is not linear. Denoting a stochastic zero-coupon-bond price by $P(s,t)$ then by Jensen's inequality we know that

$$1 = \mathbb{E}\left[\frac{P(s,t)}{P(s,t)}\right] \neq \mathbb{E}\left[P(s,t)\right]\mathbb{E}\left[\frac{1}{P(s,t)}\right] > 1.$$

It is thus important to state whether the expected return or the expected discount is modeled and in which quotation. The first case leads to classic interest rate theory (see e.g. [MB00]) while the latter case leads to the theory of deflators (introduced by Duffie in [Duf96], see also [WBF08]) which, in economic theory, is also known as the theory of state price densities.

3.3. The Market Value Model

Source: [HL12].

The MV-model proposed by Haas and Ladreiter is a true generalization[8] of the BV-model for the purpose of simple risk management. It uses deterministic best estimate input data which is a valuation basis of second order. By varying the valuation basis, this model can already be used to do a simple case study. However, in order to comply with the Solvency II single-factor-insurance stress test which is part of the core module of the stress test framework 2014 (see [IA14, ch 1.8]), a possibility to determine the time values of all asymmetric components (like the future profit participation) needs to be supplemented.

With regard to the BV model, the constant guaranteed interest rate is replaced by the prevailing forward rate, the actually observed mortalities are used instead of the loaded ones, and the model's state space is extended to include surrender, which affects the results significantly. In addition the profit account is projected into the future which provides the basis for a prospective profit reserve which is defined in analogy to the benefit reserves as the expected present value of future profit payments to the insured.

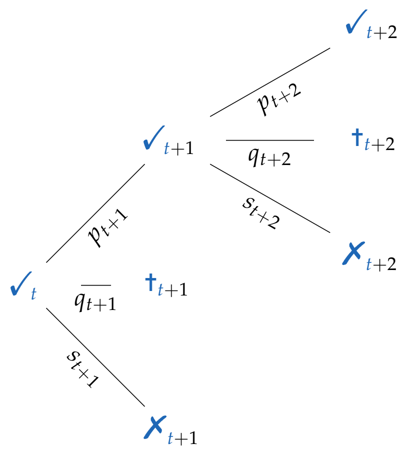


Figure 3.2.: States & transitions.

Because the premiums and all benefits are subject to the form of contract, which is specified accordingly to the BV-model they are considered to be fixed input values here. For the valuation basis has changed but the premiums have not, the equivalence principle does not hold under the MV-model any longer.

Due to the non-constancy of the new interest rate, the set of (mortality × interest) combinations refines to a level where the creation of the commutation-tables can not be justified any longer. On the plus side a vector based input regime can now be chosen for benefits, making the same formulary applicable to a whole range of tariffs. A progressive death benefit can thus be handled without alterations. It is simply a parametric change.

[8] Given the right input, it reproduces the results from the BV-model.

Input

The model needs the following input data

- premium and benefit amounts $\{\{\pi_t^{\checkmark}, b_t^{\checkmark}\}, \{b_t^{\dagger}\}, \{b_t^{\times}\}\}_{t\in[0,\dots,n]}$,
- a second order life table $[q_t^*]_{t\in\{0,\dots,100\}}$,
- a surrender table $[s_t^*]_{t\in\{0,\dots,60\}}$,
- and the prevailing forward rate $[F_t^*]_{t\in\{0,\dots,60\}}$.

The indexations of the original rates are connected as shown in figure 3.3.

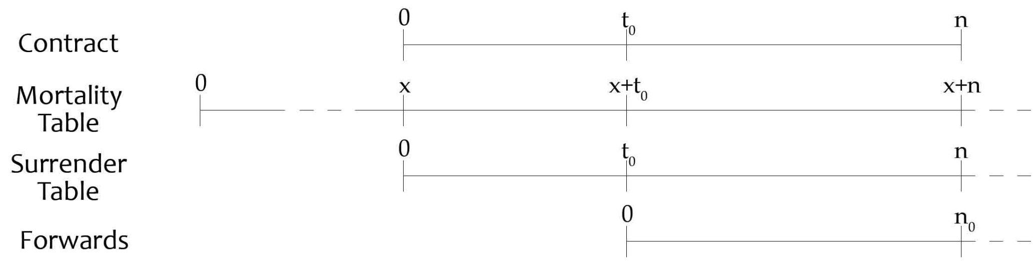


Figure 3.3.: Timelines of the input rates.

Determination of the Rates

Because the contract details are already defined by the statutory book value model and the reserves are defined as the present value of future cash flows, the contract's past is irrelevant for the MV model. Thus the input vectors are firstly trimmed to the remaining term of contract which is shown as the last interval in figure (3.3). Because society is seen as the basic population for the mortality table estimation while the surrender probabilities are usually deducted from the insurers own portfolio of contracts, the input-mortalities are then diluted by the surrender probabilities. Thus the following vectors are calculated for the remaining term of contract $n_0 = n - t_0$ for $t \in \{0, \dots, n_0\}$ and $t^* = t_0 + t$:

$$s_t := \mathbb{1}_{\{t\neq 0\}} s^*_{t^*-1} \qquad q_t := \mathbb{1}_{\{t\neq 0\}} q^*_{x+t^*-1}(1-s_t) \qquad p_t := 1-q_t-s_t \tag{3.4}$$

$$p_{0:\overline{t-1}|} := \mathbb{1}_{\{t-1>0\}} \prod_{k=0}^{t-1} p_k \qquad F_t := \mathbb{1}_{\{t\neq 0\}} F^*_{t-1} \qquad v_t := \prod_{k=0}^{t} \frac{1}{1+F_k} \tag{3.5}$$

Expected Present Values (EPVs)

The expected present values are finally calculated as

$$\begin{aligned}
A_{t:\overline{n_0-t}|} &:= \sum_{u=t+1}^{n_0} v_u\, p_{0:\overline{u-1}|}\, q_u\, b^{\dagger}[u], & E_{t:\overline{n_0-t}|} &:= \sum_{u=t}^{n_0} v_u\, p_{0:\overline{u-1}|}\, p_u\, b^{\checkmark}[u], \\
S_{t:\overline{n_0-t}|} &:= \sum_{u=t+1}^{n_0} v_u\, p_{0:\overline{u-1}|}\, s_u\, b^{\times}[u], & \ddot{a}_{t:\overline{n_0-t}|}\, P &:= \sum_{u=t}^{n_0} v_u\, p_{0:\overline{u-1}|}\, p_u\, \pi^{\checkmark}[u],
\end{aligned} \tag{3.6}$$

$$Res[t] = A_{t:\overline{n_0-t}|} + E_{t:\overline{n_0-t}|} + S_{t:\overline{n_0-t}|} - P\,\ddot{a}_{t:\overline{n_0-t}|}.$$

Remark: By writing out the formulas from the BV-model it becomes clear that they yield the same results if the same input is used.

Costs & Profits

Costs are treated analogically meaning that an inflow vector (cost premium) and an outflow vector (costs) need to be provided which are then treated like premiums and benefits.

In order to determine the necessary reserves for future profit participation to the insured, the profit account (PA) from the statutory book value model is projected into the future as respective payment amounts. We assume that the insurer shares 85% of the excess returns on the mathematical reserve, the full compounding on the PA and an additional interest-independent contribution which motivates the following equation

$$PA_{t+1} := \overbrace{(max\{0, 85\%(F_t - \bar{r})\})PB_{t+1}}^{\text{excess return on profit base}} + \underbrace{(1 + F_t)\, PA_t}_{\text{compounded profit account}} + \overbrace{AC_t}^{\text{additional contribution}}, \tag{3.7}$$

where $\bar{r}$ denotes the guaranteed interest rate and PB denotes the profit-base which typically is something like the book value reserve from the previous year. It is important to notice that the profit account defines additional payment amounts which - in analogy - need to be probability weighted, discounted and aggregated in order to result in a prospective reserve.

3.4. Extending the MV-Methodology

The MV model allows for several implications not mentioned in [HL12].

3.4.1. Expected Cash Flows

Cash flow representations are the standard in financial mathematics. In insurance mathematics, however, they were given scant attention until the last decade. In the BV-model from section (3.1), for example, the cash flows are well hidden deep within the formulary. They are not even interim results.

The basic idea is to separate the determination of the EPVs into a CF determination and a CF valuation step where the determination shall contain the insurance mathematics, while the valuation should make do with finance mathematics. The knowledge about the tariffs is thus encapsulated, while the freedom to model the financial valuation remains. To achieve this, we slightly generalize the notation from the previous section towards the notation used in chapter (2) so that we can write the formulas (3.6) in a single formula.

Let thus $S := \{\checkmark, \text{✗}, \dagger\}$ denote the biometric state space and $T := \{0, \dots, n\}$ the remaining term of the contract. Furthermore, for $j \in S$ and $u \in T$

$$a_{\checkmark j}[u] := \mathbb{1}_{\{j=\checkmark\}} \left(b^{\checkmark}[u] - \pi^{\checkmark}[u] \right) + \mathbb{1}_{\{j=\text{✗}\}} b^{\text{✗}}[u] + \mathbb{1}_{\{j=\dagger\}} b^{\dagger}[u]$$

shall denote the payments corresponding to the transition $(\checkmark \mapsto j)$ and

$$p_{\checkmark j}[u-1, u] := \mathbb{1}_{\{j=\checkmark\}} p_u + \mathbb{1}_{\{j=\text{✗}\}} s_u + \mathbb{1}_{\{j=\dagger\}} q_u$$

shall denote the respective transition probabilities in an indexed form. The volume, i.e. the reserve less current payments, can then be written as

$$\begin{aligned} \mathbb{V}[t] :=& Res[t] - b^{\checkmark}[t] + \pi^{\checkmark}[t] \\ =& \sum_{\substack{u \in T \\ u > t}} v[t,u] \sum_{j \in S} p_{\checkmark\checkmark}[t, u-1] p_{\checkmark j}[u-1, u] a_{\checkmark j}[u]. \end{aligned} \tag{3.8}$$

Splitting up the discount and the endowment probabilities into their annual effects we get

$$\mathbb{V}[t] = \sum_{\substack{u\in T\\ u>t}} \sum_{j\in S} \overbrace{\left(\prod_{k=t+1}^{u-1} v[k-1,k]\, p_{\checkmark\checkmark}[k-1,k]\right)}^{\text{discount and survival until the transition}} \underbrace{v[u-1,u]\ p_{\checkmark j}[u-1,u]}_{\text{conditional transition}}\ \overbrace{a_{\checkmark j}[u]}^{\text{amount}}, \tag{3.9}$$

where the mortality- and interest-effects until the transition are separated from the ones of the transition itself. For $u = t+1$ the product is empty and defined as 1 whereas for $t = n$ the sum is empty and defined as 0.

For the discount is deterministic in the MV model, it is independent from the biometrics. Hence, their annual effects can be separated so that we can write

$$\mathbb{V}[t] = \sum_{\substack{u\in T\\ u>t}} \sum_{j\in S} \left(\prod_{k=t+1}^{u} v[k-1,k]\right) \overbrace{\left(\prod_{k=t+1}^{u-1} p_{\checkmark\checkmark}[k-1,k]\right) \underbrace{p_{\checkmark j}[u-1,u]\, a_{\checkmark j}[u]}_{cCF}}^{uCF}, \tag{3.10}$$

where the conditional- and the unconditional cash flows as well as the separation of cash-flow-determination and cash-flow-valuation (present value calculation) are visible.

3.4.2. Thiele's Recursion [Noro2, p. 52]

Regarding equation (3.8), it is easy to see that the discount factor between t and $t+1$ is applied to every summand and can thus be factored out of the summation. Except for the death- and the surrender benefit at time $t+1$, the same thing holds true for the first endowment probability. We can thus write

$$\mathbb{V}[t] = v[t,t+1] \left(\sum_{j\in S} p_{\checkmark j}[t,t+1]\, a_{ij}[t+1] + p_{\checkmark\checkmark}[t,t+1]\, \mathbb{V}[t+1]\right) \tag{3.11}$$

which is Thiele's famous difference equation. Because $\mathbb{V}[t]$ is defined as the reserve at time t without current payments, $\mathbb{V}[n]$ is zero. This adds a terminal condition.

Using Thiele's backward recursion, the whole development of the reserve can be computed very efficiently.

3.4.3. Balancesheet Interpolation

No Sources.

So far we calculated values for the main renewal dates of a contract. In order to arrive at balance sheet values we have to interpolate either the results or the input. Given that Solvency II requires results derived from shocked rates we abandon the common practice of interpolating the results and develop our own interpolation scheme fot the rates instead.

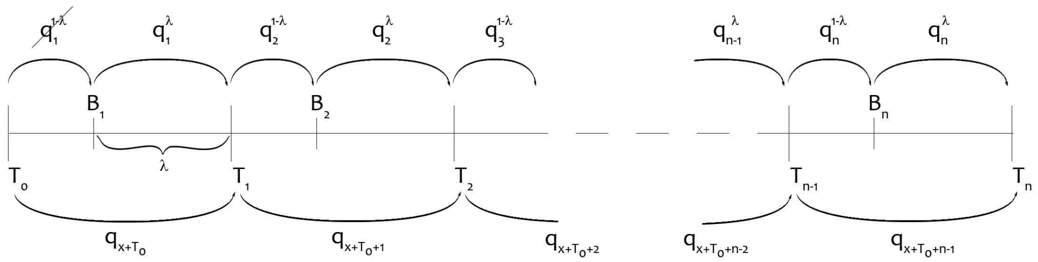


Figure 3.4.: Interpolation scheme with main renewal dates T and balance sheet dates B.

We assume to be given a set of annual conditional probabilities (p, q, s) with $p+q+s=1$. For any $\lambda \in (0,1]$ we are now looking for a set of $(0,\lambda]$- and $(\lambda,1]$-annual conditional probabilities $(p_\lambda, q_\lambda, s_\lambda, p_{1-\lambda}, q_{1-\lambda}, s_{1-\lambda})$ such that the process with the added times $B_1, \ldots, B_n$ is a Markov process whose transition probabilities between $T_0, \ldots, T_n$ coincide with those of the original process. This motivates the following restrictions

$$\begin{aligned}
&[R1] & p_\lambda p_{1-\lambda} &= p, \\
&[R2] & q_\lambda + p_\lambda q_{1-\lambda} &= q, \\
&[R3] & s_\lambda + p_\lambda s_{1-\lambda} &= s, \\
&[R4.1] & p_\lambda + q_\lambda + s_\lambda &= 1, \\
&[R4.2] & p_{1-\lambda} + q_{1-\lambda} + s_{1-\lambda} &= 1.
\end{aligned} \tag{3.12}$$

Missing at least one restriction we chose a geometric interpolation for p

$$[R0] \quad p_\lambda = p^\lambda \tag{3.13}$$

which is a quite natural choice and transforms the set of restrictions $[R1]$ to $[R4]$ into a system of linear equations. Solving equations $[R0]$ to $[R3]$ we get

$$p_{1-\lambda} = p^{1-\lambda} \qquad q_{1-\lambda} = \frac{q - q_\lambda}{p^\lambda} \qquad s_{1-\lambda} = \frac{s - s_\lambda}{p^\lambda}. \tag{3.14}$$

Inserting the results in $[R4.2]$ we see that it depends linearly on $[R4.1]$

$$\begin{aligned} p_{1-\lambda} + q_{1-\lambda} + s_{1-\lambda} &= 1 \\ p^{1-\lambda} + \frac{q - q_\lambda}{p^\lambda} + \frac{s - s_\lambda}{p^\lambda} &= 1 \\ p_\lambda + q_\lambda + s_\lambda = p + q + s &= 1 \end{aligned} \tag{3.15}$$

From an algebraic perspective we are thus missing a sixth equation, while semantically we have not specified how the annual mortality- (or surrender-) probability is distributed on the two intervals. The last restriction thus has to guarantee that the mass is evenly distributed and that neither the mortality nor the surrender can become negative on one interval. By choosing

$$[R5] \quad \frac{q_\lambda}{1 - p_\lambda} = \frac{q_{1-\lambda}}{1 - p_{1-\lambda}} \tag{3.16}$$

the ratio of the mortality over the overall dilution is fixed which guarantees the desired properties. The full system of linear equations now reads

$$\begin{array}{lcccccccc} & p_\lambda & q_\lambda & s_\lambda & p_{1-\lambda} & q_{1-\lambda} & s_{1-\lambda} & & \\ [R0] & 1 & & & & & & = & p^\lambda \\ [R1] & & & & 1 & & & = & p^{1-\lambda} \\ [R2] & & 1 & & & p^\lambda & & = & q \\ [R3] & & & 1 & & & p^\lambda & = & s \\ [R4] & 1 & 1 & 1 & & & & = & 1 \\ [R5] & & 1-p^{1-\lambda} & & & 1-p^\lambda & & = & 0 \end{array}$$

and results in the quite nice solution

$$\begin{aligned} p_\lambda &= p^\lambda, & p_{1-\lambda} &= p^{1-\lambda}, \\ q_\lambda &= \frac{1 - p_\lambda}{1 - p} q, & q_{1-\lambda} &= \frac{1 - p_{1-\lambda}}{1 - p} q, \\ s_\lambda &= 1 - p_\lambda - q_\lambda, & s_{1-\lambda} &= 1 - p_{1-\lambda} - q_{1-\lambda}. \end{aligned} \tag{3.17}$$

3.4.4. Waiver of Premiums Model Extension

No Sources.

The waiver of premium option entitles the policy holder to terminate the premium payments prematurely at the cost of reduced future benefits. More precisely, if a policy holder decides to trigger the option, the contract is converted into a single-premium contract which is financed by the current first order reserve less potential conversion costs.

Assumptions:

- The new contract shall again be subject to the original first order valuation basis.
- The probabilities $(q_{\text{STOP}}[t])_{t \in T}$ by which the policy holders trigger the option shall be independent of all previously defined components, they shall be known and there shall be no way to resume the premium payments.
- Concerning the reduction pattern, we assume that all future benefits are reduced by a single factor $R[t]$ if the option is triggered at time t.

We are now interested in measuring the impact of this option on the cash flows and the reserves of an extended MV model. However, as the contract does not terminate upon a waiver of premium, the option adds new branches to the state-tree at every node from which it can be exercised. This is illustrated in figure (3.5). The number of states which have to be considered thus becomes quadratic in time, instead of linear. This is bad news for insurance contracts have long coverage periods and the actuarial analysis is more and more heading towards monthly data instead of annual data.

However, considering a waiver of premium as a partial surrender, it is not clear why the problem should be much harder and indeed it is not. This is shown by deriving an algorithm which solves the problem within linear time.

To the authors knowledge the following approach is new.

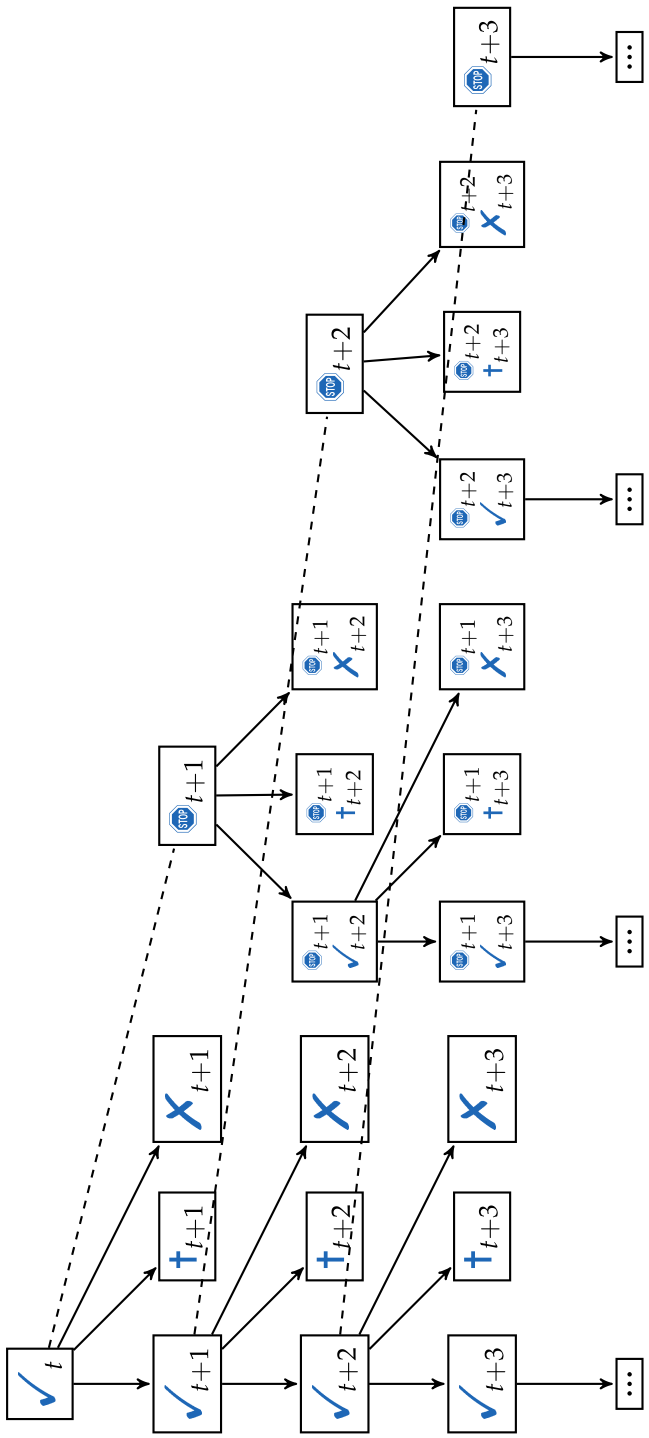


Figure 3.5.: States and transitions on an annual grid.

