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Stabilized finite element methods for Dirichlet boundary control problems in fluid mechanics

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Preface

Hereby I want to thank Prof. Dr. O. Steinbach for giving me the possibility to work on this interesting topic and who always had time for questions. Moreover, I am thankful to Dr. G. Of for his support, and to all the other members of the Institute of Computational Mathematics. Also I want to thank the Graz University of Technology for the financial support through the scholarship "KUWI–Stipendium", whereby I could go to the University of Southern Denmark in the summer term of 2010. Moreover, I am thankful for the scholarship "Förderungsstipendium" of the Graz University of Technology. Thus, I had the possibility to visit the summer school "Discontinuous Galerkin Methods" in Dobbiaco (Italy), the CISM course "Computational Fluid–Structure Interaction" in Udine (Italy), and to participate at the "23. Chemnitz FEM Symposium" in Chemnitz (Germany).

Contents

Introduction

In this thesis we consider optimal control problems constrained by partial differential equations in fluid mechanics. In particular, we treat optimal Dirichlet boundary control problems for the standard equations of fluid mechanics, such as the stationary Stokes equations and the stationary Navier–Stokes equations. We can imagine such a problem, for instance, in the following way: Let us consider the flow around an obstacle, for example an airfoil, where on a part of the boundary a control boundary is considered. On this control boundary we are able to control some inflow or outflow. The aim is to find the optimal control on the control boundary, such that the lift of the airfoil becomes maximal.
Another example would be to find the control in such a way, such that the flow around the airfoil approximates some given desired velocity best. We present such an example. There are several other interpretations of optimal control problems in fluid mechanics, which are reasonable and necessary, see for example [11, 16, 18, 19, 34]. For a mathematical overview on fluids we refer to [7, 12, 13, 15, 20, 26].
More generally, we can describe such optimal control problems by the minimization of certain cost functionals, where the constraint is a partial differential equation. In our case these partial differential equations are either the Stokes equations or the Navier–Stokes equations. This means that the control and the state, i.e. the velocity, are both coupled by a partial differential equation. Furthermore, such an optimal control problem leads to a system of coupled partial differential equations, which we are going to solve. This will be done by the finite element method, see [4, 6, 15, 26, 31]. Actually, we do so by using stabilized finite element methods, which gives us the latitude of using arbitrary elements. These methods, have been developed for the Stokes equations, see for example [3, 9, 14, 21, 22, 25], and for the Navier–Stokes equations, see for example [8, 10, 17, 24, 32, 33]. But they can be used also for other partial differential equations, for example advection–diffusion equations or convection dominated problems.
In most cases the boundary control for an optimal control problem is considered in the space $[L_2(\Gamma)]^n$. We will show, that in most cases it is more reasonable to consider the energy space $\left[H^{1/2}(\Gamma)\right]^n$ instead, see also [23]. For an overview on these Sobolev spaces we refer to [1, 15, 31]. In particular we will show how to realize the energy space in a finite element approach and discuss the difference of the two control spaces.

This thesis is organized as follows: In the first chapter we start with the description of optimal control problems in fluid mechanics and describe the problem about the different control spaces. In the second part we consider the optimal control problem for the Stokes equations in more detail. We repeat the existence and uniqueness results for a solution of the Stokes equations and prove the unique solvability of the corresponding optimal control problem, which ends by the derivation of the optimality system as an equivalent problem. In the third part we consider the optimal control problem of the Navier–Stokes equations. We comment on uniqueness results for the Navier–Stokes equations and derive the corresponding optimality system.

In the second chapter we consider stabilized finite elements methods. In the first part, these methods are considered for the Stokes equations, where we prove stability of the methods and some error estimates. Moreover, we give some related numerical results, where we focus on the choice of the stabilization parameters. These results confirm the theoretical error estimates. In the second part, we introduce a stabilized finite element formulation for the Navier–Stokes equations and give some numerical results.