We start by analyzing the first new branch from figure (3.5) which results from the possibility to waive all premiums after the next main renewal date. According to our assumptions the contract is converted by taking the current BV reserve to finance a new single premium contract with the same payment pattern and the same valuation basis of first order but a lower sum insured.

Convention: The artificial premium is considered to be payed at the end of the previous period[9].

Reduction Factor

To determine the factor by which the benefits need to be reduced, we take a look at the original BV reserve Res^{BV}_{t+1} which is the expected present value of all future cash flows within the scope of the BV model before the option is triggered. Because the premiums are negative, the expected present value of the original benefits needs to be scaled down by a reduction factor R_{t+1} in order to make the following formula work. If there are conversion costs CC_{t+1} the new benefits are reduced even further.

$$
\begin{aligned}
Res^{BV}_{t+1} &:= EPV^{BV}_{t+1}[\text{benefits, premiums, expenses}] \\
&= R_{t+1}\, EPV^{BV}_{t+1}[\text{benefits, expenses}] + CC_{t+1} =: Res^{\text{STOP}}_{t+1} + CC_{t+1}
\end{aligned}
\tag{3.18}
$$

Because the EPV is a linear functional, the reduction factor can be dragged inside the EPV where it can be interpreted as a benefit reduction factor. This argument does still hold true if different reduction factors shall be used for different payment purposes because the EPV can be separated before dragging the factors inside[10].

[9]There are two reasons for this. Firstly, there is no actual cash flow at $t+1$ and so we do not want to model one and secondly, the new BV reserve at $t+1$ would otherwise be zero due to the definition of the reserve.

[10]Different reduction factors for payments which occur at different times, however, are not within the scope of this approach.

The reduction factors for waiving at $u \in [1, n]$ is thus

$$R_u := \frac{Res_u^{BV} - CC_u}{EPV_u^{BV}[\text{benefits, expenses}]}. \tag{3.19}$$

Remark: The reduction factor can be interpreted as the percentage of the future part of the contract that has already been bought less conversion costs.

The Quadratic Algorithm

Because of the assumed independence of the MV-biometrics we can either extend the biometric MV Markov chain directly or we can leave the biometric state space unchanged and define an additional Markov chain with state space $S_{\text{STOP}} := \{\checkmark, \text{STOP}\}$ and transition probabilities $(q_{\text{STOP}}[t])_{t \in T}$ instead and deem their product to be the extended biometric Markov chain.

Additionally to the already defined allocated contract functions $a_{\checkmark j}[u]$ we define the allocated contract functions without premium payments by

$$\tilde{a}_{\checkmark j}[u] := \mathbb{1}_{\{j=\checkmark\}} b^{\checkmark}[u] + \mathbb{1}_{\{j=\times\}} b^{\times}[u] + \mathbb{1}_{\{j=\dagger\}} b^{\dagger}[u]$$

. The extended MV volume can now be written as (compare with eq. 3.8)

$$\begin{aligned}\mathbb{V}[t] := &\sum_{\substack{u \in T \\ u > t}} \sum_{j \in S} v[t,u] p_{\checkmark\checkmark}[t, u-1] p_{\checkmark j}[u-1, u] p_{\text{STOP}}[t, u-1] a_{\checkmark j}[u] + \\ &\sum_{\substack{u \in T \\ u > t}} \sum_{j \in S} \sum_{k=1}^{u-t} v[t,u] p_{\checkmark\checkmark}[t, u-1] p_{\checkmark j}[u-1, u] p_{\text{STOP}}[t, t+k-1] q_{\text{STOP}}[t+k] R_{t+k} \tilde{a}_{\checkmark j}[u]\end{aligned} \tag{3.20}$$

which is simply the sum of all discounted, probability-weighted allocated contract functions from the nodes in figure (3.5). Unfortunately this naive straight-forward algorithm is quadratic which makes it useless in practice.

Pitfall

For the following flaw has been suggested several times by different parties in practice, we explicitly want to expose a fallacy here. Trying to solve the runtime problems by considering a linear combination of the different reduced reserves weighted by the probabilities by which the insureds trigger the option at the respective points in time fails for

$$\mathbb{V}[u] \neq \left(p_{\text{STOP}}[t,u-1] + \sum_{k=1}^{u-t} p_{\text{STOP}}[t,t+k-1] q_{\text{STOP}}[t+k] R_{t+k} \right) \mathbb{V}_u^{MV} \tag{3.21}$$

This is because $\tilde{a}_{\checkmark j}[u] \neq a_{\checkmark j}[u]$ and because each reserve contains payments from different points in time. While the first problem could be solved by a distinction of cases, the latter turns out to be trouble because the probability of not having triggered the option $p_{\text{STOP}}[t,u-1]$ is always time dependent. So applying a single factor to a reserve cannot work.

Linear Algorithm

Taking a closer look at the new terms in equation (3.20), we see that the idea of a linear combination does actually work if it is applied to the cash flows instead of the reserves and if premiums are distinguished from benefits and costs. While premiums are only subject to the soujourn probabilities, the benefits and the costs are subject to the following weights $w[t,t] = 1$ and

$$w[t,u] := p_{\text{STOP}}[t,u] + \sum_{k=0}^{u-t-1} p_{\text{STOP}}[t,t+k] q_{\text{STOP}}[t+k] R_{t+k} \quad \text{for } u > t$$

which appear to be a cumulative sum. Hence, all we need is a recursion for $w[t]$ which is given by

$$\begin{aligned} w[t,t] &= 1 \\ w[t,u] &= w[t,u-1] - p_{\text{STOP}}[t,u-1] q_{\text{STOP}}[u-1](1 - R_{u-1}). \end{aligned} \tag{3.22}$$

The volume $\mathbb{V}[t]$ can now be written as

$$\sum_{\substack{u \in T \\ u > t}} \sum_{j \in S} v[t,u] p_{\checkmark\checkmark}[t,u-1] \Big(p_{\checkmark\checkmark}[u-1,u] p_{\text{STOP}}[t,u] \pi^{\checkmark}[u] + p_{\checkmark j}[u-1,u] w[t,u] \tilde{a}_{\checkmark j}[u] \Big). \tag{3.23}$$

4. Poly-Stochastic Framework

Sources: [Kol10, p. 10ff], [Jon93], [Buc11, p. 1ff] and [Nor02, p. 68ff].

In the previous chapter on certainty equivalent models we assumed to be dealing with an infinite set of identical contracts by which the biometric Markov chain degenerated. For the other components were even modeled in a deterministic way, we exclusively dealt with expected values and there was nothing to say about higher moments or dependencies.

The real world, however, is neither entirely predictable nor entirely independent and so we are very interested in possible deviations and dependencies of and between the components.

While some parts of this problem could already be tackled within the more general simply-stochastic framework from chapter (2)

- e.g. by omitting the 'infinite set of contracts' assumption one would gain the opportunity to measure the biometric idiosyncratic risks
- or by modeling the interest rate as a Markov chain one could quantify the financial idiosyncratic risk;

the dependencies between the components as well as the input uncertainties would always remain unexplained in this framework. This is bad news because unlike the biometric idiosyncratic risk which is an unsystematic risk (meaning it is hedged away by the large portfolio an insurer usually has), input uncertainties like a possible bias in the mortality probabilities pose a systematic risk to which particular attention should be devoted to.

We are therefore interested in a new class of models where all input rates can be driven by stochastic processes. Besides the possibility to model confidence bands and time values, this approach also grants the appealing possibility to model dependencies between the rates.

4.1. Framework Draft

We let the state of the insured be modeled by a **continuous**-time, decrement Markov chain

$$\mathbf{Z} = (Z[t])_{t\in[t_0,n]}, \quad Z[0] = S_0[0]$$

which is still operating on a finite state space $\mathbb{J}$ but which shall now be driven by stochastic intensities instead of deterministic probabilities. We therefore consider the short rate $r(t)$ and the g non-zero transition intensities

$$\mu_{i,j}(t) : i,j \in \mathbb{J}, t \in [t_0, n]$$

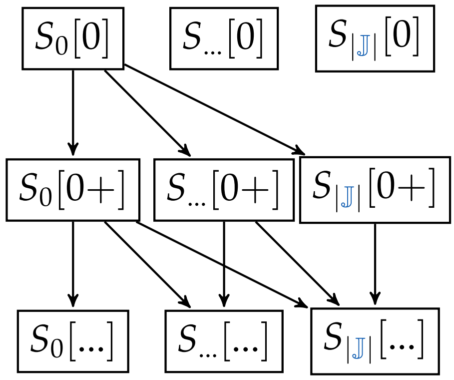


Figure 4.1.: States and transitions of a continuous, decrement Markov chain.

on a continuous state space as an affine transformation

$$Y(t) = c(t) + \Gamma(t)X(t)$$

of a d-dimensional continuous affine processes $\mathbf{X}$ with admissible functions $c : \mathbb{R}_+ \to \mathbb{R}^g$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{g\times d}$. This means that the distribution of $\mathbf{Z}$ is specified by the transition rates, conditional on $\mathbf{X}$.

Due to the continuity we also redefine the payments by the process $\mathbf{B} = (B(t))_{t\in[t_0,n]}$ which accumulates all payments up to time t. The payments at time t are assumed to be of the following form

$$dB(t) = \sum_{i\in\mathbb{J}} \mathbb{1}_{Z(t)=i}\, b_i(t)\, dt + \sum_{i\neq j\in\mathbb{J}} b_{ij}(t)\, dN_{ij}(t) \quad \in \mathbb{R}$$

for deterministic payment intensity functions b_i for the cases of sojourn, deterministic lump sum payment functions b_{ij} for each possible transition, and the process $N_{ij}(t)$ which is counting the transitions of $\mathbf{Z}$ from state i to j such that

$$N_{ij}(t) - \int_0^t \mathbb{1}_{Z(s-)=i}\, \mu_{ij}(s)\, ds$$

is a martingale, conditional on $\mathbf{X}$. We thus assume that premiums and annuities are payed continuously while the benefits are payed as lump sums immediately upon the triggering event.

4.2. Filtrations and a new Dimension

For the transition intensities can now be driven by stochastic processes, we are working with an evaluation input as depicted in figure 4.2. A nice feature of this approach is that arbitrary affine processes can be chosen for every intensity rate as well as for the short rate. Even dependencies between them can be modeled (See e.g. [Buc13]).

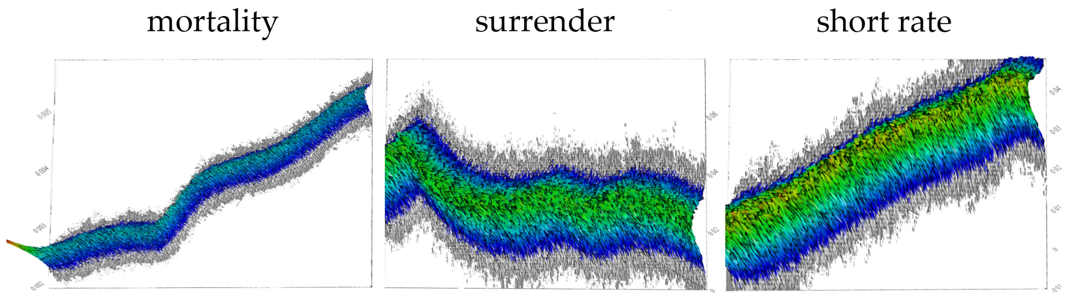


Figure 4.2.: Stochastic intensities, each modeled as a Hull-White processes $d\mu(t) = [\theta(t) - a\mu(t)]dt + \sigma dW(t)$. The mean reversion parameters $\theta(t)$ are derived from the MV-model results. The other parameters are $a_\mu = 2$, $\sigma_\mu = 0.0003$, $a_\eta = 10$, $\sigma_\eta = 0.03$, $a_r = 1.8$, $\sigma_r = 0.01$. Each rate was simulated 10.000 times and then plotted in 3D to visualize the frequencies.

In the simply stochastic approach we assumed that we do not know the future development of Z, say after t_0, meaning that we do not know how the insured will move between the states and when. We were thus working with the filtration $\mathcal{F}^Z[t_0]$. On the other hand, we assumed to know the future transition probabilities and the interest rate, so there was no need to define a σ-algebra describing the evolution of knowledge about the transition probabilities or the interest rate. This is different now. Hence, denote by $\mathcal{F}^Y$ the filtration generated by the intensities of the transition rates and the interest rate and by $\mathcal{F} = \mathcal{F}^Z \vee \mathcal{F}^Y$ the joint filtration of Z and the intensities.

To illustrate the new dimension, we take a look at the expected transition probability $p_{ij}(s,t)$ for the state space $\mathbb{J} = \{\checkmark, \dagger\}$, with $i = \checkmark$, $j = \dagger$ and think of it as the volume between s and t of the ridge depicted in figure (4.2) under 'mortality' which is a double integral.

The stochastic transition probability $p_{ij}^X(s,t)$ on the other hand is a simple integral just like the expected intensity, but over the other dimension, respectively. It is thus the average cross-section area of the respective part of the ridge and not a path like the expected intensity. The connection between them is

$$\begin{aligned}
p_{ij}(s,t) &= \mathbb{P}\left[Z(t)=j|Z(s)=i\right] \\
&= \mathbb{E}\left[\mathbb{1}_{Z(t)=j}\middle|\mathcal{F}^X(s), Z(s)=i\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{Z(t)=j}\middle|\mathcal{F}^X(\infty), Z(s)=i\right]\middle|\mathcal{F}^X(s)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[Z(t)=j\middle|\mathcal{F}^X(\infty), Z(s)=i\right]\middle|\mathcal{F}^X(s)\right] = \mathbb{E}\left[p_{ij}^X(s,t)\middle|\mathcal{F}^X(s)\right].
\end{aligned}$$

Furthermore consider the expected loss as the difference in the expected present values of future liabilities arising from conditioning on the two different states of knowledge where we can now (at time t_0) separate the risk which looms at time $T > t_0$ (first line) into an unsystematic (second line) and a systematic component (third line) in which we are interested:

$$\begin{aligned}
L[T] &= \mathbb{E}[PV[t_0]|\mathcal{F}[T]] - \mathbb{E}[PV[t_0]|\mathcal{F}[t_0]] \\
&= \mathbb{E}[PV[t_0]|\mathcal{F}[T]] - \mathbb{E}[PV[t_0]|\mathcal{F}^Z[t_0] \vee \mathcal{F}^Y[T]] \qquad (4.1) \\
&+ \mathbb{E}[PV[t_0]|\mathcal{F}^Z[t_0] \vee \mathcal{F}^Y[T]] - \mathbb{E}[PV[t_0]|\mathcal{F}[t_0]].
\end{aligned}$$

4.3. Markov Chains & Processes

Affine processes are Markov processes. We are therefore investigating a few important characteristics of this superset before dealing with the specialties of affine processes themselves.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a Kolmogorov probability space consisting of a sample space Ω, a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$.

Definition 4.1 (*Stochastic process*)

A stochastic process is a family of random variables $\{X_t : t \in \mathbb{T}\}$ mapping into a measurable state space $(\mathbb{J}, \mathcal{J})$ where $\mathbb{T}$ is a totally ordered index set

$$X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{J}, \mathcal{J}), \quad \omega \mapsto X_t(\omega).$$

Definition 4.2 (*Markov Property*)

An adapted $\mathbb{J}$-valued stochastic process X is said to possess the Markov property if, for each $\mathcal{A} \in \mathcal{J}$ and each $s < t \in \mathbb{T}$

$$\mathbb{P}[X_t \in \mathcal{A} | \mathcal{F}_s] = \mathbb{P}[X_t \in \mathcal{A} | X_s].$$

This means the conditional probability of a future event depends exclusively on the current[1] state and it is irrelevant how the process reached its current state. The Markov property is thus also known as the *no memory* property.

In the previous chapters we have used a discrete-time Markov process and its conditional transition probabilities to model the state of the insured. Now we are going to use a continuous-time Markov chain $\mathbf{Z}$ and an affine multidimensional Markov processes $\mathbf{Y}$ to model the transition intensities (and the short rate).

In the case of a Markov chain with a finite state space, the transition probabilities can be written in matrix form for every admissible interval.

[1] More precisely: the time the conditioning refers to, which might as well be in the future.

Definition 4.3 (*Conditional transition matrix*)

The conditional transition matrix is defined by

$$\begin{aligned} P(s,t) &:= (p_{ij}(s,t))_{i,j=1,\dots,|\mathbb{J}|}, \\ p_{ij}(s,t) &:= \mathbb{P}[Z(t)=j \mid Z(s)=i], \quad s \leq t \text{ and } i,j \in \mathbb{J}. \end{aligned} \tag{4.2}$$

Lemma 4.1 (*Chapman-Kolmogorov*)

For all $s \leq t \leq u \in \mathbb{R}_+$, $i,j,k \in \mathbb{J}$ and $\mathbb{P}[Z(s)=i] > 0$ the Chapman-Kolmogorov equation states

$$\begin{aligned} P(s,u) &= P(s,t) \cdot P(t,u), \\ p_{ik}(s,u) &= \sum_{j \in S} p_{ij}(s,t) p_{jk}(t,u). \end{aligned} \tag{4.3}$$

Regarding the Chapman-Kolmogorov equation in terms of the states, it points out how to deal with recombining trees in discrete-time like in figure 4.1. Regarding it in terms of the time instants, it adumbrates why a discrete distribution of **Z** can be fully specified by the conditional probabilities.

However, the intensities and the short rate are modeled in continuous time and there is no smallest time interval. Thus, a smoothness assumption has to be made before their distributions can be characterized. We thus assume that the following transition intensities exist and are continuous.

Definition 4.4 (*Transition intensities*)

$$\begin{aligned} \Lambda(t) &:= \begin{pmatrix} \mu_{11}(t) & \dots & \mu_{1n}(t) \\ \vdots & \ddots & \vdots \\ \mu_{n1}(t) & \dots & \mu_{nn}(t) \end{pmatrix}, \\ \mu_{ij}(t) &:= \lim_{\Delta t \downarrow 0} \frac{p_{ij}(t,t+\Delta t) - \mathbb{1}_{i=j}}{\Delta t} \text{ for all } i,j \in \mathbb{J}, t \in [0,n]. \end{aligned} \tag{4.4}$$

A comprehensible connection between intensities and probabilities can be established by taking a look at the derivatives of $p_{ij}(s,t)$ which results in the Kolmogorov differential equation.

Derivation 4.1 (*Kolmogorov differential equation*)

$$\begin{aligned}
\frac{d}{dt}p_{ij}(s,t) &= \lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\left(p_{ij}(s,t+\Delta t)-p_{ij}(s,t)\right)\\
&\overset{4.3}{=} \lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\left(\sum_{k\in\mathbb{J}}p_{ik}(s,t)p_{kj}(t,t+\Delta t)-p_{ij}(s,t)\right)\\
&= \lim_{\Delta t\downarrow 0}\sum_{k\in\mathbb{J}}p_{ik}(s,t)\frac{p_{kj}(t,t+\Delta t)-\mathbb{1}_{k=j}}{\Delta t}.
\end{aligned}\tag{4.5}$$

Inserting the definition (4.4) completes the derivation of the forward case.

Lemma 4.2 (*Kolmogorov differential equations*)

- *The forward case*

$$\begin{aligned}
\frac{d}{dt}P(s,t) &= P(s,t)\Lambda(t),\\
\frac{d}{dt}p_{ij}(s,t) &= \sum_{k\in\mathbb{J}}p_{ik}(s,t)\mu_{kj}(t).
\end{aligned}\tag{4.6}$$

- *The backward case is derived analogously resulting in*

$$\begin{aligned}
\frac{d}{ds}P(s,t) &= -\Lambda(s)P(s,t),\\
\frac{d}{ds}p_{ij}(s,t) &= -\sum_{k\in\mathbb{J}}\mu_{ik}(s)p_{kj}(s,t).
\end{aligned}\tag{4.7}$$

This provides the necessary tools to formulate the residence probability in terms of the intensities

$$p_{\overline{ii}}(s,t) := \mathbb{P}\left[\bigcap_{\xi\in[s,t]\cap\mathbb{Q}}\{Z_\xi = i | Z_s = i\}\right] = e^{-\sum\limits_{k\neq i}\int\limits_s^t \mu_{ik}(\tau)d\tau}.$$

Furthermore the set of differential equations can be rewritten as an equivalent set of integral equations.

Derivation 4.2 (*Probabilities are Intensity-Integrals*)

Starting from the Kolmogorov backwards equation

$$\frac{d}{ds} p_{ij}(s,t) = -\mu_{ii}(s) p_{ij}(s,t) - \sum_{k \neq i} \mu_{ik}(s) p_{kj}(s,t),$$

the first step is to multiply both sides with the integration factor $e^{-\int_s^t \mu_{ii}}$*, move the* μ_{ii} *term to the left, and apply the rule of integrating by parts reversely. This results in*

$$d_s \left(e^{-\int_s^t \mu_{ii}} p_{ij}(s,t) \right) = -\sum_{k \neq i} e^{-\int_s^t \mu_{ii}} \mu_{ik}(s) p_{kj}(s,t) ds.$$

The next step is to integrate both sides over $(s,t]$

$$\mathbb{1}_{i=j} - e^{-\int_s^t \mu_{ii}} p_{ij}(s,t) = -\int_s^t \sum_{k \neq i} e^{-\int_\tau^t \mu_{ii}} \mu_{ik}(\tau) p_{kj}(\tau,t) d\tau.$$

Removing the integration factor $e^{\int_s^t \mu_{ii}}$ *and rearranging terms results in*

$$p_{ij}(s,t) = \mathbb{1}_{i=j} e^{\int_s^t \mu_{ii}} + \int_s^t \sum_{k \neq i} e^{\int_s^\tau \mu_{ii}} \mu_{ik}(\tau) p_{kj}(\tau,t) d\tau.$$

Using the residence probabilities completes the derivation of the backward case. An analogous procedure applied to the forward Kolmogorov equation results in the respective forward integral equation.

Theorem 4.1 (*Probabilities are Intensity-Integrals*)

- *The forward case*

$$p_{ij}(s,t) = \mathbb{1}_{i=j} p_{\overline{ii}}(s,t) + \int_s^t \sum_{k \neq j} p_{ik}(s,\tau) \mu_{kj}(\tau) p_{\overline{jj}}(\tau,t) d\tau$$

- *The backward case*

$$p_{ij}(s,t) = \mathbb{1}_{i=j} p_{\overline{ii}}(s,t) + \int_s^t \sum_{k \neq i} p_{\overline{ii}}(s,\tau) \mu_{ik}(\tau) p_{kj}(\tau,t) d\tau.$$

Even though this is not the way to actually calculate the probabilities (at least not if one of the rates is stochastic), this result is still important because it establishes a very comprehensible connection between intensities and probabilities.

4.4. Affine Processes

Sources: time-homogeneous: [Fil09]
inhomogeneous: [Buc11; Buc12; Buc13]

Consider the class of d-dimensional, adapted, processes $\boldsymbol{X}\colon \Omega \times \mathbb{R}_+ \to \mathcal{X}$,

$$dX(t) := \delta(t, X(t))dt + \rho(t, X(t))dW(t), \qquad X(0) := x \in \mathcal{X} \tag{4.8}$$

on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in\mathbb{R}}, \mathbb{P})$ which,

for measurable functions

$$\delta : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}, \quad \rho : \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}^{d\times d}, \tag{4.10}$$

where the drift vector δ and the diffusion matrix $\rho\rho^\intercal$ are locally Lipschitz continuous,

is driven by the filtration generating d-dimensional Wiener process $\mathbf{W}$ and mapping into the canonical state space

$$\mathcal{X} := \mathbb{R}_+^m \times \mathbb{R}^n, \quad d = m+n, \quad I := \{1, ..., m\}, \quad J := \{m+1, ..., d\}. \tag{4.11}$$

Moreover assume that $\boldsymbol{X}$ exists for all start values $x \in \mathcal{X}$ at any time $t \geq 0$.

Definition 4.5 (*Affine process*)

The process $\boldsymbol{X}$, introduced above, is called affine if there exist $\mathbb{C}$- and $\mathbb{C}^d$-valued functions ϕ and ψ with jointly continuous derivatives allowing for an exponentially affine representation of the $\mathcal{F}_t$-conditional characteristic function of $X(T)$ i.e.

$$\mathbb{E}\left[e^{z^\intercal \boldsymbol{X}(T)} \,\middle|\, \mathcal{F}(t)\right] = e^{\phi(t,T,z)+\psi(t,T,z)^\intercal \boldsymbol{X}(t)} \tag{4.12}$$

for all $x \in \mathcal{X}$, $0 \leq t \leq T$, and $z \in i\mathbb{R}^d$.

As shown in [Fil09, p. 144] for the time-homogeneous case and in [Buc11] for the time-inhomogeneous case, affine drift and diffusion parameter functions of the form

$$\begin{aligned} \delta(t,x) &= \delta_0(t) + \sum_{i=1}^{d} \delta_i(t)x_i = \delta(t) + \mathfrak{D}(t)x, \\ \rho(t,x)\rho_0(t,x)^\intercal &= \rho(t) + \sum_{i=1}^{d} \rho_i(t)x_i, \end{aligned} \tag{4.13}$$

where the vector functions $\delta_0(t)$ and $\delta_i(t)\colon \mathbb{R}_+ \to \mathbb{R}^d$ and the matrix functions $\rho_0(t)$ and $\rho_i(t)\colon \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ are all assumed to be bounded and piecewise continuous, are a necessary condition for $\boldsymbol{X}$ to be affine.

Furthermore, sufficient conditions can be given. For time-inhomogeneous affine processes this is proven in [Buc11, sec. 2.2]. A comprehensible illustration is found in [Buc12, p. 6].

Theorem 4.2 (*Admissibility Conditions*)

The stochastic process $\boldsymbol{X}$, as introduced in 4.8 is affine iff $\delta(t,x)$ and $\rho(t,x)$ are affine of the form 4.13 with admissible parameters in the following sense:

$$\begin{gathered} \delta_0(t) \in \mathbb{R}^m_+ \times \mathbb{R}^n, \qquad [\delta_i(t)]_k \geq 0, \quad {\scriptstyle i \in I, \quad k \in I \setminus \{i\}}, \qquad [\mathfrak{D}(t)]_{IJ} = 0, \\ \rho_0(t), \rho_i(t) \textit{ are symmetric and positive semi-definite}, {\scriptstyle i \in I}, \end{gathered} \tag{4.14}$$

$$\begin{aligned} [\rho_0(t)]_{II} &= 0, & [\rho_0(t)]_{IJ} &= 0, \\ \rho_j(t) &= 0, \quad {\scriptstyle j \in J}, & [\rho_i(t)]_{k\cdot} &= 0, \quad {\scriptstyle i \in I, \quad k \in I \setminus \{i\}}, \end{aligned} \tag{4.15}$$

for all $t \in \mathbb{R}_+$.

Theorem 4.2 thus provides not only sufficient conditions, but an explicit construction for affine processes. The following theorem 4.3 ensures that the characteristic function of an affine process is always, at least numerically, manageable.

Theorem 4.3 (*Appendant Riccati Equations*)

If the conditions of theorem 4.2 are met, the functions $\phi(t,T,z)$ and $\psi(t,T,z)$ from equation 4.12 solve the Riccati equations,

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,T,z) &= -\frac{1}{2}[\psi(t,T,z)]_J^\intercal [\rho_0(t)]_{JJ} [\psi(t,T,z)]_J - \delta_0(t)^\intercal \psi(t,T,z), \\ \phi(T,T,z) &= 0, \\ \frac{\partial}{\partial t}[\psi(t,T,z)]_i &= -\frac{1}{2}\psi(t,T,z)^\intercal [\rho_i(t)] \psi(t,T,z) - \delta_i(t)^\intercal \psi(t,T,z), {\scriptstyle i \in I}, \\ \frac{\partial}{\partial t}[\psi(t,T,z)]_J &= -[\mathfrak{D}(t)]_{JJ}^\intercal [\psi(t,T,z)]_J, \\ \psi(T,T,z) &= z, \end{aligned} \tag{4.16}$$

and there exists a unique solution $t \mapsto (\phi(t,T,z), \psi(t,T,z)) : \mathbb{R}_+ \to \mathbb{C}_- \times \mathbb{C}^m_- \times i\mathbb{R}^n$ for all $z \in \mathbb{C}^m_- \times i\mathbb{R}^n$ and $T > 0$.

Affine processes allow for admissible affine transformations without loosing the affinity property (see e.g. [Buc12]).

Lemma 4.3 (*Affine transformation*)

For $p \geq 1$, a component-by-component integrable vector function $c : \mathbb{R}_+ \to \mathbb{R}^p$, and an everywhere-injective matrix function $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{p \times d}$, where every component of the latter two is bounded, with limits everywhere, and discontinuous in at most finitely many points, the affine transformation

$$Y(t) := c(t) + \Gamma(t)X(t) \tag{4.17}$$

passes the affinity property of $\mathbf{X}$ on to $\mathbf{Y}$.

Notation 4.1 (*Column sums*)

Let the column sums of Γ be denoted by the vector function $\gamma(t) = \mathbb{1}^\intercal \Gamma(t)$.

The definition and the characterization of affine processes required the use of the characteristic function which means that z was restricted to imaginary values in the affine transformation formula 4.12. However, when pricing becomes an issue, one has to deal with real z as well, leading to the moment generating function which does not always exist.

Theorem 4.4 (*Existence of the Moment Generating Function*)

If there exist a solution to the Riccati equations 4.16 for an affine process $\mathbf{X}$, $u \in \mathbb{R}^d$, and $T > 0$ in $u \in [0, T]$ and either of the following holds,

1. $e^{\phi(t,T,u)+\psi(t,T,u)^\intercal \mathbf{X}(\mathbf{t})}$ *is uniformly bounded on $[0, T]$,*
2. $\mathbb{E}\left[\int_0^T e^{2\phi(t,T,u)+2\psi(t,T,u)^\intercal \mathbf{X}(t)} (\psi(t,T,u)^\intercal \rho(t, \mathbf{X}(t)))^2 dt\right] < \infty$,

then the affine transformation formula 4.12 holds for u, for all $t \in [0, T]$.

Theorem 4.5 (*Basic Pricing Theorem*)

Let $\mathbf{X}, \mathbf{Y}$ *be affine processes in the canonical state space as defined above and assume that either of the following is the case*

1. $n = 0$ *and* $\gamma_i \geq 0$ *for* $i = 1, ..., d$,
2. *there exists a combination of functions* (ϕ, ψ) *solving*

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,T) &= -\frac{1}{2}[\psi(t,T)]_J^\intercal [\rho_0(t)]_{JJ}[\psi(t,T)]_J - \delta_0(t)^\intercal \psi(t,T) + \mathbb{1}^\intercal c(t), \\
\phi(T,T) &= 0, \\
\frac{\partial}{\partial t}[\psi(t,T)]_i &= -\frac{1}{2}\psi(t,T)^\intercal \rho_i(t)\psi(t,T) - \delta_i(t)^\intercal \psi(t,T) + [\gamma(t)]_i, {}_{i \in I}, \\
\frac{\partial}{\partial t}[\psi(t,T)]_J &= -[\mathfrak{D}(t)]_{JJ}^\intercal [\psi(t,T)]_J + [\gamma(t)]_J, \\
\psi(T,T) &= 0,
\end{aligned} \tag{4.18}$$

in a way that $t \mapsto e^{-\int_0^t \mathbb{1}^\intercal Y(s)ds + \phi(t,T) + \psi(t,T)^\intercal X(t)}$ *is a martingale.*

Then

$$\mathbb{E}\left[e^{\int_t^T \mathbb{1}^\intercal Y(s)ds} \middle| \mathcal{F}(t) \right] = e^{\phi(t,T) + \psi(t,T)^\intercal X(t)} \tag{4.19}$$

where ϕ, ψ *are given by the equations 4.18.*

As already pointed out by Buchardt the simple choice of

$$X(t) = (r(t), \mu(t)), c \equiv 0 \text{ and } \Gamma \in \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right\} \tag{4.20}$$

under a risk-neutral measure leads to bond prices, survival probabilities and the price of a pure endowment contract when inserted in equation 4.19. The two rates can therein be dependent. If they are not, the discounting and the surviving part can be separated i.e.

$$\phi(t,T) = \phi^r(t,T) + \phi^\mu(t,T), \tag{4.21}$$

$$\psi(t,T) = \psi^r(t,T) + \psi^\mu(t,T). \tag{4.22}$$

If the policy includes a benefit upon death the basic pricing formula is not sufficient. One is then encountering the expected value of the partial derivative of the term structure which can be handled by solving an additional system of differential equations.

Theorem 4.6 (*Extended Pricing Theorem*)

For $k \in \{1,..,p\}$, $u \in [t,T]$, under the conditions of Theorem 4.5, and the additional condition of

$$t \mapsto e^{-\int_t^T \mathbb{1}^\intercal Y(s)ds + \phi(t,T) + \psi(t,T)X(t)} \left(A_k(t,T,u) + B_k(t,T,u)^\intercal X(t)\right)$$

being a martingale, it holds that

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^T \mathbb{1}^\intercal Y(s)ds}[Y(u)]_k \middle| \mathcal{F}(t)\right]\\ =&e^{\phi(t,T)+\psi(t,T)^\intercal X(t)} \left(A_k(t,T,u) + B_k(t,T,u)^\intercal X(t)\right)\end{aligned} \tag{4.23}$$

where (ϕ,ψ) solves the equations 4.18 and (A_k, B_k) solves the following system of linear differential equations

$$\begin{aligned}\frac{\partial}{\partial t}A_k(t,T,u) &= -[\psi(t,T)]_J^\intercal [\rho_0(t)]_{JJ}[B_k(t,T,u)]_J - \delta_0(t)^\intercal B_k(t,T,u),\\ A_k(u,T,u) &= [c(u)]_k,\\ \frac{\partial}{\partial t}[B_k(t,T,u)]_i &= -\psi(t,T)^\intercal[\rho_i(t)]B_k(t,T,u) - \delta_i(t)^\intercal B_k(t,T,u),_{i\in I},\\ \frac{\partial}{\partial t}[B_k(t,T,u)]_J &= -[\mathfrak{D}(t)]_{JJ}^\intercal [B_k(t,T,u)]_J,\\ B_k(T,T) &= e_k^\intercal \Gamma(u).\end{aligned} \tag{4.24}$$

In combination with the linearity of the expectation the following result follows immediately

$$\begin{aligned}\mathbb{E}\left[e^{-\int_t^T \mathbb{1}^\intercal Y(s)ds}([Y(u)]_k + [Y(v)]_l)\middle|\mathcal{F}(t)\right] =& \mathbb{E}\left[e^{-\int_t^T \mathbb{1}^\intercal Y(s)ds}[Y(u)]_k\middle|\mathcal{F}(t)\right]\\ &+\mathbb{E}\left[e^{-\int_t^T \mathbb{1}^\intercal Y(s)ds}[Y(v)]_l\middle|\mathcal{F}(t)\right].\end{aligned} \tag{4.25}$$

As soon as invalidity becomes an issue one is likely to stumble upon the second derivative as well. As the number of differential equation-systems increases exponentially with the order of differentiation we note that these terms are still manageable but refer the interested reader to [Buc12].

4.5. Examples of affine processes

Source: [Buc11]

Let us start by considering a very simple affine processes for the mortality intensity of somebody being x-years old. Probably the simplest thing to do is adding an F^Y-adapted Wiener process $(W(t))_{t\in\mathbb{R}_+}$ to the deterministic intensity $\mu^\circ(t)$ i.e.

$$\mu_x(t) = \mu_x^\circ(t) + \sigma W(t). \tag{4.26}$$

For $t \leq u \leq n$ and $(x+t) \leq (x+n)$ and conditional upon $F^Y(u)$ this results in the $(n-t)$-year survival probability of an $(x+t)$-year old of

$$\begin{aligned}
{}_{n-t}p_{x+t} &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{T_x\geq n\}}\middle|T_x \geq t, F^Y(\infty)\right]\middle|T_x \geq t, F^Y(u)\right] \\
&= \mathbb{E}\left[e^{-\int_t^n \mu_x(s)ds}\middle|F^Y(u)\right] \\
&= e^{-\int_t^n \mu_x^\circ(s)ds - \int_t^u \sigma W(s)ds}\,\mathbb{E}\left[e^{-\int_u^n \sigma W(s)ds}\middle|F^Y(u)\right] \\
&= e^{-\int_t^n \mu_x^\circ(s)ds - \int_t^u \sigma W(s)ds - (n-u)\sigma W(u) + \frac{\sigma^2(n-u)^3}{6}}
\end{aligned} \tag{4.27}$$

4.5.1. Vasiček Model

Source: [MB00, p. 58f]

Let us now consider the standard Vasiček model for the short rate which uses an Ornstein-Uhlenbeck process with constant, positive coefficients i.e.

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \qquad \text{for } k, \theta, \sigma \geq 0. \tag{4.28}$$

Due to its mean reversion term $(\theta - r_t)$ its paths do not stray as much and stay closer to the expectation. Once again, however, the diffusion coefficient does not depend on r_t and so the volatility cannot diminish as r_t goes to zero. Thus, the process allows for negative values with positive probability. By integrating equation (4.28), the intensity is explicitly found for each $s \leq t$ to be

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)} dW(u). \tag{4.29}$$

Recognizing $r_t|\mathcal{F}_s$ to be Gaussian, the first moments are easily derived as

$$\begin{aligned}\mathbb{E}[r_t|\mathcal{F}_s] &= r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right), \\ Var[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right).\end{aligned} \tag{4.30}$$

The bond price is given by

$$\begin{aligned}P(t,T) &= e^{A(t,T)-B(t,T)r(t)}, \\ A(t,T) &= \left(\theta - \frac{\sigma^2}{2k^2}\right)(B(t,T) - (T-t)) - \frac{\sigma^2}{4k}B(t,T)^2, \\ B(t,T) &= \frac{1}{k}\left(1 - e^{-k(T-t)}\right).\end{aligned} \tag{4.31}$$

4.5.2. Hull-White Model

Sources: [Boh11, p. 26ff], [YYT08]

Considering the following extension of the Vasiček model where θ is replaced by a time-inhomogeneous mean reversion parameter θ_t, results in the well known Hull-White model which thereby allows for an incorporation of the initial forward rate. The differential equation now reads

$$dr_t = k(\theta_t - r_t)dt + \sigma dW_t \tag{4.32}$$

and has the following solution

$$r(t) = r(s)e^{-k(t-s)} + k\int_s^t \theta(u)e^{-k(t-u)}du + \sigma\int_s^t e^{-k(t-u)}dW(u). \tag{4.33}$$

The first moments are

$$\begin{aligned}\mathbb{E}\left[r(t)|F(s)\right] &= r(s)e^{-k(t-s)} + k\int_s^t \theta(u)e^{-k(t-u)}du, \\ \mathbb{V}\left[r(t)|F(s)\right] &= \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right).\end{aligned} \tag{4.34}$$

The bond prices are now given by

$$\begin{aligned} P(t,T) &= \bar{A}(t,T)e^{-r(t)B(t,T)}, \\ \bar{A}(t,T) &= e^{-k\int_t^T \theta(u)B(u,T)du} A(t,T) \end{aligned} \tag{4.35}$$

with $A(t,T)$ and $B(t,T)$ as in the Vasiček model with $\theta = 0$. In order to calibrate θ_t to the current interest rate it needs to chosen as

$$\theta(t) = f^M(0,t) + \frac{1}{\lambda}\frac{\partial f^M}{\partial T}(0,t) + \frac{\sigma^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right). \tag{4.36}$$

where f^M denotes the given interest rate from the market. In doing so the bond prices from the model satisfy

$$P(t,T) = e^{-r(t)B(t,T)}\frac{P^M(0,T)}{P^M(0,t)}e^{B(t,T)f^M(0,t)-\frac{\sigma^2}{4k}(1-e^{-2kt})B^2(t,T)} \tag{4.37}$$

where P^M denote currently observable bond prices from the market.

4.6. Categorization

Sources: [DS00], [MB00, p. 53ff]

First of all, affine models are usually classified by their number of random sources which are called factors. One-factor models are further separated into Gaussian and square root processes while multi-Factor models can contain both.