In the third chapter we combine the ideas of the first and second chapter. First we consider a stabilized finite element method for the optimal control problem of the Stokes equations. We present some related numerical results and discuss the difference of a realization of the control in $[L_2(\Gamma)]^n$, and in the energy space $\left[H^{1/2}(\Gamma)\right]^n$. In the second part we consider a stabilized finite element method for the optimal control problem of the Navier–Stokes equations. We give some related numerical results and again focus on the difference of the control spaces. Afterwards, we consider a more realistic example, where the optimal control of an airfoil is considered. Again we focus on the difference of the realization of the control in $[L_2(\Gamma)]^n$, and in the energy space $\left[H^{1/2}(\Gamma)\right]^n$.

1 An optimal control problem

In this chapter we consider some optimal control problems in fluid mechanics. We already mentioned in the introduction the meaning of these kinds of problems. In the first section we describe optimal control problems for different state equations, such as the Stokes or the Navier–Stokes equations. In the second part we treat the optimal control problem for the Stokes equations. We first give a short review on results on the unique solvability of the Stokes equations itself. Afterwards we prove existence and uniqueness of the solution for the optimal control problem. Furthermore we derive the adjoint equations and the corresponding optimality system. In the third section we consider the Navier–Stokes equations. Here we comment on the existence and uniqueness of the solution under certain assumptions. In addition we derive the corresponding optimality system.

1.1 Model problems

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. We consider the stationary Navier–Stokes equations, with four different types of boundary conditions, which are formulated at mutually different parts of the boundary: Γ_{ns} for a noslip boundary, Γ_{in} for an inflow boundary, Γ_{out} for an outflow boundary, and Γ_{c} for a control boundary, where we assume $\Gamma = \overline{\Gamma}_{\mathrm{ns}} \cup \overline{\Gamma}_{\mathrm{in}} \cup \overline{\Gamma}_{\mathrm{out}} \cup \overline{\Gamma}_{\mathrm{c}}$. In the following we denote by $\underline{u}$ and p the velocity and the pressure, respectively, and the control by $\underline{z}$. Our aim is to determine the control in such a way, such that the velocity, i.e. the state, is the best possible approximation of a given desired velocity $\overline{\underline{u}}$. Such an optimal control problem is given as follows: Minimize the cost functional

$$\mathcal{J}(\underline{u}, \underline{z}) := \frac{1}{2} \, \|\underline{u} - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n} + \frac{1}{2}\varrho \langle A\underline{z}, \underline{z}\rangle_{\Gamma_{\mathrm{c}}} \tag{1.1}$$

under the constraint

$$\begin{aligned} -\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{0} && \text{on } \Gamma_{\mathrm{ns}}, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_{\mathrm{in}}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{out}}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_{\mathrm{c}}, \end{aligned} \tag{1.2}$$

and where the control satisfies the box constraints

$$z_{a,i} \le z_i \le z_{b,i} \qquad \text{a.e. on } \Gamma_{\mathrm{c}},\ i = 1, \ldots, n. \tag{1.3}$$

The constraints $\underline{z}_a$ and $\underline{z}_b$ are assumed to be smooth enough. The given constant $\varrho > 0$ denotes the cost coefficient, and $\nu > 0$ is the viscosity constant. The former can be understood as measure of the costs to realize the control, or from the mathematical point of view as a regularization. The operator A will be chosen in such a way, that the duality product induces either the $[L_2(\Gamma_c)]^n$ norm, or the $\left[H^{1/2}(\Gamma_c)\right]^n$ semi–norm, this will be discussed later in more detail.
Since the constraint, the Navier–Stokes equations (1.2), is nonlinear, we will introduce a linearization of this problem. If we consider instead of the nonlinear term $(\underline{u}_0 \cdot \nabla)\underline{u}$, where $\underline{u}_0$ is given, we obtain the Oseen equations. If we neglect the nonlinearity, the state equations become the Stokes equations:

$$\begin{aligned} -\nu\Delta\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla \cdot \underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{0} && \text{on } \Gamma_{\mathrm{ns}}, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_{\mathrm{in}}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{out}}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_{\mathrm{c}}. \end{aligned} \tag{1.4}$$

This linearization makes sense when we consider fluids with slow motion and with high viscosity, i.e. a low Reynolds number. An overview on different fluid flow models can be found, for example, in [7].

Remark 1.1. *If we consider the weak formulation of (1.2) or (1.4), then $\underline{u}$ is considered in $[H^1(\Omega)]^n$ or in an appropriate subspace, depending on the boundary conditions. By the trace theorem, see for example [31], the trace of a $[H^1(\Omega)]^n$ function is in $[H^{1/2}(\Gamma)]^n$. From this point of view, we need to find an optimal control in $[H^{1/2}(\Gamma)]^n$, and that is why the operator A in the cost functional (1.1) should be a map $A : [H^{1/2}(\Gamma)]^n \to [H^{-1/2}(\Gamma)]^n$, such that the duality product on the boundary is well defined. Such a mapping can be realized, for e.g., by the so called Steklov–Poincaré operator S, see for example [31]. This operator induces the following semi–norm*

$$|\underline{z}|^2_{\left[H^{1/2}(\Gamma)\right]^n} = \langle S\underline{z}, \underline{z}\rangle_\Gamma .$$

In many cases the operator A is just realized by the identity and so we find the optimal control in $[L_2(\Gamma)]^n$. The difference of considering the control in $[L_2(\Gamma)]^n$ or $\left[H^{1/2}(\Gamma)\right]^n$ is also discussed in the current paper [23], where the Poisson equation is considered. For a polygonal or polyhedral bounded domain it turns out that, if the control is going to be considered in $L_2(\Gamma)$, it vanishes in each corner point of the domain, but not for a control in $H^{1/2}(\Gamma)$. So there is also a difference from a physical point of view. Moreover, this effect has an influence on the order of convergence of the control, when we discretize this problem, see [23]. We will see that there are similar effects for optimal control problems for the Stokes and the Navier–Stokes equations. Furthermore, we describe the Steklov–Poincaré operator in the case of an open subset of the boundary, which we need to consider mixed boundary value problems.

With reference to the cost functional (1.1), there are several other choices possible and reasonable, for example:

$$\mathcal{J}(\underline{u}, \underline{z}) = \nu \int_\Omega \sum_{i=1}^n |\nabla u_i|^2 \, dx + \frac{1}{4}\varrho \int_\Gamma |\underline{z} - \underline{g}|^2 \underline{z} \cdot \underline{n} \, ds_x,$$

see [11], and

$$\mathcal{J}(\underline{u}, \underline{z}) = \mathcal{F}(\underline{u}, p) + \frac{1}{2}\varrho(|\underline{z}|^2_{[H^2(\Gamma_c)]^n} + \|\underline{z}\|^2_{[L_2(\Gamma_c)]^n}),$$

with some appropriate functional $\mathcal{F}(\cdot,\cdot)$, see [18]. More details can be found in [11, 16, 18, 19].
In the following we will consider the optimal control problem for the Stokes and Navier–Stokes equations in more detail.

1.2 Stokes equations

In this section we focus on the linearized optimal control problem, where the state is described by the Stokes equations. We consider the boundary $\Gamma = \overline{\Gamma}_\mathrm{D} \cup \overline{\Gamma}_\mathrm{N} \cup \overline{\Gamma}_\mathrm{c}$ with a Dirichlet boundary Γ_D, a Neumann boundary Γ_N, and a control boundary Γ_c. For this case the optimal control problem for the Stokes equations is given as follows: Minimize the cost functional

$$\mathcal{J}(\underline{u}, \underline{z}) := \frac{1}{2} \|\underline{u} - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n} + \frac{1}{2}\varrho\langle S\underline{z}, \underline{z}\rangle_{\Gamma_\mathrm{c}} \tag{1.5}$$

under the constraint

$$\begin{aligned} -\nu\Delta\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}, \end{aligned} \tag{1.6}$$

where the control satisfies the box constraints

$$z_{a,i} \le z_i \le z_{b,i} \qquad \text{a.e. on } \Gamma_\mathrm{c}, \; i = 1, \dots, n. \tag{1.7}$$

The aim of this section is to prove the unique solvability of the optimal control problem (1.5)–(1.7) and to derive the corresponding optimality system.