Another way to classify affine models is to separate endogenous models from exogenous ones. If the bond prices $P(0,T)$ are rather output than input because the model is specified internally by a small set of parameters, the model is said to be an endogenous model. These models are nice ad hoc solutions because they often allow for closed form solutions, even for many derivatives. On the other hand they lack the necessary flexibility when it comes to calibrating the model to the initial zero-coupon curve $T \to P^M(0,T)$ from the market so that not all curves are attainable. The problem can be tackled by using exogenous models with time-inhomogeneous parameters along with numerical methods.

4.6.1. Endogenous One-Factor Models

X is said to be a Gaussian affine process or an Ornstein-Uhlenbeck process if it satisfies the SDE

$$dX(t) = (\alpha + \beta X(t))dt + \Sigma dW_t^*$$

for $\alpha \in \mathbb{R}^n$, $\beta, \Sigma \in \mathbb{R}^{n \times n}$ and an n-dimensional Wiener process. In this case the diffusion coefficient does not contain X which is why the volatility does not diminish if the process tends toward zero and so the process can become negative as well. This problem has been addressed by Cox, Ingersoll and Ross (CIR). Within a CIR model the process is assumed to satisfy the dynamics

$$dX(t) = (\alpha + \beta X(t))dt + \Sigma\sqrt{X(t)}dW_t^*$$

where $2\alpha\beta \geq \sigma^2$ needs to be chosen to guarantee a strictly positive rate and the components of dW^* need to be independent to not get in trouble with the affinity property. Famous endogenous one-factor short rate models are e.g.

$$\begin{array}{cl} \text{Merton 1973} & dr_t = a dt + \sigma dW_t^*. \\[1em] \text{Vasiček 1977} & dr_t = (\theta - \alpha r_t)dt + \sigma dW_t \\ \iff & dr_t = a(b - r_t)dt + \sigma dW_t. \\[1em] \text{Rendleman-Bartter 1980} & dr_t = \theta r_t\, dt + \sigma r_t\, dW_t. \\[1em] \text{Cox-Ingersoll-Ross (CIR) 1985} & dr_t = (\theta - \alpha r_t)\, dt + \sqrt{r_t}\,\sigma\, dW_t \\ \iff & dr_t = a(b - r_t)\, dt + \sqrt{r_t}\,\sigma\, dW_t. \end{array}$$

4.6.2. Exogenous One-Factor Models

By allowing the parameters to be functions of time the necessary modeling freedom can be bought at the cost of adding a potentially infinite number of new parameters to the model. Famous examples are

$$\begin{array}{cl} \text{Ho-Lee 1986} & dr_t = \theta_t\, dt + \sigma\, dW_t. \\ \text{Hull-White 1990} & dr_t = (\theta_t - \alpha r_t)\, dt + \sigma_t\, dW_t. \end{array}$$

The Hull-White model is often called 'extended Vasiček model' and it is not always used with a time-inhomogeneous diffusion coefficient (DC). This is because an exact calibration to market bond prices is desirable, while an exact calibration to the volatility term structure can be dangerous - especially in less liquid markets. Furthermore the implied future volatility structures of the model deviate from typical market shapes.

4.6.3. Multi-Factor Models

The most famous multi-factor models are three factor models which consist of three processes. In [Dai98], Qiang Dai categorizes them in four families according to the number of factors appearing in the DC. The first and the last families are special cases. The first family $\mathbb{A}_0(3)$ results in homoskedastic state variables for none of the factors appear in the DC and the last family $\mathbb{A}_3(3)$ consists of correlated square-root diffusion models, so no factor can be modeled as a Gaussian process anymore. The second family $\mathbb{A}_1(3)$ is a heteroskedastic set of models, where one of the factors affects the DC. A famous example is the BDFS-model where

$$\begin{aligned}
dr_t &= \kappa(\theta_t - r_t)dt &&+ \sqrt{v_t}dW_t^r && \text{is the short rate process,}\\
d\theta_t &= \alpha(\bar{\theta} - \theta_t)dt &&+ \zeta dW_t^\theta && \text{is the drift process, and}\\
dv_t &= \mu(\bar{v} - v_t)dt &&+ \eta\sqrt{v_t}dW_t^v && \text{is the volatility process.}
\end{aligned} \tag{4.38}$$

For details see e.g. [Bal+96]. The class contains furthermore all models which allow for the following representation

$$\begin{aligned}
dr(t) &= \kappa_{vr}(\bar{v} - v(t))dt + \kappa(\theta(t) - r(t))dt &&+ \sqrt{\alpha_r + v(t)}dW^r(t),\\
d\theta(t) &= \nu(\bar{\theta} - \theta(t))dt &&+ \sqrt{\zeta^2 + \beta_\theta v(t)}dW^\theta(t),\\
dv(t) &= \mu(\bar{v} - v(t))dt &&+ \eta\sqrt{v(t)}dW^v(t).
\end{aligned} \tag{4.39}$$

In the $\mathbb{A}_2(3)$ family there may be feedback between θ and v and both may affect the drift and the DC of the short rate:

$$
\begin{aligned}
dr(t) &= \kappa_{rv}(\bar{v} - v(t))dt \ + \kappa(\bar{r} + \theta(t) - r(t))dt \\
&\quad + \sigma_{rv}\eta\sqrt{v(t)}dW_1(t) + \sigma_{r\tau}\zeta\sqrt{\theta(t)}dW_2(t) + \sqrt{\alpha_r + \beta_\theta\theta(t)v(t)}dW_3(t), \\
d\theta(t) &= \nu(\bar{\theta} - \theta(t))dt + \kappa_{\theta v}(\bar{v} - v(t))dt + \zeta\sqrt{\theta(t)}dW_2(t), \\
dv(t) &= \mu(\bar{v} - v(t))dt + \kappa_{v\theta}(\bar{\theta} - \theta(t))dt + \eta\sqrt{v(t)}dW_1(t).
\end{aligned}
\tag{4.40}
$$

Famous members of the $\mathbb{A}_2(3)$ family are

Longstaff-Schwartz 1992	$dr_t = (\mu X + \theta Y)dt + \sigma_t\sqrt{Y}dW_{3,t}$, $dX_t = (a_t - bX_t)\,dt + \sqrt{X_t}\,c_t\,dW_{1,t}$, $dY_t = (d_t - eY_t)\,dt + \sqrt{Y_t}\,f_t\,dW_{2,t}$.
Chen model 1996	$dr_t = (\theta_t - \alpha_t)\,dt + \sqrt{r_t}\,\sigma_t\,dW_{3,t}$, $d\alpha_t = (\zeta_t - \alpha_t)\,dt + \sqrt{\alpha_t}\,\sigma_t\,dW_{1,t}$, $d\sigma_t = (\beta_t - \sigma_t)\,dt + \sqrt{\sigma_t}\,\eta_t\,dW_{2,t}$.

4.6.4. Criticism

Affine models are a quite tractable way to model the stochastic environment but endogenous models clearly lack the possibility for a proper calibration. Their simple structure makes due with a very small set of parameters by which it stints itself to an extent where the freedom for calibration is not sufficient. In [MB00, p. XX] the use of time-inhomogeneous diffusion coefficients is promoted and corroborated by the argument that the calibration to bond prices from the market can be made perfect, and the remaining model parameters can be used to calibrate the volatility structures if it is appropriate. However, the evolution of the term structure in one-factor models is still subject to only a single stochastic factor, namely the short-rate. This can be a dangerous assumption. In risk management multi-factor models are therefore often preferred for the resulting scenarios are said to be more consistent with the observed market behavior. Multi-factor models are nevertheless not the answer to everything. Even though Dai establishes a connection between the three factor-models and the factor loadings from a principal component analysis in [Dai98] which eases some interpretations, they continue to be way more complex and less intuitive than their predecessors and they are still not able to reflect every desirable behavior.

4.7. Buchardt's Life Model

Sources: [Buc13]

In his paper, Kristian Buchardt takes a look at a mixed life insurance contract similar to the one treated in (section 3.3 - The Market Value Model on page 31) but with continuous premium payments and with immediate payments upon the event of death and surrender. In figure (4.7) the respective tree structure is illustrated together with the tree structure of his invalidity model for which we refer the interested reader to his paper.

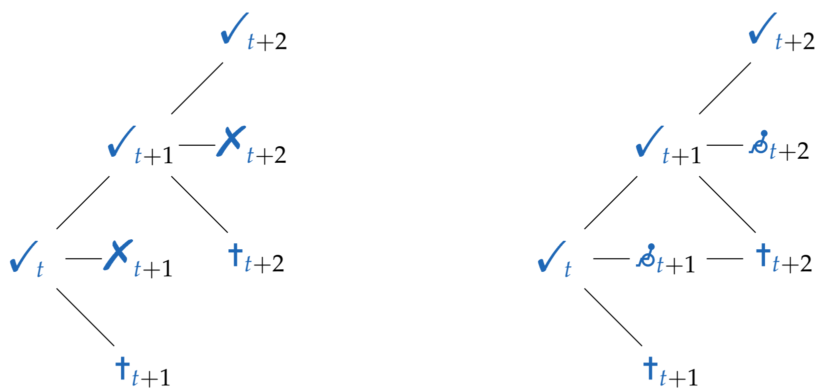


Figure 4.3.: Buchardt's mixed life insurance and recombining invalidity model.

Denoting the sum of all past and present payments by $B(t)$ the payments at time t can be written as

$$dB(t) = \underbrace{b_e(t)\mathbb{1}_{Z(t)=0}\,dt}_{\text{premiums}} + \underbrace{b_d(t)\,dN_{01}(t)}_{\text{death benefit}} + \underbrace{U(t)\,dN_{02}(t)}_{\text{surrender benefit}}$$

The mortality rate $\mu(t)$, is assumed to be continuous but still deterministic. Now the main difference is made by using a stochastic force of interest $r(t)$ and a stochastic surrender intensity $\eta(t)$ which offers the possibility to model a dependency between these two. The two stochastic intensities are modeled as components of the $p = 2$-dimensional process Y which is an affine transformation (as in equation 4.17) of the $d=2$-dimensional process X satisfying

$$Y(t) := \begin{cases} r(t) = X_1(t) \\ \eta(t) = \eta^0(t)X_2(t) \end{cases} \quad \Rightarrow \quad c(t) = 0, \quad \Gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & \eta^0(t) \end{pmatrix}$$

$$dX(t) := \begin{cases} (b_1(t) - \beta_1 X_1(t))\, dt + \sigma_1 \left(\sqrt{1-\rho^2} dW_1(t) + \rho \sqrt{X_2(t)} dW_2(t) \right) \\ (b_2 - \beta_2 X_2(t))\, dt + \sigma_2 \sqrt{X_2(t)} dW_2(t) \end{cases}.$$

This means the interest rate is assumed to behave like a mix of a Hull-White extension of a Vasicek process and a Heston process and it can thus become negative as well. The surrender rate is modeled by relative deviations from an estimated surrender rate $\eta^0(t)$ where the deviations are modeled as a CIR-process ensuring non-negativity. The state space $\mathcal{X}$ is thus $\mathbb{R} \times \mathbb{R}_+$ which means $d = m + n = 1 + 1$, $I = \{1\}$, $J = \{2\}$, the drift is given by

$$\delta_0(t) = \begin{pmatrix} b_1(t) \\ b_2 \end{pmatrix} \quad \delta_1(t) = \begin{pmatrix} -\beta_1 \\ 0 \end{pmatrix} \quad \delta_2(t) = \begin{pmatrix} 0 \\ -\beta_2 \end{pmatrix},$$

and the diffusion coefficient satisfies

$$\rho(t, X(t)) = \begin{pmatrix} \sigma_1 \sqrt{1-\rho^2} & \sigma_1 \rho \sqrt{X_2(t)} \\ 0 & \sigma_2 \sqrt{X_2(t)} \end{pmatrix}.$$

Taking a look at

$$\rho(t, X(t)) \rho(t, X(t))^\intercal = \begin{pmatrix} \sigma_1^2 (1-\rho^2) + \sigma_1^2 \rho^2 X_2(t) & \sigma_1 \sigma_2 \rho X_2(t) \\ \sigma_1 \sigma_2 \rho X_2(t) & \sigma_2^2 X_2(t) \end{pmatrix}$$

reveals the models affinity in X. Splitting it up into a linear combination of its X components finally unsheathes the diffusion parameters

$$\rho_0(t) = \begin{pmatrix} \sigma_1^2 (1-\rho^2) & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_2(t) = \begin{pmatrix} \sigma_1^2 \rho^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{pmatrix},$$

requested by the differential equations 4.19 and 4.23.

In his setup the reserve is simply called the present value. It can now be written as

$$PV(t) = \int_t^T e^{-\int_t^s r(\tau) d\tau} dB(s).$$

Since the interest rate and the surrender rate are now stochastic, the present value is a random variable. The expected present value is calculated by

$$
\begin{aligned}
&EPV(t) = \mathbb{E}\left[PV(t)|\mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[PV(t)\middle|\mathcal{F}^X(\infty), Z(t)=0\right]\middle|\mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^T e^{-\int_t^s r(\tau)d\tau} dB(s)\middle|\mathcal{F}^X(\infty), Z(t)=0\right]\middle|\mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^T e^{-\int_t^s r(\tau)d\tau}(\mathbb{1}_{Z(s)=0} b_e(s)\,ds + b_d(s)\,dN_{01}(s) + U(t)\,dN_{02}(s))\middle|...\right]\middle|...\right] \\
&= \mathbb{E}\left[\int_t^T e^{-\int_t^s r(\tau)+\mu(\tau)+\eta(\tau)d\tau}(b_e(s) + \mu(s)b_d(s) + \eta(s)U(s))ds\middle|\mathcal{F}(t)\right] \\
&= \int_t^T e^{-\int_t^s \mu(\tau)d\tau}\Big(\ \mathbb{E}\left[e^{-\int_t^s r(\tau)+\eta(\tau)d\tau}\middle|\mathcal{F}(t)\right](b_e(s) + \mu(s)b_d(s)) \\
&\qquad\qquad + \mathbb{E}\left[e^{-\int_t^s r(\tau)+\eta(\tau)d\tau}\eta(s)\middle|\mathcal{F}(t)\right]U(s)\qquad\qquad\Big)\,ds.
\end{aligned}
$$

Using the theorems 4.5 and 4.6 the expected values can be calculated (at least numerically) by solving two appendant systems of ODEs (for every supporting point of the numerical integral) resulting in

$$
\begin{aligned}
EPV(t) = \int_t^T & e^{-\int_t^s \mu(\tau)d\tau + \phi(t,s) + \psi(t,s)^\intercal X(t)} \\
& (\ b_e(s) + \mu(s)b_d(s) + (A^\eta(t,s,s) + B^\eta(t,s,s)^\intercal X(t))U(s)\ )\,ds.
\end{aligned}
\tag{4.41}
$$

Unfortunately the calculation of the profit reserve in the previously introduced setup is forlorn hope due to the path dependance. Even for a simple pure endowment policy, it makes one deal with the term

$$
\begin{aligned}
\mathbb{E}\left[e^{-\int_t^n r(\tau)+\mu(\tau)d\tau}(b_{\checkmark}(n) + b_{\$}(n))\middle|\mathcal{F}(t)\right] \\
db_{\$}(t) = \max\{r(t), \bar{r}\}b_{\$}(t)dt
\end{aligned}
\tag{4.42}
$$

which can not be solved explicitly. The use of a simulation technique seems to be inevitable at this point.

5. Results

Let us consider the two widely used mixed life insurance **tariffs** from section (3.1). This means that the risks insured are death as well as endowment. Both contracts shall pay the nominal amount in case of endowment at the end of coverage. While the first tariff shall pay the nominal amount also in case of death, the second policy shall only pay $1/m$ of the nominal amount, where m denotes the number of premiums payed up to the year of death. The guaranteed interest rate shall be 2% and the guaranteed mortality table shall be as denoted in column $q^*(x)$ in table (5.1). After the first year, the insured shall be granted the right to surrender at 95% of the previous statutory book value reserve and should participate in 85% of the insurers profits.

We assume that a 40 year old person has concluded a **policy** with a nominal amount of 20.000 in each of them and that both policies have a covered period of 15 years. The premiums are agreed to be payed annually in advance throughout the covered period while the benefits should be payed in arrears. We summarized the data as the green values in table (5.2) and (5.3), respectively.

Table (5.1) shows the development of the mathematical statutory book value reserve at the main renewal dates of the policies and the derivation according to the algorithm from section (3.1). This means that surrender and waiver are not considered here.

As can be seen from column 'E' and 'A', the developments of the present values for endowment are the same while the present values for death are lower for the second policy. For the annuity 'ä' only depends upon the guaranteed interest rate and the mortality table it is also the same. The only difference in the premium is thus a lower risk premium within the second policy. Because both policies save up to the same nominal amount, their development is very similar non the less. The difference is caused by the time-inhomogeneous mortality table in combination with a constant premium which implicitly builds and reverses a risk reserve.

Statutory Book Value Model

Commutation Table

x	q*(x)	v(x)	l(x)	D(x)	N(x)	d(x)	C(x)	M(x)	R(x)
40	0,08039%	45,28904%	98.308	44.523	1.258.455	79	36	18.627	712.506
41	0,09094%	44,40102%	98.218	43.610	1.213.932	89	40	18.587	693.879
42	0,09371%	43,53041%	98.126	42.715	1.170.322	92	40	18.547	675.292
43	0,11221%	42,67688%	98.016	41.830	1.127.608	110	47	18.500	656.745
44	0,12920%	41,84007%	97.889	40.957	1.085.777	127	53	18.447	638.245
45	0,13043%	41,01968%	97.762	40.102	1.044.820	128	52	18.395	619.798
46	0,16538%	40,21537%	97.600	39.250	1.004.719	162	65	18.330	601.403
47	0,16333%	39,42684%	97.441	38.418	965.469	159	63	18.267	583.073
48	0,19578%	38,65376%	97.250	37.591	927.051	191	74	18.193	564.806
49	0,20333%	37,89584%	97.052	36.779	889.460	198	75	18.118	546.613
50	0,26044%	37,15279%	96.799	35.964	852.681	253	94	18.024	528.495
51	0,29216%	36,42430%	96.517	35.155	816.718	283	103	17.921	510.471
52	0,29873%	35,71010%	96.228	34.363	781.562	288	103	17.818	492.549
53	0,29453%	35,00990%	95.945	33.590	747.199	283	99	17.719	474.731
54	0,40430%	34,32343%	95.557	32.798	713.609	388	133	17.586	457.012
55	0,41490%	33,65042%	95.160	32.022	680.810	396	133	17.453	439.426

Tariff 1							Tariff 2					
E	A	AE	ä	P	Res		E	A	AE	ä	P	Res
14.384,57	527,51	14.912,08	12,97	1.149,37	0,00	2010	14.384,57	338,16	14.722,73	12,97	1.134,77	0,00
14.685,61	520,35	15.205,96	12,22		1.155,21	2011	14.685,61	344,03	15.029,64	12,22		1.157,31
14.993,38	512,50	15.505,87	11,46		2.334,12	2012	14.993,38	348,73	15.342,11	11,46		2.337,61
15.310,42	500,87	15.811,29	10,68		3.534,68	2013	15.310,42	351,61	15.662,04	10,68		3.541,31
15.636,83	485,67	16.122,51	9,89		4.758,03	2014	15.636,83	352,21	15.989,05	9,89		4.768,88
15.970,40	469,91	16.440,31	9,08		6.007,29	2015	15.970,40	351,02	16.321,42	9,08		6.020,88
16.316,80	**446,97**	**16.763,77**	**8,25**		**7.278,75**	**2016**	**16.316,80**	**345,38**	**16.662,17**	**8,25**		**7.297,60**
16.670,36	423,94	17.094,30	7,41		8.578,02	2017	16.670,36	337,59	17.007,95	7,41		8.599,82
17.037,12	394,03	17.431,15	6,55		9.902,17	2018	17.037,12	324,10	17.361,22	6,55		9.927,83
17.413,27	361,98	17.775,25	5,67		11.254,78	2019	17.413,27	306,80	17.720,07	5,67		11.282,40
17.807,91	317,96	18.125,88	4,78		12.633,04	2020	17.807,91	278,94	18.086,85	4,78		12.663,77
18.217,30	266,67	18.483,96	3,87		14.040,64	2021	18.217,30	242,38	18.459,67	3,87		14.072,77
18.637,32	212,89	18.850,21	2,93		15.480,31	2022	18.637,32	200,02	18.837,34	2,93		15.510,23
19.066,22	158,71	19.224,93	1,98		16.953,29	2023	19.066,22	153,42	19.219,65	1,98		16.976,85
19.526,49	81,35	19.607,84	1,00		18.458,48	2024	19.526,49	81,35	19.607,84	1,00		18.473,07
20.000,00	0,00	20.000,00	0,00		20.000,00	2025	20.000,00	0,00	20.000,00	0,00		20.000,00

Table 5.1.: Commutation table and present values according to the statutory book value model for both contracts derived by the algorithm from section (3.1).

Tariff 1: Recursive Stautory Book Value Model												
Benefits (BFT), Mortality (qx), Guaranteed Return (gR)												
Input and easily deduced values												
			BFTs				P!		Interest	Single values		
ultimo	age	t*	D	E	P-01	S-01	q*x-1	k1px	gR	term	code	value
2010	40	0	0	0	1	0	0,000%	100,00%	0,00%	Start of coverage	t0	2010
2011	41	1	20000	0	1	0	0,091%	100,00%	2,00%	Balance year	tt	2016
2012	42	2	20000	0	1	1	0,094%	99,91%	2,00%	End of coverage	tn	2025
2013	43	3	20000	0	1	1	0,112%	99,82%	2,00%			
2014	44	4	20000	0	1	1	0,129%	99,70%	2,00%	Face value	FV	20000
2015	45	5	20000	0	1	1	0,130%	99,57%	2,00%	Profit Account	PA	600
2016	46	6	20000	0	1	1	0,165%	99,44%	2,00%			
2017	47	7	20000	0	1	1	0,163%	99,28%	2,00%	Age	x	40
2018	48	8	20000	0	1	1	0,196%	99,12%	2,00%	Period of coverage	n	15
2019	49	9	20000	0	1	1	0,203%	98,92%	2,00%	Premium payment period	m	15
2020	50	10	20000	0	1	1	0,260%	98,72%	2,00%			
2021	51	11	20000	0	1	1	0,292%	98,47%	2,00%	Surrender charge	SurrPenalty	95%
2022	52	12	20000	0	1	1	0,299%	98,18%	2,00%	Sharing rule	ShareRule	85%
2023	53	13	20000	0	1	1	0,295%	97,88%	2,00%	GuaranteedReturn	gR	2,00%
2024	54	14	20000	0	1	1	0,404%	97,60%	2,00%	Profit rate	ProfitRate	3,00%
2025	55	15	20000	20000	0	1	0,415%	97,20%	2,00%	Added profit	AddedProfit	0,33%

			cCFs		cPVs		Premium determination		cPV	cNetRes
ultimo	age	t*	D	E	D	E	Annuity-terms	P	P	cNetRes
2010	40	0	0,00	0,00	527,51	14.384,57	1	-1149,37	-13.762,72	0,00
2011	41	1	18,19	0,00	520,35	14.685,61	0,979500573	-1149,37	-12.901,38	1.155,21
2012	42	2	18,74	0,00	512,50	14.993,38	0,959394815	-1149,37	-12.022,39	2.334,12
2013	43	3	22,44	0,00	500,87	15.310,42	0,93952773	-1149,37	-11.127,25	3.534,68
2014	44	4	25,84	0,00	485,67	15.636,83	0,919915559	-1149,37	-10.215,11	4.758,03
2015	45	5	26,09	0,00	469,91	15.970,40	0,900701638	-1149,37	-9.283,66	6.007,29
2016	**46**	**6**	**33,08**	**0,00**	**446,97**	**16.316,80**	**0,881580426**	**-1149,37**	**-8.335,65**	**7.278,75**
2017	47	7	32,67	0,00	423,94	16.670,36	0,862882923	-1149,37	-7.366,91	8.578,02
2018	48	8	39,16	0,00	394,03	17.037,12	0,844307397	-1149,37	-6.379,62	9.902,17
2019	49	9	40,67	0,00	361,98	17.413,27	0,826069286	-1149,37	-5.371,11	11.254,78
2020	50	10	52,09	0,00	317,96	17.807,91	0,807762585	-1149,37	-4.343,47	12.633,04
2021	51	11	58,43	0,00	266,67	18.217,30	0,789610428	-1149,37	-3.293,96	14.040,64
2022	52	12	59,75	0,00	212,89	18.637,32	0,77181528	-1149,37	-2.220,54	15.480,31
2023	53	13	58,91	0,00	158,71	19.066,22	0,754452974	-1149,37	-1.122,27	16.953,29
2024	54	14	80,86	0,00	81,35	19.526,49	0,736669337	-1149,37	0,00	18.458,48
2025	55	15	82,98	19917,02	0,00	0,00	0	0,00	0,00	20.000,00

Table 5.2.: Statutory book values for the first contract derived by the conditional version of the algorithm from section (3.4).

The same results can be obtained by using a conditional version of the extended market value algorithm from section (3.4). This is shown in tables (5.2) and (5.3), respectively.

On top it shows the input benefits (BFTs) for death (D) and endowment (E), boolean values where premiums (P-01) and surrender possibilities (S-01) shall be scheduled, the mortality table as well as the sojourn probabilities, the guaranteed interest rate and finally some single values. At the bottom it shows conditional cash flows, conditional present values and the conditional net reserves.

Tariff 2: Recursive Stautory Book Value Model

Benefits (BFT), Mortality (qx), Guaranteed Return (gR)

Input and easily deduced values

			BFTs				P!		Interest	Single values		
ultimo	**age**	**t***	**D**	**E**	**P-01**	**S-01**	**q*x-1**	**k1px**	**gR**	**term**	**code**	**value**
2010	40	0	0	0	1	0	0,000%	100,00%	0,00%	Start of coverage	t0	2010
2011	41	1	1333	0	1	0	0,091%	100,00%	2,00%	Balance year	tt	2016
2012	42	2	2667	0	1	1	0,094%	99,91%	2,00%	End of coverage	tn	2025
2013	43	3	4000	0	1	1	0,112%	99,82%	2,00%			
2014	44	4	5333	0	1	1	0,129%	99,70%	2,00%	Face value	FV	20000
2015	45	5	6667	0	1	1	0,130%	99,57%	2,00%	Profit Account	PA	600
2016	46	6	8000	0	1	1	0,165%	99,44%	2,00%			
2017	47	7	9333	0	1	1	0,163%	99,28%	2,00%	Age	x	40
2018	48	8	10667	0	1	1	0,196%	99,12%	2,00%	Period of coverage	n	15
2019	49	9	12000	0	1	1	0,203%	98,92%	2,00%	Premium payment period	m	15
2020	50	10	13333	0	1	1	0,260%	98,72%	2,00%			
2021	51	11	14667	0	1	1	0,292%	98,47%	2,00%	Surrender charge	SurrPenalty	95%
2022	52	12	16000	0	1	1	0,299%	98,18%	2,00%	Sharing rule	ShareRule	85%
2023	53	13	17333	0	1	1	0,295%	97,88%	2,00%	GuaranteedReturn	gR	2,00%
2024	54	14	18667	0	1	1	0,404%	97,60%	2,00%	Profit rate	ProfitRate	3,00%
2025	55	15	20000	20000	0	1	0,415%	97,20%	2,00%	Added profit	AddedProfit	0,33%

			cCFs		cPVs		Premium determination		cPV	cNetRes
ultimo	**age**	**t***	**D**	**E**	**D**	**E**	**Annuity-terms**	**P**	**P**	**cNetRes**
2010	40	0	0,00	0,00	338,16	14.384,57	1	-1134,77	-13.587,96	0,00
2011	41	1	1,21	0,00	344,03	14.685,61	0,979500573	-1134,77	-12.737,56	1.157,31
2012	42	2	2,50	0,00	348,73	14.993,38	0,959394815	-1134,77	-11.869,73	2.337,61
2013	43	3	4,49	0,00	351,61	15.310,42	0,93952773	-1134,77	-10.985,95	3.541,31
2014	44	4	6,89	0,00	352,21	15.636,83	0,919915559	-1134,77	-10.085,40	4.768,88
2015	45	5	8,70	0,00	351,02	15.970,40	0,900701638	-1134,77	-9.165,77	6.020,88
2016	**46**	**6**	**13,23**	**0,00**	**345,38**	**16.316,80**	**0,881580426**	**-1134,77**	**-8.229,80**	**7.297,60**
2017	47	7	15,24	0,00	337,59	16.670,36	0,862882923	-1134,77	-7.273,36	8.599,82
2018	48	8	20,88	0,00	324,10	17.037,12	0,844307397	-1134,77	-6.298,61	9.927,83
2019	49	9	24,40	0,00	306,80	17.413,27	0,826069286	-1134,77	-5.302,90	11.282,40
2020	50	10	34,73	0,00	278,94	17.807,91	0,807762585	-1134,77	-4.288,32	12.663,77
2021	51	11	42,85	0,00	242,38	18.217,30	0,789610428	-1134,77	-3.252,13	14.072,77
2022	52	12	47,80	0,00	200,02	18.637,32	0,77181528	-1134,77	-2.192,34	15.510,23
2023	53	13	51,05	0,00	153,42	19.066,22	0,754452974	-1134,77	-1.108,02	16.976,85
2024	54	14	75,47	0,00	81,35	19.526,49	0,736669337	-1134,77	0,00	18.473,07
2025	55	15	82,98	19917,02	0,00	0,00	0	0,00	0,00	20.000,00

Table 5.3.: Statutory book values for the second contract derived by the conditional version of the algorithm from section (3.4).

The main advantage of this algorithm is that the cash flows are now explicitly part of the calculation. An interesting thing to observe is that the cash flows of the final year add up to the reserve because the final reserve does not contain any death benefit. This is a spurious relation stemming from the fact that the endowment benefit equals the death benefit in our examples.

Within the statutory book value model the profit participation is not modeled into the future. According to the data, however, both policies already participated in 600 up to the current balance sheet date.

q_x factor = 60%			Tariff 1: Unconditional Market Value Model with 2. Order Input									
s_x factor = 100%			Benefits, Mortality, Surrender (st), Spotrate									
			Additional input and easily deduced values									
			BFT	P!					Interest			
ultimo	age	t	S	st*	q*x-1	qx	px	k1px	Spot	Forward	Excess Return	kDisc
2010	40											
2011	41											
2012	42						history irrelevant					
2013	43											
2014	44											
2015	45											
2016	46	0	6.914,81	0,000%	0,000%	0,000%	100,000%	100,000%	0,000%	0,000%	0,000%	1,0000
2017	47	1	8.149,12	6,396%	0,098%	0,092%	93,512%	100,000%	0,386%	1,945%	0,000%	0,9809
2018	48	2	9.407,06	5,603%	0,117%	0,111%	94,286%	93,512%	0,492%	1,024%	0,000%	0,9710
2019	49	3	10.692,04	5,197%	0,122%	0,116%	94,687%	88,169%	0,658%	1,660%	0,000%	0,9551
2020	50	4	12.001,39	4,249%	0,156%	0,150%	95,601%	83,485%	0,876%	2,415%	0,415%	0,9326
2021	51	5	13.338,61	3,547%	0,175%	0,169%	96,284%	79,813%	1,117%	3,066%	1,066%	0,9049
2022	52	6	14.706,29	2,942%	0,179%	0,174%	96,884%	76,847%	1,354%	3,512%	1,512%	0,8742
2023	53	7	16.105,63	2,179%	0,177%	0,173%	97,648%	74,452%	1,567%	3,722%	1,722%	0,8428
2024	54	8	17.535,55	2,179%	0,243%	0,237%	97,584%	72,701%	1,763%	3,944%	1,944%	0,8108
2025	55	9	19.000,00	2,179%	0,249%	0,244%	97,577%	70,945%	1,763%	1,763%	0,000%	0,7968

			unconditional Cash Flows				unconditional Volumes				uNetRes
ultimo	age	t	D	E	S	P	D	E	S	P	sum
2010	40										
2011	41										
2012	42			history irrelevant					history irrelevant		
2013	43										
2014	44										
2015	45										
2016	46	0	-	-	-	- 1.149,37	205,91	11.031,40	3.166,63	- 6.722,93	6.531,63
2017	47	1	18,35	-	521,22	- 1.074,80	191,57	11.245,95	2.707,00	- 5.778,89	7.290,83
2018	48	2	20,74	-	492,88	- 1.013,38	172,79	11.361,08	2.241,83	- 4.824,66	7.937,64
2019	49	3	20,39	-	489,93	- 959,55	155,26	11.549,64	1.789,11	- 3.945,20	8.589,27
2020	50	4	24,98	-	425,72	- 917,34	134,03	11.828,60	1.406,60	- 3.123,14	9.328,75
2021	51	5	26,99	-	377,61	- 883,25	111,15	12.191,25	1.072,11	- 2.335,64	10.155,62
2022	52	6	26,74	-	332,48	- 855,73	88,31	12.619,42	777,28	- 1.561,94	11.067,35
2023	53	7	25,74	-	261,28	- 835,60	65,86	13.089,09	544,93	- 784,47	12.079,80
2024	54	8	34,50	-	277,79	- 815,41	33,95	13.605,34	288,63	-	13.112,51
2025	55	9	34,55	13.845,20	293,72	-	-	-	-	-	13.845,20

			Profit-Account				uCFs			uProRes
ultimo	age	t	Added	Excess	Terminal	Account	D	E	S	cProRes
2016	46	0	0,00	0,00		600,00	0,00	0,00	0,00	1.769,98
2017	47	1	66,00	72,79		750,46	0,69	0,00	48,00	1.755,72
2018	48	2	66,00	85,78		909,92	0,94	0,00	47,68	1.725,07
2019	49	3	66,00	0,00		991,02	1,01	0,00	45,41	1.707,28
2020	50	4	66,00	39,73		1.120,69	1,40	0,00	39,75	1.707,36
2021	51	5	66,00	114,45		1.335,50	1,80	0,00	37,81	1.720,10
2022	52	6	66,00	180,47		1.628,87	2,18	0,00	36,83	1.741,51
2023	53	7	66,00	226,56		1.982,05	2,55	0,00	32,16	1.771,62
2024	54	8	66,00	280,15		2.406,38	4,15	0,00	38,12	1.799,22
2025	55	9	66,00	0,00	66,00	2.580,80	4,46	1.786,59	39,90	1.786,59

Table 5.4.: Best estimate values for the first contract derived by the algorithm from sec. (3.4) without interpolation and waiver of premiums.

In tables (5.4) and (5.5), respectively, the same contracts are now evaluated unconditionally under a valuation basis of second order. This means that benefits and premiums are deemed to be fixed input values, the mortality

q_x factor = 60%

s_x factor = 100%

Tariff 2: Unconditional Market Value Model with 2. Order Input

Benefits, Mortality, Surrender (st), Spotrate

Additional input and easily deduced values

			BFT	P!					Interest			
ultimo	age	t	S	st*	q*x-1	qx	px	k1px	Spot	Forward	Excess Return	kDisc
2010	40		history irrelevant									
2011	41											
2012	42											
2013	43											
2014	44											
2015	45											
2016	46	0	6.932,72	0,000%	0,000%	0,000%	100,000%	100,000%	0,000%	0,000%	0,000%	1,0000
2017	47	1	8.169,83	6,396%	0,098%	0,092%	93,512%	100,000%	0,386%	0,386%	0,000%	0,9962
2018	48	2	9.431,44	5,603%	0,117%	0,111%	94,286%	93,512%	0,492%	0,598%	0,000%	0,9902
2019	49	3	10.718,28	5,197%	0,122%	0,116%	94,687%	88,169%	0,658%	0,991%	0,000%	0,9805
2020	50	4	12.030,58	4,249%	0,156%	0,150%	95,601%	83,485%	0,876%	1,533%	0,000%	0,9657
2021	51	5	13.369,13	3,547%	0,175%	0,169%	96,284%	79,813%	1,117%	2,087%	0,087%	0,9460
2022	52	6	14.734,72	2,942%	0,179%	0,174%	96,884%	76,847%	1,354%	2,547%	0,547%	0,9225
2023	53	7	16.128,01	2,179%	0,177%	0,173%	97,648%	74,452%	1,567%	2,854%	0,854%	0,8969
2024	54	8	17.549,42	2,179%	0,243%	0,237%	97,584%	72,701%	1,763%	3,146%	1,146%	0,8695
2025	55	9	19.000,00	2,179%	0,249%	0,244%	97,577%	70,945%	1,763%	1,763%	0,000%	0,8545

			unconditional Cash Flows				unconditional Volumes				uNetRes
ultimo	age	t	D	E	S	P	D	E	S	P	sum
2010	40		history irrelevant				history irrelevant				
2011	41										
2012	42										
2013	43										
2014	44										
2015	45										
2016	46	0	-	-	-	- 1.134,77	163,55	11.830,15	3.296,25	- 6.895,58	7.259,60
2017	47	1	8,56	-	522,54	- 1.061,15	155,62	11.875,81	2.786,43	- 5.861,05	7.895,67
2018	48	2	11,06	-	494,16	- 1.000,52	145,49	11.946,84	2.308,94	- 4.895,58	8.505,17
2019	49	3	12,24	-	491,13	- 947,36	134,70	12.065,21	1.840,69	- 3.996,73	9.096,51
2020	50	4	16,66	-	426,76	- 905,69	120,11	12.250,15	1.442,15	- 3.152,30	9.754,41
2021	51	5	19,79	-	378,47	- 872,04	102,82	12.505,79	1.093,77	- 2.346,05	10.484,29
2022	52	6	21,39	-	333,13	- 844,86	84,05	12.824,35	788,50	- 1.560,94	11.291,09
2023	53	7	22,31	-	261,65	- 824,99	64,14	13.190,42	549,36	- 780,51	12.198,41
2024	54	8	32,20	-	278,01	- 805,06	33,95	13.605,34	288,63	-	13.122,86
2025	55	9	34,55	13.845,20	293,72	-	-	-	-	-	13.845,20

			Profit-Account				uCFs			uProRes
ultimo	age	t	Added	Excess	Terminal	Account	D	E	S	cProRes
2016	46	0	0,00	0,00		600,00	0,00	0,00	0,00	1.349,63
2017	47	1	30,80	72,98		706,09	0,65	0,00	45,16	1.309,03
2018	48	2	35,20	86,00		831,51	0,86	0,00	43,57	1.272,43
2019	49	3	39,60	0,00		879,35	0,90	0,00	40,29	1.243,85
2020	50	4	44,00	0,00		936,83	1,17	0,00	33,23	1.228,51
2021	51	5	48,40	9,34		1.014,12	1,37	0,00	28,71	1.224,07
2022	52	6	52,80	65,47		1.158,23	1,55	0,00	26,19	1.227,52
2023	53	7	57,20	112,65		1.361,14	1,75	0,00	22,08	1.238,72
2024	54	8	61,60	165,32		1.630,87	2,81	0,00	25,84	1.249,04
2025	55	9	66,00	0,00	66,00	1.791,62	3,10	1.240,27	27,70	1.240,27

Table 5.5.: Best estimate values for the second contract derived by the algorithm from sec. (3.4) without interpolation and waiver of premiums.

is reduced by 60%, mortality and sojourn are diluted by surrender, the guaranteed interest rate is replaced by a market spot rate and the profit account is modeled into the future according to equation (3.7) in order to determine the prospective profit reserve as well.

Due to the unconditional representation the cash flows, present values and reserves are now per contract which is in the portfolio at the balance sheet date no matter which year it belongs to. In the conditional representation the development showed the values per contract that is in the portfolio at the beginning of the respective year. The book value perspective is thus rather the insured's point of view while the unconditional perspective is more natural from an insurers point of view.

In our examples the market values at the current balance sheet date are higher than the book values. While the different effects coincidentally annihilate each other within the mathematical net reserve this is not the case when it comes to the value for the profit participation. In case of the net reserve this mainly roots in the spot rate being below the guaranteed interest rate while the introduction of surrender leads to easing surrender profits. The profit participation, however, has three different issues. Firstly, it is fed by additional and terminal bonuses which are interest-independent discounts. Secondly, within the first two years the profit rate is already fixed at 3% so excess return on the statutory book value reserve needs to be granted. Thirdly, further excess return has to be granted once the forward rate exceeds the guaranteed interest rate. All of this accumulates to an amount that cannot be compensated by the respective discounting effects. This is a very good example why the prospective perspective needs to be considered instead of the retrospective one when it comes to best estimates.

In table (5.6) we consider the last certainty equivalence model where we included the possibility to waive future premiums starting from the year 2017. Using the results from the statutory book value model we first calculated the reduction factors R which denote how much of the contract is left if the option is triggered at this point in time. We then calculate the reduction factor W which denotes the factor by which the benefits need to be reduced given the waiver probabilities and the reduction factors R. We then applied W to the benefit cash flows and no-waiver probabilities to the premiums to incorporate the option to the algorithm. For this approach is new, we also depicted the interim results of the naive algorithm in tables (5.7) to (5.10). The results are of course the same but it is a more tangible representation.