1.2.1 Unique solvability of the state equations

In the following we derive the variational formulation for the Stokes equations and prove the existence and uniqueness of the solution. This will be done for a mixed boundary value problem, where $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$ with $\mathrm{meas}(\Gamma_{\mathrm{D}}) > 0$ and $\mathrm{meas}(\Gamma_{\mathrm{N}}) > 0$, which is given by

$$\begin{aligned} -\nu\Delta \underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla \cdot \underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_{\mathrm{D}}, \\ \nu(\nabla \underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{1.8}$$

Remark 1.2. *Note that in (1.6) the control boundary condition* $\underline{u} = \underline{z}$ on Γ_{c} *can be interpreted as a Dirichlet boundary condition. Due to this reason it is enough to consider problem (1.8).*

The derivation of the corresponding variational formulation can be found for example in [4, 6, 15, 26, 31] and is given as follows: Find $(\underline{u}, p) \in [H^1(\Omega)]^n \times L_2(\Omega)$ with $\underline{u} = \underline{g}$ on Γ_{D}, such that

$$\begin{aligned} &a(\underline{u}, \underline{v}) - b(\underline{v}, p) = \langle \underline{f}, \underline{v} \rangle_\Omega, \\ &b(\underline{u}, q) \qquad\quad\;\; = 0 \end{aligned} \tag{1.9}$$

is satisfied for all test functions $(\underline{v}, q) \in [H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n \times L_2(\Omega)$, where the related bilinear forms are given by

$$a(\underline{u}, \underline{v}) = \nu \int_\Omega \sum_{i=1}^n \nabla u_i \cdot \nabla v_i \; dx \quad \text{and} \quad b(\underline{v}, p) = \int_\Omega p \; \nabla \cdot \underline{v} \; dx. \tag{1.10}$$

These bilinear forms are both bounded, i.e.

$$a(\underline{u}, \underline{v}) \leq c_2^A \, \|\underline{u}\|_{[H^1(\Omega)]^n} \, \|\underline{v}\|_{[H^1(\Omega)]^n} \quad \text{and} \quad b(\underline{v}, q) \leq c_2^B \, \|\underline{v}\|_{[H^1(\Omega)]^n} \, \|q\|_{L_2(\Omega)}$$

are satisfied for all $\underline{u}, \underline{v} \in [H^1(\Omega)]^n$ and $q \in L_2(\Omega)$. In addition the bilinear form $a(\cdot, \cdot)$ is $[H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n$–elliptic, see for example [31], i.e.

$$a(\underline{v}, \underline{v}) \geq c_1^A \, \|\underline{v}\|^2_{[H^1(\Omega)]^n}$$

for all $\underline{v} \in [H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n := \left\{ \underline{v} \in [H^1(\Omega)]^n \;\middle|\; \underline{v} = \underline{0} \quad \text{on } \Gamma_{\mathrm{D}} \right\}$.

Remark 1.3. *If we consider the Dirichlet problem, i.e.* $\Gamma = \Gamma_{\mathrm{D}}$ *in (1.8), we have to assume the solvability condition*

$$\int_\Gamma \underline{g} \cdot \underline{n} \; ds_x = 0.$$

Moreover, the pressure p *is only unique up to an additive constant. For this reason we introduce* $L_{2,0}(\Omega) \subset L_2(\Omega)$*, where* $p \in L_{2,0}(\Omega)$ *satisfies the scaling condition* $\int_\Omega p \; dx = 0$*, see [31].*

In the following we prove the unique solvability of the problem (1.9). Therefore we need the following result on the inf–sup condition.