Tarif 2: Unconditional Market Value Model with 2. Order Input

Benefits, Mortality, Surrender (st), Waiver, Spotrate

Additional input and easily deduced values

q_x factor = 60%

s_x factor = 100%

w_x factor = 100%

			P!			Reduction Factors	
ultimo	**age**	**t**	**waiver table**	**waiver P!**	**k(1-waiver)**	**R**	**W**
2010	40						
2011	41						
2012	42			history irrelevant		history irrelevant	
2013	43						
2014	44						
2015	45						
2016	46	0	0,00%	0,00%	100,00%	43,80%	100,00%
2017	47	1	3,00%	3,00%	100,00%	50,56%	100,00%
2018	48	2	2,50%	2,50%	97,00%	57,18%	98,52%
2019	49	3	2,00%	2,00%	94,58%	63,67%	97,48%
2020	50	4	1,70%	1,70%	92,68%	70,02%	96,79%
2021	51	5	1,40%	1,40%	91,11%	76,24%	96,32%
2022	52	6	1,00%	1,00%	89,83%	82,34%	96,02%
2023	53	7	0,80%	0,80%	88,93%	88,33%	95,86%
2024	54	8	0,70%	0,70%	88,22%	94,21%	95,77%
2025	55	9	-	-	87,61%	100,00%	95,74%

			unconditional Cash Flows				unconditional Volumes				uNetRes
ultimo	**age**	**t**	**D**	**E**	**S**	**P**	**D**	**E**	**S**	**P**	**sum**
2010	40										
2011	41										
2012	42			history irrelevant					history irrelevant		
2013	43										
2014	44										
2015	45										
2016	46	0	-	-	-	- 1.134,77	157,82	11.326,00	3.207,57	- 6.320,35	7.236,28
2017	47	1	8,56	-	522,54	- 1.029,32	149,87	11.369,72	2.697,41	- 5.315,43	7.872,26
2018	48	2	10,90	-	486,83	- 946,24	139,87	11.437,72	2.226,72	- 4.400,99	8.457,09
2019	49	3	11,93	-	478,74	- 878,05	129,33	11.551,05	1.770,03	- 3.566,54	9.005,82
2020	50	4	16,12	-	413,07	- 825,16	115,19	11.728,11	1.384,10	- 2.796,06	9.606,19
2021	51	5	19,06	-	364,54	- 783,37	98,53	11.972,85	1.048,44	- 2.071,03	10.265,42
2022	52	6	20,54	-	319,86	- 751,37	80,50	12.277,84	755,29	- 1.372,42	10.989,85
2023	53	7	21,38	-	250,81	- 727,83	61,42	12.628,30	526,04	- 683,76	11.804,17
2024	54	8	30,84	-	266,26	- 705,27	32,51	13.025,54	276,33	-	12.629,11
2025	55	9	33,08	13.255,18	281,20	-	-	-	-	-	13.255,18

			Profit-Account				uCFs			uProRes
ultimo	**age**	**t**	**Added**	**Excess**	**Terminal**	**Account**	**D**	**E**	**S**	**cProRes**
2016	46	0	0,00	0,00		600,00	0,00	0,00	0,00	1.303,36
2017	47	1	29,88	72,98		705,17	0,65	0,00	45,10	1.262,64
2018	48	2	33,29	84,72		827,40	0,86	0,00	43,35	1.225,98
2019	49	3	36,70	0,00		872,30	0,89	0,00	39,97	1.197,27
2020	50	4	40,09	0,00		925,76	1,16	0,00	32,84	1.181,63
2021	51	5	43,48	9,00		997,55	1,35	0,00	28,24	1.176,70
2022	52	6	46,96	62,87		1.132,79	1,51	0,00	25,61	1.179,55
2023	53	7	50,46	107,98		1.323,56	1,70	0,00	21,47	1.190,04
2024	54	8	53,96	158,33		1.577,49	2,72	0,00	24,99	1.199,77
2025	55	9	57,82	0,00	57,82	1.720,94	2,97	1.191,34	26,60	1.191,34

Table 5.6.: Best estimate values for the second contract derived by the algorithm from sec. (3.4) without interpolation but with waiver of premiums.

The values are slightly lower with the option in our example. This, however, is not an intrinsic property of the waiver option because the reduction factors are based on book values while the net reserve here is subject to the valuation basis of second order. A tangible counter-example is a pure risk policy which usually has a negative best estimate because the expected premiums exceed the expected benefits under a best estimate valuation basis.

			Reduced - Death - Benefits									
			No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
ultimo	**age**	**t***										
2016	46	6	8000									
2017	47	7	9333	4087,76								
2018	48	8	10667	4671,72	5393,44							
2019	49	9	12000	5255,69	6067,62	6862,08						
2020	50	10	13333	5839,65	6741,80	7624,53	8489,35					
2021	51	11	14667	6423,62	7415,98	8386,98	9338,29	10269,07				
2022	52	12	16000	7007,58	8090,16	9149,44	10187,22	11202,63	12197,64			
2023	53	13	17333	7591,55	8764,34	9911,89	11036,16	12136,18	13214,11	14271,87		
2024	54	14	18667	8175,51	9438,52	10674,34	11885,10	13069,73	14230,58	15369,70	16488,40	
2025	55	15	20000	8759,48	10112,70	11436,79	12734,03	14003,28	15247,05	16467,54	17666,15	18842,53

			Reduced - Endowment - Benefits									
			No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
ultimo	**age**	**t***										
2016	46	6	0									
2017	47	7	0	0,00								
2018	48	8	0	0,00	0,00							
2019	49	9	0	0,00	0,00	0,00						
2020	50	10	0	0,00	0,00	0,00	0,00					
2021	51	11	0	0,00	0,00	0,00	0,00	0,00				
2022	52	12	0	0,00	0,00	0,00	0,00	0,00	0,00			
2023	53	13	0	0,00	0,00	0,00	0,00	0,00	0,00	0,00		
2024	54	14	0	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	
2025	55	15	20000	8759,48	10112,70	11436,79	12734,03	14003,28	15247,05	16467,54	17666,15	18842,53

			Reduced - Surrender - Benefits									
			No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
ultimo	**age**	**t***										
2016	46	6	6.932,72									
2017	47	7	8.169,83	3578,17								
2018	48	8	9.431,44	4130,73	4768,87							
2019	49	9	10.718,28	4694,33	5419,54	6129,14						
2020	50	10	12.030,58	5269,08	6083,08	6879,56	7659,89					
2021	51	11	13.369,13	5855,33	6759,90	7645,00	8512,15	9360,59				
2022	52	12	14.734,72	6453,42	7450,39	8425,90	9381,62	10316,72	11233,05			
2023	53	13	16.128,01	7063,65	8154,89	9222,64	10268,73	11292,25	12295,23	13279,43		
2024	54	14	17.549,42	7686,19	8873,60	10035,46	11173,74	12287,47	13378,84	14449,79	15501,53	
2025	55	15	19.000,00	8321,51	9607,07	10864,96	12097,33	13303,12	14484,69	15644,16	16782,84	17900,41

			Reduced - Premiums									
			No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
ultimo	**age**	**t***										
2016	46	6	-1134,77									
2017	47	7	-1134,77	0,00								
2018	48	8	-1134,77	0,00	0,00							
2019	49	9	-1134,77	0,00	0,00	0,00						
2020	50	10	-1134,77	0,00	0,00	0,00	0,00					
2021	51	11	-1134,77	0,00	0,00	0,00	0,00	0,00				
2022	52	12	-1134,77	0,00	0,00	0,00	0,00	0,00	0,00			
2023	53	13	-1134,77	0,00	0,00	0,00	0,00	0,00	0,00	0,00		
2024	54	14	-1134,77	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	
2025	55	15	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00

Table 5.7.: Best estimate values for the second contract derived by the naive waiver-algorithm from sec. (3.4) without interpolation part 1/4.

Unconditional - Mortality - Effects (k-1px)												
ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
2016	46	6	100,00%									
2017	47	7	100,00%	100,00%								
2018	48	8	99,90%	99,90%	99,90%							
2019	49	9	99,78%	99,78%	99,78%	99,78%						
2020	50	10	99,66%	99,66%	99,66%	99,66%	99,66%					
2021	51	11	99,51%	99,51%	99,51%	99,51%	99,51%	99,51%				
2022	52	12	99,33%	99,33%	99,33%	99,33%	99,33%	99,33%	99,33%			
2023	53	13	99,15%	99,15%	99,15%	99,15%	99,15%	99,15%	99,15%	99,15%		
2024	54	14	98,98%	98,98%	98,98%	98,98%	98,98%	98,98%	98,98%	98,98%	98,98%	
2025	55	15	98,74%	98,74%	98,74%	98,74%	98,74%	98,74%	98,74%	98,74%	98,74%	98,74%

Unconditional - Surrender - Effects (k-1px)												
ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
2016	46	6	100,00%									
2017	47	7	100,00%	100,00%								
2018	48	8	93,60%	93,60%	93,60%							
2019	49	9	88,36%	88,36%	88,36%	88,36%						
2020	50	10	83,77%	83,77%	83,77%	83,77%	83,77%					
2021	51	11	80,21%	80,21%	80,21%	80,21%	80,21%	80,21%				
2022	52	12	77,36%	77,36%	77,36%	77,36%	77,36%	77,36%	77,36%			
2023	53	13	75,09%	75,09%	75,09%	75,09%	75,09%	75,09%	75,09%	75,09%		
2024	54	14	73,45%	73,45%	73,45%	73,45%	73,45%	73,45%	73,45%	73,45%	73,45%	
2025	55	15	71,85%	71,85%	71,85%	71,85%	71,85%	71,85%	71,85%	71,85%	71,85%	71,85%

Unconditional - Waiver - Effects (k-1px)												
ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020
2016	46	6	100,00%									
2017	47	7	100,00%	100,00%								
2018	48	8	97,00%	100,00%	100,00%							
2019	49	9	94,58%	100,00%	100,00%	97,00%						
2020	50	10	92,68%	100,00%	100,00%	97,00%	94,58%					
2021	51	11	91,11%	100,00%	100,00%	97,00%	94,58%	92,68%				
2022	52	12	89,83%	100,00%	100,00%	97,00%	94,58%	92,68%	91,11%			
2023	53	13	88,93%	100,00%	100,00%	97,00%	94,58%	92,68%	91,11%	89,83%		
2024	54	14	88,22%	100,00%	100,00%	97,00%	94,58%	92,68%	91,11%	89,83%	88,93%	
2025	55	15	87,61%	100,00%	100,00%	97,00%	94,58%	92,68%	91,11%	89,83%	88,93%	88,22%

Unconditional - Waiver - Effects (k-1px)*pf*R													
ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	W as sum
2016	46	6	100,00%										100,000%
2017	47	7	100,00%	0,00%									100,000%
2018	48	8	97,00%	0,00%	1,52%								98,517%
2019	49	9	94,58%	0,00%	1,52%	1,39%							97,479%
2020	50	10	92,68%	0,00%	1,52%	1,39%	1,20%						96,791%
2021	51	11	91,11%	0,00%	1,52%	1,39%	1,20%	1,10%					96,319%
2022	52	12	89,83%	0,00%	1,52%	1,39%	1,20%	1,10%	0,97%				96,016%
2023	53	13	88,93%	0,00%	1,52%	1,39%	1,20%	1,10%	0,97%	0,74%			95,857%
2024	54	14	88,22%	0,00%	1,52%	1,39%	1,20%	1,10%	0,97%	0,74%	0,63%		95,774%
2025	55	15	87,61%	0,00%	1,52%	1,39%	1,20%	1,10%	0,97%	0,74%	0,63%	0,58%	95,738%

Table 5.8.: Best estimate values for the second contract derived by the naive waiver-algorithm from sec. (3.4) without interpolation part 2/4.

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Diff. to MV CFs
Unconditional - Death - Cash - Flows													
2016	46	6	-										
2017	47	7	8,56	-									-
2018	48	8	10,73	-	0,17								- 0,16
2019	49	9	11,57	-	0,19	0,17							- 0,31
2020	50	10	15,44	-	0,25	0,23	0,20						- 0,53
2021	51	11	18,03	-	0,30	0,27	0,24	0,22					- 0,73
2022	52	12	19,22	-	0,32	0,30	0,26	0,24	0,21				- 0,85
2023	53	13	19,84	-	0,34	0,31	0,27	0,25	0,22	0,17			- 0,92
2024	54	14	28,41	-	0,49	0,45	0,39	0,36	0,31	0,24	0,20		- 1,36
2025	55	15	30,27	-	0,52	0,48	0,42	0,38	0,34	0,26	0,22	0,20	- 1,47

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Diff. to MV CFs
Unconditional - Endowment - Cash - Flows													
2016	46	6	-										
2017	47	7	-	-									-
2018	48	8	-	-	-								-
2019	49	9	-	-	-	-							-
2020	50	10	-	-	-	-	-						-
2021	51	11	-	-	-	-	-	-					-
2022	52	12	-	-	-	-	-	-	-				-
2023	53	13	-	-	-	-	-	-	-	-			-
2024	54	14	-	-	-	-	-	-	-	-	-		-
2025	55	15	12.129,09	-	210,02	191,99	166,74	152,74	134,63	102,41	87,01	80,55	- 590,02

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Diff. to MV CFs
Unconditional - Surrender - Cash - Flows													
2016	46	6	-										
2017	47	7	522,54	-									-
2018	48	8	479,33	-	7,50								- 7,33
2019	49	9	464,48	-	7,45	6,81							- 12,38
2020	50	10	395,53	-	6,47	5,92	5,14						- 13,69
2021	51	11	344,82	-	5,74	5,25	4,56	4,18					- 13,93
2022	52	12	299,26	-	5,05	4,62	4,01	3,68	3,24				- 13,27
2023	53	13	232,69	-	3,97	3,63	3,15	2,89	2,54	1,94			- 10,84
2024	54	14	245,27	-	4,22	3,86	3,35	3,07	2,70	2,06	1,75		- 11,75
2025	55	15	257,31	-	4,46	4,07	3,54	3,24	2,86	2,17	1,85	1,71	- 12,52

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Diff. to MV CFs
Unconditional - Premium - Cash - Flows													
2016	46	6	- 1.134,77										-
2017	47	7	- 1.029,32	-									31,83
2018	48	8	- 946,24	-	-								54,28
2019	49	9	- 878,05	-	-	-							69,31
2020	50	10	- 825,16	-	-	-	-						80,54
2021	51	11	- 783,37	-	-	-	-	-					88,67
2022	52	12	- 751,37	-	-	-	-	-	-				93,49
2023	53	13	- 727,83	-	-	-	-	-	-	-			97,16
2024	54	14	- 705,27	-	-	-	-	-	-	-	-		99,79
2025	55	15	-	-	-	-	-	-	-	-	-	-	-

Table 5.9.: Best estimate values for the second contract derived by the naive waiver-algorithm from sec. (3.4) without interpolation part 3/4.

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Death Reserve = row sum	Total Reserve
			Unconditional - Death - Volumes											
2016	46	6	148,55	-	2,35	2,00	1,59	1,28	0,95	0,57	0,36	0,17	157,82	7.236,28
2017	47	7	140,56	-	2,36	2,01	1,60	1,28	0,95	0,58	0,36	0,17	149,87	7.872,26
2018	48	8	130,68	-	2,21	2,02	1,61	1,29	0,96	0,58	0,37	0,17	139,87	8.457,09
2019	49	9	120,40	-	2,04	1,87	1,62	1,30	0,96	0,58	0,37	0,18	129,33	9.005,82
2020	50	10	106,81	-	1,82	1,67	1,45	1,32	0,98	0,59	0,37	0,18	115,19	9.606,19
2021	51	11	91,00	-	1,56	1,43	1,24	1,13	1,00	0,61	0,38	0,18	98,53	10.265,42
2022	52	12	74,11	-	1,27	1,17	1,01	0,93	0,82	0,62	0,39	0,19	80,50	10.989,85
2023	53	13	56,38	-	0,97	0,89	0,77	0,71	0,62	0,47	0,40	0,19	61,42	11.804,17
2024	54	14	29,74	-	0,52	0,47	0,41	0,37	0,33	0,25	0,21	0,20	32,51	12.629,11
2025	55	15	-	-	-	-	-	-	-	-	-	-	-	13.255,18

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Endowment Reserve = row sum
			Unconditional - Endowment - Volumes										
2016	46	6	10.363,80	-	179,45	164,05	142,47	130,51	115,03	87,50	74,35	68,83	11.326,00
2017	47	7	10.403,81	-	180,14	164,68	143,02	131,01	115,48	87,84	74,63	69,10	11.369,72
2018	48	8	10.466,03	-	181,22	165,67	143,88	131,80	116,17	88,37	75,08	69,51	11.437,72
2019	49	9	10.569,73	-	183,02	167,31	145,30	133,10	117,32	89,24	75,82	70,20	11.551,05
2020	50	10	10.731,75	-	185,82	169,87	147,53	135,14	119,12	90,61	76,99	71,27	11.728,11
2021	51	11	10.955,70	-	189,70	173,42	150,61	137,96	121,60	92,50	78,59	72,76	11.972,85
2022	52	12	11.234,78	-	194,53	177,84	154,45	141,48	124,70	94,86	80,59	74,61	12.277,84
2023	53	13	11.555,47	-	200,09	182,91	158,85	145,52	128,26	97,56	82,90	76,74	12.628,30
2024	54	14	11.918,96	-	206,38	188,67	163,85	150,09	132,30	100,63	85,50	79,16	13.025,54
2025	55	15	-	-	-	-	-	-	-	-	-	-	13.255,18

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Surrender Reserve = row sum
			Unconditional - Surrender - Volumes										
2016	46	6	3.076,67	-	42,11	31,71	21,74	15,36	10,06	5,38	3,10	1,46	3.207,57
2017	47	7	2.566,00	-	42,27	31,83	21,82	15,42	10,10	5,40	3,11	1,47	2.697,41
2018	48	8	2.102,01	-	35,02	32,02	21,95	15,52	10,16	5,43	3,13	1,47	2.226,72
2019	49	9	1.658,36	-	27,92	25,53	22,17	15,67	10,26	5,49	3,16	1,49	1.770,03
2020	50	10	1.288,24	-	21,88	20,00	17,37	15,91	10,42	5,57	3,21	1,51	1.384,10
2021	51	11	970,30	-	16,59	15,17	13,17	12,07	10,64	5,69	3,27	1,54	1.048,44
2022	52	12	695,76	-	11,96	10,93	9,50	8,70	7,67	5,83	3,36	1,58	755,29
2023	53	13	482,93	-	8,33	7,62	6,62	6,06	5,34	4,06	3,45	1,63	526,04
2024	54	14	252,85	-	4,38	4,00	3,48	3,18	2,81	2,13	1,81	1,68	276,33
2025	55	15	-	-	-	-	-	-	-	-	-	-	-

ultimo	age	t*	No Waiver	Waiver at 2012	Waiver at 2013	Waiver at 2014	Waiver at 2015	Waiver at 2016	Waiver at 2017	Waiver at 2018	Waiver at 2019	Waiver at 2020	Premium Reserve = row sum
			Unconditional - Premium - Volumes										
2016	46	6	- 6.320,35	-	-	-	-	-	-	-	-	-	- 7.455,12
2017	47	7	- 5.315,43	-	-	-	-	-	-	-	-	-	- 6.344,75
2018	48	8	- 4.400,99	-	-	-	-	-	-	-	-	-	- 5.347,22
2019	49	9	- 3.566,54	-	-	-	-	-	-	-	-	-	- 4.444,59
2020	50	10	- 2.796,06	-	-	-	-	-	-	-	-	-	- 3.621,21
2021	51	11	- 2.071,03	-	-	-	-	-	-	-	-	-	- 2.854,40
2022	52	12	- 1.372,42	-	-	-	-	-	-	-	-	-	- 2.123,79
2023	53	13	- 683,76	-	-	-	-	-	-	-	-	-	- 1.411,59
2024	54	14	-	-	-	-	-	-	-	-	-	-	- 705,27
2025	55	15	-	-	-	-	-	-	-	-	-	-	-

Table 5.10.: Best estimate values for the second contract derived by the naive waiver-algorithm from sec. (3.4) without interpolation part 4/4.

5.1. Affine Models

Let us now consider the second contract but without the waiver-option and in a time-continuous framework where the premiums shall be payed as a continuous payment stream and the surrender and death benefits shall be payed immediately upon the respective events. While the endowment benefit is still the same lump sum payment at the end, the annual premium payments are now replaced by a constant premium intensity, and the death- as well as the surrender benefits are now continuously increasing claims.

Before we let the transition intensities be driven by affine processes we take a look at the results of a time-continuous model with *deterministic* intensities.