Lemma 1.1. *Let Ω be a bounded and simply connected Lipschitz domain. Then the inf–sup condition*

$$c_s \|q\|_{L_2(\Omega)} \le \sup_{\underline{0}\neq\underline{v}\in\left[H_0^1(\Omega)\right]^n} \frac{b(\underline{v},q)}{\|\underline{v}\|_{[H^1(\Omega)]^n}} \tag{1.11}$$

for all $q \in L_{2,0}(\Omega)$ is satisfied with a positive constant $c_s > 0$.

Proof. From the Nečas inequality, see [5],

$$\|q\|_{L_2(\Omega)} \le c_N \|\nabla q\|_{[H^{-1}(\Omega)]^n}$$

for all $q \in L_{2,0}(\Omega)$, and the norm definition of the space $[H^{-1}(\Omega)]^n$ we get

$$\|q\|_{L_2(\Omega)} \le c_N \sup_{\underline{0}\neq\underline{w}\in\left[H_0^1(\Omega)\right]^n} \frac{\langle \nabla q, \underline{w}\rangle_\Omega}{\|\underline{w}\|_{[H^1(\Omega)]^n}} = c_N \sup_{\underline{0}\neq\underline{w}\in\left[H_0^1(\Omega)\right]^n} \frac{-b(\underline{w},q)}{\|\underline{w}\|_{[H^1(\Omega)]^n}}$$

for all $q \in L_{2,0}(\Omega)$. We obtain the inf–sup condition (1.11) by setting $\underline{v} = -\underline{w}$. □

Remark 1.4. *The inf–sup condition is also valid for spaces $[H_0^1(\Omega,\Gamma_{\mathrm{D}})]^n$ and $L_2(\Omega)$ as needed for the mixed boundary value problem (1.9), see for example [6].*

Lemma 1.1 and Remark 1.4 are important results to show the unique solvability of the variational problem (1.9).

Theorem 1.1. *Let Ω be a bounded, simply connected Lipschitz domain, $\underline{f} \in \left[\widetilde{H}^{-1}(\Omega)\right]^n$ and $\underline{g} \in \left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n$. Then there exists a unique solution $(\underline{u},p) \in [H^1(\Omega)]^n \times L_2(\Omega)$ with $\underline{u} = \underline{g}$ on Γ_{D} of the problem (1.9) and the stability estimates*

$$\|\underline{u}\|_{[H^1(\Omega)]^n} \le \frac{1}{c_1^A} \|\underline{f}\|_{\left[\widetilde{H}^{-1}(\Omega)\right]^n} + \left(1 + \frac{c_2^A}{c_1^A}\right) c_g \|\underline{g}\|_{\left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n} \tag{1.12}$$

and

$$\|p\|_{L_2(\Omega)} \le \frac{1}{c_s}\left(1 + \frac{c_2^A}{c_1^A}\right)\left(\|\underline{f}\|_{\left[\widetilde{H}^{-1}(\Omega)\right]^n} + c_g c_2^A \|\underline{g}\|_{\left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n}\right) \tag{1.13}$$

are satisfied.

Proof. For $\underline{g} \in \left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n$ there exists an extension $\widetilde{\underline{g}} \in \left[H^{1/2}(\Gamma)\right]^n$ with

$$\|\widetilde{\underline{g}}\|_{\left[H^{1/2}(\Gamma)\right]^n} \le \tilde{c}\, \|\underline{g}\|_{\left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n}.$$

By the inverse trace theorem there exists a bounded extension $\underline{u}_{\widetilde{g}} \in [H^1(\Omega)]^n$ such that

$$\|\underline{u}_{\widetilde{g}}\|_{\left[H^{1/2}(\Gamma)\right]^n} \le c_g \|\underline{g}\|_{\left[H^{1/2}(\Gamma_{\mathrm{D}})\right]^n}$$

is satisfied. For the variational problem (1.9) we now obtain the following formulation: Find $(\underline{u}_0, p) \in [H^1_0(\Omega, \Gamma_\mathrm{D})]^n \times L_2(\Omega)$, such that

$$\begin{aligned} a(\underline{u}_0, \underline{v}) - b(\underline{v}, p) &= \langle \underline{f}, \underline{v}\rangle_\Omega - a(\underline{u}_{\widetilde{g}}, \underline{v}), \\ b(\underline{u}_0, q) &= -b(\underline{u}_{\widetilde{g}}, q) \end{aligned} \tag{1.14}$$