In order to have a little validation we do not use the pricing formulas from the affine framework but we proceed accordingly to ([Noro2, p. 79ff]) where Thiele's differential equation is applied directly and we provide the resulting probabilities to allow for a comparison with the time-discrete models. We then compare these results to those of the affine approach from section (4.4) with zero volatility.

For the deterministic mortality intensities $\mu^\circ(t)$ we take the parameters from ([Kol10, p. 18 ff]) and apply the same idea to come up with deterministic surrender intensities $\eta^\circ(t)$ ourselves. The explicit choice is

$$\begin{aligned} \mu^\circ(t) &:= Exp[-9.13275 + 8.09438 * 10^{-2} * t - 1.10180 * 10^{-5} * t^2], \\ \eta^\circ(t) &:= Exp[-5 + 0.2 * (t + 10) - 0.01 * (t + 5)^2 - 0.001 * (t + 0)^3]. \end{aligned} \tag{5.1}$$

For the deterministic force of interest $r^\circ(t)$ we take the EIOPA risk free spot rate as of October 31, 2016 and convert it into intensities accordingly to appendix (A). The reserve is finally derived by solving Thiele's differential equation

$$\begin{aligned} Res'_t &= (r^\circ_t + \mu^\circ_{x+t} + \eta^\circ_t)\, Res_t - \mu^\circ_{x+t}\, dBFT_t - \eta^\circ_t\, sBFT_t + pBFT, \\ Res_n &= eBFT, \end{aligned} \tag{5.2}$$

where $dBFT$ denotes the death benefit, $sBFT$ denotes the surrender benefit, $pBFT$ denotes the premium intensity, and $eBFT$ denotes the endowment benefit.

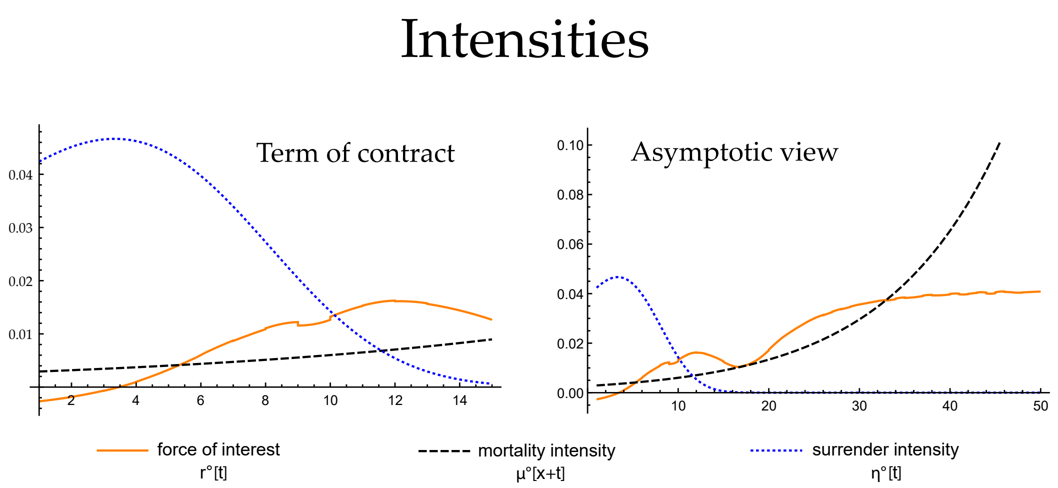


Figure 5.1.: Deterministic intensities used as input for the time-continuous model.

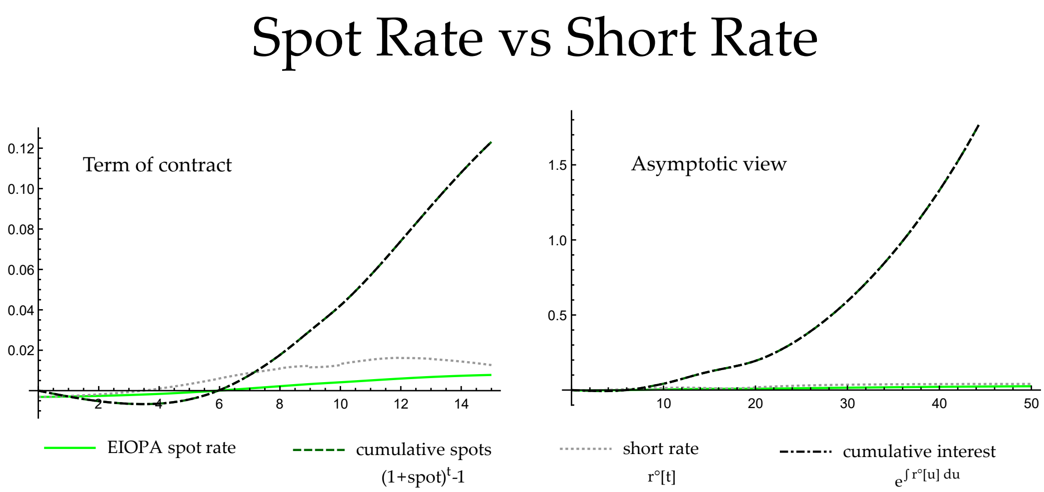


Figure 5.2.: Deterministic spot rates, derived intensities and coinciding accumulated interest rates thereof.

For the Kolmogorov backwards equation is a special case of Thiele's differential equation we could use it to determine the resulting bond prices and probabilities by simply setting the irrelevant benefits and intensities to zero and the endowment benefit to one. But because we are using this approach for the reserves anyway we take the opportunity to add another redundancy and use equations (5.3) instead.

$$
\begin{aligned}
P(0,t) &:= e^{-\int_0^t r^\circ(u)du} \\
p00(0,t) &:= e^{-\int_0^t \mu^\circ(x+u)+\eta^\circ(u)du} \\
p01(0,t) &:= e^{-\int_0^t \mu^\circ(x+u)+\eta^\circ(u)du} \int_{t-1}^{t} e^{-\int_{t-1}^{u} \mu^\circ(x+\xi)+\eta^\circ(\xi)d\xi} \mu^\circ(x+u)du \\
p02(0,t) &:= e^{-\int_0^t \mu^\circ(x+u)+\eta^\circ(u)du} \int_{t-1}^{t} e^{-\int_{t-1}^{u} \mu^\circ(x+\xi)+\eta^\circ(\xi)d\xi} \eta^\circ(x+u)du
\end{aligned}
\tag{5.3}
$$

Year	Bond Prices	Probabilities			Mathematical
		Sojourn	Death	Surrender	
t	P(0,t)	p00(0,t)	p01(0,t)	p02(0,t)	Reserve
0	1,000	100,00%	0,00%	0,00%	867,37
1	1,003	95,75%	0,28%	3,98%	2.036,88
2	1,005	91,36%	0,56%	8,08%	3.209,93
3	1,007	86,97%	0,86%	12,18%	4.387,79
4	1,006	82,71%	1,16%	16,13%	5.572,34
5	1,004	78,75%	1,47%	19,78%	6.768,23
6	1,000	75,20%	1,80%	23,01%	7.981,32
7	0,992	72,14%	2,13%	25,73%	9.218,89
8	0,983	69,62%	2,48%	27,90%	10.481,90
9	0,971	67,62%	2,85%	29,54%	11.771,90
10	0,960	66,07%	3,23%	30,70%	13.071,60
11	0,946	64,90%	3,64%	31,46%	14.413,60
12	0,931	64,00%	4,08%	31,92%	15.794,70
13	0,916	63,28%	4,55%	32,18%	17.196,80
14	0,903	62,64%	5,05%	32,31%	18.602,60
15	0,890	62,04%	5,58%	32,37%	20.000,00

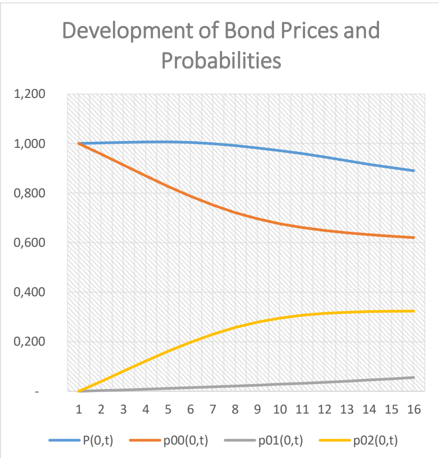


Table 5.11.: Bond Prices and Probabilities derived from the deterministic intensities.

5.1.1. Buchardt's Model

Having built the continuous reference model according to Norberg's lecture we are now interested in the results from Buchardt's model introduced in section (4.7).

We recall that the mortality is modeled by the deterministic intensity $\mu^\circ(x+t)$ while the force of interest $r(t)$ and the surrender intensity $\eta(t)$ are modeled by the two-dimensional process Y which is given by

$$Y(t) := \begin{cases} r(t) = X_1(t), \\ \eta(t) = \eta^\circ(t)\, X_2(t) \end{cases} \tag{5.4}$$

with

$$dX(t) := \begin{cases} (b_1(t) - \beta_1 X_1(t))\, dt + \sigma_1 \left(\sqrt{1-\rho^2} dW_1(t) + \rho \sqrt{X_2(t)} dW_2(t) \right), \\ (b_2 - \beta_2 X_2(t))\, dt + \sigma_2 \sqrt{X_2(t)} dW_2(t). \end{cases} \tag{5.5}$$

We chose the parameters to be

$$\begin{matrix} b_1(t) := r^\circ(t) & \beta_1 := -1 & \sigma_1 := 0.05 & \rho := 0.45 \\ b_2 := 1 & \beta_2 := -1 & \sigma_2 := 0.20 & \end{matrix} \tag{5.6}$$

While these parameter choices are more or less arbitrary, the choices for $X(n)$ must follow from solving

$$\begin{matrix} X_1'(t) := b_1(t) & -X_1(t), & X_1(t_0) := 0, \\ X_2'(t) := b_2 & -X_2(t), & X_2(t_0) := 0, \end{matrix} \tag{5.7}$$

for every $t_0 \in [0, n]$ which e.g. for $t_0 = 0$ results in $X(n) = \{0.014087, 1\}$.

Using an annual grid to approximate the integral in equation (4.41), we find the affine factors to take the values depicted in figures (5.12) and (5.13), respectively.

ϕ	$\psi 1$	$\psi 2$	A	B1	B2
0.	0.	0.	0.	0.	0.0387742
-0.01403	-0.63212	-0.02549	0.0267914	0.	0.015592
-0.04569	-0.86466	-0.035639	0.0390304	0.	0.00610896
-0.08590	-0.95021	-0.03956	0.0442425	0.	0.00231812
-0.12997	-0.98168	-0.04102	0.045389	0.	0.000846837
-0.17508	-0.99326	-0.041555	0.0436409	0.	0.000296044
-0.21923	-0.99752	-0.041735	0.0396176	0.	0.000098446
-0.26094	-0.99908	-0.041793	0.0339145	0.	0.0000309547
-0.29895	-0.99966	-0.041811	0.02726	0.	9.14798×10^{-6}
-0.33249	-0.99987	-0.041817	0.0204633	0.	2.52454×10^{-6}
-0.36085	-0.99995	-0.041818	0.0142636	0.	6.47111×10^{-7}
-0.38490	-0.99998	-0.041818	0.00917735	0.	1.53324×10^{-7}
-0.40635	-0.99999	-0.041818	0.00541812	0.	3.33645×10^{-8}
-0.42584	-0.99999	-0.041818	0.00291758	0.	6.62574×10^{-9}
-0.44343	-0.99999	-0.041818	0.00142441	0.	1.33955×10^{-9}
-0.45911	-1.	-0.041818	0.000626726	0.	2.07768×10^{-10}

Table 5.12.: Affine factors for $\sigma_1 = 0.00$, $\sigma_2 = 0.00$, $\rho = 0.00$.

Using the results with zero volatility from figure (5.12) in equation (4.41), we find that we get the same results as with Thiele's differential equation (5.2) if we only consider the endowment benefit in both of them. In order to get the same results for all benefits we would have to use a finer grid for the integral in equation (4.41). Like this the annual grid results in $Res_0 = 732.06$ instead of $Res_0 = 867.37$.

ϕ	$\psi 1$	$\psi 2$	A	B1	B2
0.	0.	0.	0.	0.	0.0387742
-0.0138451	-0.632121	-0.0254344	0.0267746	0.	0.0155568
-0.044781	-0.864665	-0.0354289	0.0389684	0.	0.00606601
-0.0838652	-0.950213	-0.0392251	0.04414	0.	0.00228854
-0.126607	-0.981684	-0.0406145	0.0452644	0.	0.00083091
-0.170297	-0.993262	-0.0411016	0.0435121	0.	0.000288674
-0.213006	-0.997521	-0.0412638	0.0394985	0.	0.0000954081
-0.253268	-0.999088	-0.0413146	0.0338137	0.	0.0000298209
-0.289863	-0.999665	-0.0413293	0.0271814	0.	8.76295×10^{-6}
-0.322019	-0.999877	-0.0413332	0.0204066	0.	2.40561×10^{-6}
-0.349031	-0.999955	-0.0413341	0.0142259	0.	6.13449×10^{-7}
-0.371764	-0.999983	-0.0413343	0.00915415	0.	1.44865×10^{-7}
-0.391923	-0.999994	-0.0413343	0.00540494	0.	3.14202×10^{-8}
-0.410135	-0.999998	-0.0413343	0.00291069	0.	6.18864×10^{-9}
-0.426464	-0.999999	-0.0413342	0.00142112	0.	1.24493×10^{-9}
-0.44089	-1.	-0.0413342	0.000625299	0.	1.89288×10^{-10}

Table 5.13.: Affine factors for $\sigma_1 = 0.05$, $\sigma_2 = 0.20$, $\rho = 0.45$.

Out of curiosity we found that one can also get a pretty good approximation by shifting the death- and surrender benefits by 0.35 years back instead of refining the grid in this particular situation. The resulting reserve values with volatility are presented in table (5.14).

Deterministic Mathematical Reserve	Stochastic Mathematical Reserve	Absolute Difference	Relative Difference
867,37	994,78	127,42	15%
2.036,88	2.162,29	125,41	6%
3.209,93	3.335,15	125,22	4%
4.387,79	4.513,89	126,10	3%
5.572,34	5.699,14	126,80	2%
6.768,23	6.894,34	126,11	2%
7.981,32	8.104,07	122,75	2%
9.218,89	9.334,75	115,86	1%
10.481,90	10.586,70	104,80	1%
11.771,90	11.862,00	90,10	1%
13.071,60	13.143,10	71,50	1%
14.413,60	14.468,60	55,00	0%
15.794,70	15.837,40	42,70	0%
17.196,80	17.240,30	43,50	0%
18.602,60	18.664,10	61,50	0%
20.000,00	20.000,00	0,00	0%

Table 5.14.: Development of the reserve for $\sigma_1 = 0.05$, $\sigma_2 = 0.20$, $\rho = 0.45$.

MATHEMATICA – CODE;

Contract Parameters;

```
x  = 40;  (* age *)
n  = 15;  (* term insured *)
t0 =  0;  (* balance year *)

(* benefit payments in case of endowment, death, surrender, premium *)

eBFT = 20 000.00;

dBFT = Interpolation[
       Interpolation
   {0.00, 1333.33, 2666.67, 4000.00, 5333.33,
    6666.67, 8000.00, 9333.33, 10 666.67, 12 000.00, 13 333.33,
    14 666.67, 16 000.00, 17 333.33, 18 666.67, 20 000.00}];

sBFT = Interpolation[
       Interpolation
   {0.00, 1099.44, 2220.73, 3364.25, 4530.43,
    5719.83, 6932.72, 8169.83,  9431.44 , 10 718.28, 12 030.58,
    13 369.13, 14 734.72, 16 128.01, 17 549.42, 19 000.00}];

pBFT = 1134.77;

(* ContinuousMortalityTable *)
μ[t_] := Exp[-9.13275 + 8.09438 * 10^(-2) * t - 1.10180 * 10^(-5) * t^2];
         Exponentialfunktion

(* ContinuousSurrenderTable *)
η[t_] := Exp[-5 + 0.2 * (t + 10) - 0.01 * (t + 5)^2 - 0.001 * (t + 0)^3];
         Exponentialfunktion
```

```
In[62]:= (* Continuous EIOPA Spot Rate *)

spots := Interpolation[
           Interpolation
  {-0.00293, -0.00261, -0.00217, -0.00161, -0.00087, 0.00004, 0.00110,
   0.00219, 0.00326, 0.00413, 0.00506, 0.00597, 0.00675, 0.00735, 0.00777,
   0.00801, 0.00816, 0.00833, 0.00857, 0.00892, 0.00941, 0.01000, 0.01065,
   0.01134, 0.01206, 0.01279, 0.01351, 0.01423, 0.01493, 0.01562, 0.01629,
   0.01694, 0.01757, 0.01817, 0.01876, 0.01932, 0.01986, 0.02039, 0.02089,
   0.02137, 0.02184, 0.02228, 0.02271, 0.02313, 0.02352, 0.02391, 0.02428,
   0.02463, 0.02497, 0.02530, 0.02562, 0.02593, 0.02622, 0.02651, 0.02678,
   0.02705, 0.02731, 0.02756, 0.02780, 0.02803, 0.02826, 0.02848, 0.02869,
   0.02889, 0.02909, 0.02929, 0.02947, 0.02966, 0.02983, 0.03001, 0.03017,
   0.03034, 0.03050, 0.03065, 0.03080, 0.03095, 0.03109, 0.03123, 0.03136,
   0.03150, 0.03163, 0.03175, 0.03187, 0.03199, 0.03211, 0.03223, 0.03234,
   0.03245, 0.03255, 0.03266, 0.03276, 0.03286, 0.03296, 0.03305, 0.03315,
   0.03324, 0.03333, 0.03342, 0.03350, 0.03359, 0.03367, 0.03375, 0.03383,
   0.03391, 0.03399, 0.03406, 0.03414, 0.03421, 0.03428, 0.03435, 0.03442,
   0.03449, 0.03455, 0.03462, 0.03468, 0.03474, 0.03481, 0.03487, 0.03493,
   0.03499, 0.03504, 0.03510, 0.03516, 0.03521, 0.03526, 0.03532, 0.03537,
   0.03542, 0.03547, 0.03552, 0.03557, 0.03562, 0.03567, 0.03572, 0.03576,
   0.03581, 0.03585, 0.03590, 0.03594, 0.03598, 0.03603, 0.03607, 0.03611,
   0.03615, 0.03619, 0.03623, 0.03627, 0.03631, 0.03635, 0.03638}];

(* force of interest r[t] *)

   R[t_] := t * Log[1 + spots[t]]
               Logarithmus
   r[t_] := R'[t];

(* mortality intensity μ01 *)

   μ01[t_] := μ[t];

(* surrender intensity μ02 *)

   μ02[t_] := η[t];
```

```
In[67]:= Deterministic Transition Probabilities;

(*Define Functions *)

P[s_, t_]   := Exp[-NIntegrate[  r[u]                , {u, s,   t }]];
               Exp··· integriere numerisch

P00[s_, t_] := Exp[-NIntegrate[μ01[x + u]  + μ02[u]   , {u, s,   t  }]];
               Exp··· integriere numerisch

P01[s_, t_] := Exp[-NIntegrate[μ01[x + u]  + μ02[u]   , {u, s, t - 1}]]
               Exp··· integriere numerisch
   NIntegrate[Exp[-NIntegrate[μ01[x + xi] + μ02[xi] , {xi, t - 1, u}]]
               Exp··· integriere numerisch
     μ01[x + u], {u, t - 1, t}];

P02[s_, t_] := Exp[-NIntegrate[μ01[x + u]   + μ02[u]   , {u, s, t - 1}]]
               Exp··· integriere numerisch
   NIntegrate[Exp[-NIntegrate[μ01[x + xi] + μ02[xi] , {xi, t - 1, u}]]
   integriere n··· Exp··· integriere numerisch
     μ02[u], {u, t - 1, t}];

(* Use Functions *)

p   = P[0, #]     & /@ Range[1, n - t0]
                       Liste aufeinanderfolgender Zahlen
p00 = P00[t0, #] & /@ Range[t0 + 1, n]
                       Liste aufeinanderfolgender Zahlen
p01 = P01[t0, #] & /@ Range[t0 + 1, n]
                       Liste aufeinanderfolgender Zahlen
p02 = P02[t0, #] & /@ Range[t0 + 1, n]
                       Liste aufeinanderfolgender Zahlen
error  = p00 + p01 + p02 - Prepend[Delete[p00, -1], 1]
                           stelle vo··· lösche
```

Deterministic Reserves;

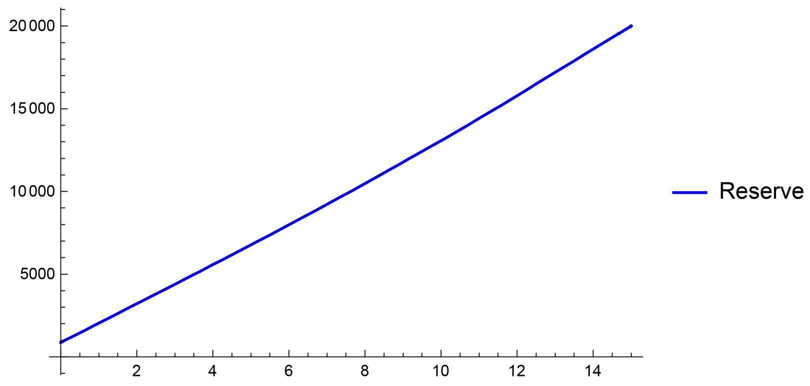
```
(* Solve Thiele's differential equation *)
   ⌊löse

sol = NDSolve[{
      ⌊löse Differentialgleichung numerisch
   v0'[t] == (r[t] + μ01[x + t] + μ02[t]) * v0[t] -
     μ01[x + t] * dBFT[t + 1] - μ02[t] * sBFT[t + 1] + pBFT,
   v0[n] == eBFT},
  {v0}, {t, 0, n}]


mvReserve = Evaluate[v0[#] /. sol] & /@ Range[0, n]
            ⌊werte aus                    ⌊Liste aufeinanderfolgender Zahlen
Plot[Evaluate[{v0[t]} /. sol], {t, 0, n},
⌊stell··· ⌊werte aus
 PlotStyle → {Blue}, PlotLegends → {"Reserve"}]
 ⌊Darstellungsstil ⌊blau   ⌊Legenden der Graphik
```

```
Out[79]= {{867.365}, {2036.88}, {3209.93}, {4387.79}, {5572.34},
 {6768.23}, {7981.32}, {9218.89}, {10481.9}, {11771.9},
 {13071.6}, {14413.6}, {15794.7}, {17196.8}, {18602.6}, {20000.}}
```

Out[80]=



Stochastic Transition Intensities;

```
(*  Model
    Y1(t)  = r(t)  = X1(t);
    Y2(t)  = η(t)  = η0(t)*X2(t);

    dX1(t)  =  (b1(t)+β1*X1(t))dt  +  σ1*(√(1-ρ^2)dW1(t)+ρ*√(X2(t))dW2(t));
    dX2(t)  =  (b2(t)+β2*X2(t))dt  +  σ2*(                   √(X2(t))dW2(t));
 *)

μ0[t_] := μ01[x + t]      (*deterministic mortality intensity           *);
η0[t_] := μ02[t]          (*deterministic surrender intensity           *);


b1[t_] := r[t]            (*mean reversion: interest-rate               *);
b2     := 1               (*mean reversion: surrender-deviation-rate    *);

β1     := -1              (*mean reversion: interest-rate               *);
β2     := -1              (*mean reversion: surrender-deviation-rate    *);

ρ      := 0.45 * 0;
σ1     := 0.05 * 0;
σ2     := 0.20 * 0;

(*initial values of interest and surrender-deviation-processes*)

ClearAll[XX, YY]
lösche alle
eqn1 = XX'[t] == +b1[t] - XX[t];
eqn2 = YY'[t] == b2 - YY[t];
sol  =
  NDSolve[{eqn1, eqn2, XX[t0] == 0, YY[t0] == 0}, {XX[t], YY[t]}, {t, t0, n}];
  löse Differentialgleichung numerisch

XX1[t]  = XX[t]   /. sol[[1]];
sol1    = XX1[t]  /. t → n;

YY1[t]  = YY[t]   /. sol[[1]];
sol2    = YY1[t]  /. t → n;
X       = {sol1, sol2}

Out[99]= {0.014087, 1.}
```

In[100]:=

Define functions to solve the systems of ODEs;

```
Intμtill[tk_] := NIntegrate[μ0[t], {t, t0, tk}];
                 integriere numerisch

Intηtill[tk_] := NIntegrate[η0[t], {t, t0, tk}];
                 integriere numerisch


stdPricing = Function[{η0, b1, b2, β1, β2, ρ, σ1, σ2, t0, tk},
             Funktion
   ClearAll[ϕ, ψ1, ψ2];
   lösche alle

   eqn1 =  ϕ'[t] == -1 / 2 (σ1^2 (1 - ρ^2) ψ1[t]^2) - b1[t] ψ1[t]  - b2 ψ2[t];
   eqn2 = ψ1'[t] == -β1 * ψ1[t] + 1;
   eqn3 =
    ψ2'[t] == -1 / 2 (σ1^2 ρ^2 ψ1[t]^2 + 2 σ1 σ2 ρ ψ1[t] ψ2[t] + σ2^2 ψ2[t]^2) -
      β2 ψ2[t] + η0[t];

   sol = NDSolve[{eqn1, eqn2, eqn3, ϕ[tk] == 0, ψ1[tk] == 0, ψ2[tk] == 0},
         löse Differentialgleichung numerisch
     {ϕ[t], ψ1[t], ψ2[t]}, {t, t0, tk}];

   ϕ[t]  = ϕ[t]  /. sol[[1]];
   ψ1[t] = ψ1[t] /. sol[[1]];
   ψ2[t] = ψ2[t] /. sol[[1]];

   sol1 = ϕ[t]  /. t → t0;
   sol2 = ψ1[t] /. t → t0;
   sol3 = ψ2[t] /. t → t0;                                    ];
```

```
xtdPricing = Function[{η0, b1, b2, β1, β2, ρ, σ1, σ2, t0, tk},
                     Funktion
    ClearAll[A, B1, B2];
    lösche alle

    eqn4 = A'[t]  == -σ1^2 (1-ρ^2) ψ1[t] B1[t] - b1[t] B1[t] - b2 B2[t];
    eqn5 = B1'[t] == -β1 B1[t];
    eqn6 = B2'[t] == -(σ1^2 ρ^2 ψ1[t] B1[t] +
           σ1 σ2 ρ (ψ1[t] B2[t] + ψ2[t] B1[t]) + σ2^2 ψ2[t] B2[t]) - β2 B2[t];

    sol = NDSolve[{eqn4, eqn5, eqn6, A[tk] == 0, B1[tk] == 0, B2[tk] == η0[tk]},
          löse Differentialgleichung numerisch
      {A[t], B1[t], B2[t]}, {t, t0, tk}];

    sol4 = A[t] /. sol[[1]]  /. t → t0;
    sol5 = B1[t] /. sol[[1]] /. t → t0;
    sol6 = B2[t] /. sol[[1]] /. t → t0;                          ];
```

Use the functions;

```
solution = {{"ϕ", "ψ1", "ψ2", "A", "B1", "B2"}};

For[tk = t0, tk ≤ n, tk++,
For-Schleife
  stdPricing[η0, b1, b2, β1, β2, ρ, σ1, σ2, t0, tk];
  xtdPricing[η0, b1, b2, β1, β2, ρ, σ1, σ2, t0, tk];
  solution = Append[solution, {sol1, sol2, sol3, sol4, sol5, sol6}]; ];
             hänge an

ϕ = Rest[solution[[All, 1]]]; (*column without the name*)
    alle außer erstes A··· alle
ψ = Transpose[{Rest[solution[[All, 2]]], Rest[solution[[All, 3]]]}];
    transponiere  alle außer erstes A··· alle    alle außer erstes A··· alle
A = Rest[solution[[All, 4]]];
    alle außer erstes A··· alle
B = Transpose[{Rest[solution[[All, 5]]], Rest[solution[[All, 6]]]}];
    transponiere  alle außer erstes A··· alle    alle außer erstes A··· alle
```

```
In[149]:= epv = Total[Exp[-Intμtill /@ Range[t0, n] + ϕ + ψ.X]
               Gesa··· Exponentialfunktion   Liste aufeinanderfolgender Zahlen
             ((eBFT * Boole[# == n] - pBFT * Boole[# ≠ n] + μ0[#] * dBFT[# + 1] & /@
                      charakteristische Booles···  charakteristische Boolesche Funktion
                Range[t0, n]) + (A + B.X) * (sBFT[# + 1] & /@ Range[t0, n]))];
                                                             Liste aufeinanderfolgende

         (* Output the results *)

         MatrixForm[solution]
         Matritzenform
         epv
```

Out[150]//MatrixForm=

ϕ	ψ1	ψ2	A	B1	B2
0.	0.	0.	0.	0.	0.0387742
-0.0140311	-0.632121	-0.025495	0.0267914	0.	0.015592
-0.0456958	-0.864665	-0.0356397	0.0390304	0.	0.00610896
-0.0859019	-0.950213	-0.039563	0.0442425	0.	0.00231812
-0.129973	-0.981684	-0.041029	0.045389	0.	0.000846837
-0.175083	-0.993262	-0.0415551	0.0436409	0.	0.000296044
-0.219239	-0.997521	-0.0417353	0.0396176	0.	0.000098446
-0.260942	-0.999088	-0.0417939	0.0339145	0.	0.0000309547
-0.298953	-0.999665	-0.0418119	0.02726	0.	9.14798×10^{-6}
-0.332491	-0.999877	-0.0418171	0.0204633	0.	2.52454×10^{-6}
-0.360851	-0.999955	-0.0418185	0.0142636	0.	6.47111×10^{-7}
-0.384901	-0.999983	-0.0418188	0.00917735	0.	1.53324×10^{-7}
-0.406354	-0.999994	-0.0418189	0.00541812	0.	3.33645×10^{-8}
-0.425841	-0.999998	-0.0418189	0.00291758	0.	6.62574×10^{-9}
-0.443434	-0.999999	-0.0418189	0.00142441	0.	1.33955×10^{-9}
-0.459118	-1.	-0.0418189	0.000626726	0.	2.07768×10^{-10}

Out[151]= 732.637

```
(* A little adaption for the grid on which
 the numerical integration is performed is quite
 wide: Try shifting continuous payments back in time by 0.35 *)

ϕ2 = Interpolation[Rest[solution[[All, 1]]]]; (*column without the name*)
     Interpolation      alle außer erstes A⋯ alle
ψ21 = Interpolation[Rest[solution[[All, 2]]]];
      Interpolation      alle außer erstes A⋯ alle
ψ22 = Interpolation[Rest[solution[[All, 3]]]];
      Interpolation      alle außer erstes A⋯ alle
A2 = Interpolation[Rest[solution[[All, 4]]]];
     Interpolation      alle außer erstes A⋯ alle
B21 = Interpolation[Rest[solution[[All, 5]]]];
      Interpolation      alle außer erstes A⋯ alle
B22 = Interpolation[Rest[solution[[All, 6]]]];
      Interpolation      alle außer erstes A⋯ alle
epv2 =
 Total[Exp[(-Intμtill /@ Range[t0 + 0.35, n - 0.65]) + (ϕ2[# + 0.35] + ψ21[# + 0.35]
 Gesa⋯ Exponentialfunktion    Liste aufeinanderfolgender Zahlen
           X[[1]] + ψ22[# + 0.35] X[[2]] & /@ Range[1, n - t0]) ]
                                          Liste aufeinanderfolgender Zahlen
    ((-pBFT + μ0[# - 0.35] * dBFT[# + 0.35] & /@ Range[t0 + 1, n]) +
                                          Liste aufeinanderfolgender Zahlen
      (A2[# + 0.35] + B21[# + 0.35] X[[1]] + B22[# + 0.35] X[[2]] & /@
        Range[1, n - t0]) * (sBFT[# + 0.35] & /@ Range[t0 + 1, n]))] + Total[
        Liste aufeinanderfolgender Zahlen                 Liste aufeinanderfolgende⋯ Gesamtsumme
   Exp[-Intμtill /@ Range[t0, n] + ϕ + ψ.X] (eBFT * Boole[# == n] & /@ Range[t0, n])]
   Exponentialfunktion  Liste aufeinanderfolgender Zahlen   charakteristische Bo⋯ Liste aufeinanderfolg

Out[125]= 865.547
```

6. Discussion

Book Value Model

Despite its crudeness, the statutory book value model is the most important model in life insurance business because all contract details like premiums, benefits, but also options and guarantees are defined accordingly to its values. Moreover, it is used for drawing up a balance sheet accordingly to the local accounting standards which makes it a crucial part within every simulation of future profit participation.

In a risk management context, however, it simply is a blunt tool. Although, due to the principle of precaution[1], the statutory book value model usually leads to quite risk averse policy parameters when the policy is issued, the economic situation can change dramatically during the term of contract turning an initially good portfolio into a toxic one. The model is then completely blind for the market situation. This means it does not only ignore future **systematic risks** but also systematic risks which have already realized since the policy was written.

Nonetheless, it has proven quite sufficient concerning **idiosyncratic risks** in the context of mortality. This is due to the large number of contracts and the independence between the events of death, which make the law of large numbers applicable. This, however, does not hold true when it comes to investment risk. Even though there are many investment possibilities as well, they are highly correlated and therefore the risk does not diminish in the same way. Hence market risks are not hedged away to the same extend and should therefore be modeled in a more sophisticated way.

The **book value algorithm** with its commutation tables on the other hand can be deemed a pure anachronism. The same goes for the implementation approach targeting reserve problems directly (without cash flows as interim results) by implementing separate expected present value functions for each tariff and for different purposes. The notion of a framework covering far

[1]It usually contains a deluge of loading parameters in the domains of costs, interest, mortality, profits, reinsurance, etc. to cushion all kinds of risks.

more in a single implementation while still being resource-friendly simply places a ban on it.

Extended Market Value Model

The market value model introduced in section (3.3) with its extensions from section (3.4) already deals with many of the statutory book value model's flaws. It calculates the prospective reserves using the prevailing interest rate from the market instead of the guaranteed interest rate, it uses unloaded parameters, it includes policy holder behavior like surrender and waiver of premiums and it determines the prospective profit reserve which is a pretty good approximation of the inner value of the profit participation guarantee.

It is still blind for future systematic and idiosyncratic risks but it considers the systematic risks which realized since the policies were issued and it can very successfully be used as a data clustering algorithm to support sophisticated Monte Carlo simulations in order to determine the inner value as well as the time value of the profit participation.

Affine Models

Poly-stochastic models with multidimensional time-inhomogeneous continuous affine processes as driving sources are rather new and so many questions will have to be answered before actuaries will add them to their standard repertoire but they already look very promising for the purpose of evaluating model assumptions in less sophisticated models because they allow for an astonishing modeling flexibility without Monte Carlo simulations.

Nonetheless they are way more complex than their certainty equivalence counterparts. This makes them harder to explain and thus harder to justify in front of the board. Furthermore it is questionable whether the approach will soon be applicable to whole portfolios because they do not allow for a direct modeling of management rules which is not only criticized by the board but also by the supervisory authorities.

Appendix A.

Interest Rates

[Bjö09], [MB00], [Cuc06],and [Fil09].

For $t \leq T \leq S \in \mathbb{R}_+$ let $\tau(T,S)$ denote the time between T and S and let $P(t,T)$ denote the value of a risk-free zero coupon bond at time t, which pays 1 at time T. It can thus be seen as a cash transport factor from time T back to time t applied to a nominal amount of 1. To ease the comparability of the following rates, the corresponding expression has been added to each interest rate definition.

Definition A.1 (*The simply compounded Spot Rate or Yield Curve*)

$$F_s(t;T) := \frac{1}{\tau(t,T)}\left(\frac{1}{P(t,T)} - 1\right) \iff P(t,T) = \frac{1}{1+\tau(t,T)F_s(t;T)}$$

Definition A.2 (*The simply compounded Forward Rate*)

$$F_s(t;T,S) := \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \iff P(t,S) = \frac{P(t,T)}{1+\tau(T,S)F_s(t;T,S)}$$

Definition A.3 (*The annually-compounded Forward Rate*)

$$F_a(t;T,S) := \left(\frac{P(t,T)}{P(t,S)}\right)^{1/\tau(T,S)} - 1 \iff P(t,S) = \frac{P(t,T)}{(1+F_a(t;T,S))^{\tau(T,S)}}$$

Definition A.4 (*The continuously-compounded Forward Rate*)

$$F_c(t;T,S) := \frac{\ln P(t,T) - \ln P(t,S)}{\tau(T,S)} \iff P(t,S) = P(t,T)e^{-\tau(T,S)F_c(t;T,S)}$$

year k	$F_c(0,k)$	$F_c(0,k-1,k)$	$F_c(0,k-2,k)$
1	3.0		
2	4.0	5.0	
3	4.6	5.8	5.4
4	5.0	6.2	6.0
5	5.3	6.5	6.35

Table A.1.: Numeric example for Spot and Forward Rates.

Infinitesimally Valid Interest Rates

In the sequel we are, however, exclusively using short-term interest rates. They are a theoretical concept and thus not directly observable.

Definition A.5 (*Instantaneous Forward Rate*)

$$\begin{aligned} f(t,T) :&= \lim_{S \downarrow T} F_s(t;T,S) = \lim_{S \downarrow T} -\frac{1}{P(t,S)} \frac{P(t,S) - P(t,T)}{\tau(S-T)} \\ &= -\frac{\partial \ln P(t,T)}{\partial T} \\ P(t,T) &= e^{-\int_t^T f(t,u)du} \end{aligned}$$

Definition A.6 (*Short Rate*)

$$r(t) := \lim_{T \downarrow t} f(t,T)$$

Here the bond price $P(t,T)$ can not be stated immediately. Assuming that t is the current time, all interest rates defined above are time t-measurable. This means they are stochastic as soon as t is replaced by $t^* \in (t, \infty)$. While the forward rate as a function of T is known at t, the short rate is only known in a single point at time t.

Lemma A.1

Consider the short rate r_t to be an adapted process and $\mathcal{F}_t$ to be the filtration which is generated by it. Hence, using an arbitrage argument, the price of a zero coupon bond satisfies

$$P(t,T) = \mathbb{E}^{Q}\left[\exp\left\{\left(-\int_t^T r_s ds\right)\right\}\middle|\mathcal{F}_t\right].$$

Thus, the bond prices are fixed once the short rate has been specified under a risk neutral measure Q. This is not true under the real world measure ([Bjö09] page 365)!

We can now define a stochastic compounding factor and a stochastic discount factor, which evolve accordingly to the short rate.

Definition A.7 (*Bank Account*)

Let the bank account be defined as the adapted process which satisfies

$$B(0) := 1 \quad \textit{and} \quad dB(t) := r(t)B(t)dt \tag{A.1}$$
$$\Rightarrow B(t) = e^{\int_0^t r(s)ds} \quad \forall t \in [0, T]. \tag{A.2}$$

Definition A.8 (*Stochastic Discount Factor*)

The stochastic discount factor is the value at time t of receiving one Euro at time T. It is thus defined by

$$D(t, T) := \frac{B(t)}{B(T)} = e^{-\int_t^T r(s)ds}.$$

There is a close relation between compounding, discounting and the bond price.

Definition A.9 (*Arbitrage-free family of bond prices*)

A family of bond prices $\{P(t, T) : t \leq T \leq T^*\}$ relative to the short rate r is called arbitrage free if

- *$P(T, T) = 1$ for all $T \in [0, T^*]$ and*
- *it exists an equivalent probability measure $\mathbb{Q}^*$ such that the discounted bond price*
$$\tilde{P}(t, T) = D(0, t)P(t, T) = \frac{P(t, T)}{B(t)}$$
is a martingale under $\mathbb{Q}^*$ i.e. measurable, with bounded expectation and
$$\tilde{P}(t, T) = \mathbb{E}_{\mathbb{Q}^*}[\tilde{P}(T, T)|\mathcal{F}_t] \quad \forall t \leq T.$$

Lemma A.2 (*Bond Prices and Discount Factors*)

An arbitrage-free bond price is the expectation of the corresponding discount factor taken under the martingale measure $\mathbb{Q}^*$:

$$
\begin{aligned}
P(t,T) &= \frac{\mathbb{E}_{\mathbb{Q}^*}[D(0,T)|\mathcal{F}_t]}{D(0,t)} && \text{(A.3)}\\
&= e^{\int_0^t r(s)ds}\mathbb{E}_{\mathbb{Q}^*}[e^{-\int_0^T r(s)ds}|\mathcal{F}_t] && \text{(A.4)}\\
&= \mathbb{E}_{\mathbb{Q}^*}[e^{-\int_t^T r(s)ds}|\mathcal{F}_t] && \text{(A.5)}\\
&= \mathbb{E}_{\mathbb{Q}^*}[D(t,T)|\mathcal{F}_t]. && \text{(A.6)}
\end{aligned}
$$

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