is satisfied for all test functions $(\underline{v}, q) \in [H^1_0(\Omega, \Gamma_\mathrm{D})]^n \times L_2(\Omega)$. The bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are bounded and $a(\cdot,\cdot)$ is in addition $[H^1_0(\Omega, \Gamma_\mathrm{D})]^n$–elliptic. The bilinear form $b(\cdot,\cdot)$ induces an opterator $B : [H^1_0(\Omega, \Gamma_\mathrm{D})]^n \to L_2(\Omega)$ with

$$\ker B := \left\{ \underline{v} \in \left[H^1_0(\Omega, \Gamma_\mathrm{D})\right]^n : \nabla \cdot \underline{v} = 0 \right\} \subset \left[H^1_0(\Omega, \Gamma_\mathrm{D})\right]^n,$$

from which we conclude the $\ker B$–ellipticity of $a(\cdot,\cdot)$. Now, all assumptions of the abstract theorem of saddle point problems are satisfied from which we conclude existence and uniqueness of the solution, see [31, Theorem 3.11]. Furthermore, we get by this theorem the desired stability estimates (1.12)–(1.13), see also [4, 6, 15, 26]. □

1.2.2 Unique solvability of the optimal control problem

Now we are in a position to prove the unique solvability of the optimal control problem for the Stokes equations, which is given by (1.5)–(1.7) for a control in $\left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$. Let $\underline{f} \in \left[\widetilde{H}^{-1}(\Omega)\right]^n$, $\overline{\underline{u}} \in [L_2(\Omega)]^n$ and $\underline{z}_a, \underline{z}_b \in \left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$ be satisfied. Moreover we set the operator $A = S$ in the cost functional (1.5), where S denotes the Steklov–Poincaré operator, see Remark 1.1. More precisely, we realize this operator by solving the following homogeneous boundary value problem

$$\begin{aligned} -\Delta \underline{u} &= \underline{0} && \text{in } \Omega, \\ \underline{u} &= \underline{0} && \text{on } \Gamma_\mathrm{D}, \\ (\nabla \underline{u})\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}. \end{aligned} \tag{1.15}$$

The corresponding variational formulation is given as follows: Find $\underline{u} \in [H^1_0(\Omega, \Gamma_\mathrm{D})]^n$ with $\underline{u} = \underline{z}$ on Γ_c, such that

$$\int_\Omega \sum_{i=1}^n \nabla u_i \cdot \nabla v_i \, dx = 0$$

is satisfied for all $\underline{v} \in [H^1_0(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n$. Thus the Steklov–Poincaré operator is given by

$$S\underline{z} = (\nabla \underline{u})\underline{n} \qquad \text{on } \Gamma_\mathrm{c},$$

which realizes the Dirichlet to Neumann map.

Remark 1.5. *Another possibility would be to realize the Steklov–Poincaré operator via the solution of a homogeneous boundary value problem of the Stokes equations. In this case the operator is given by*

$$S\underline{z} = \nu(\nabla\underline{u})\underline{n} - p\underline{n} \qquad \text{on } \Gamma_{\mathrm{c}}$$

with some additional pressure term.

For the mixed boundary value problem (1.15) we have to distinguish different situations, which leads to different spaces for the control:

(i) If the complete boundary is considered as a control boundary, i.e. $\Gamma = \Gamma_{\mathrm{c}}$, we consider the control in $[H^{1/2}(\Gamma)]^n$ and the Steklov–Poincaré operator is given by the mapping $S : [H^{1/2}(\Gamma)]^n \to [H^{-1/2}(\Gamma)]^n$.

(ii) If the control boundary is an open subset of the boundary, i.e. $\Gamma_{\mathrm{c}} \subset \Gamma$ and $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}}$ such that $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{c}} \neq \emptyset$ and $\overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}} = \emptyset$ holds, we consider the control in $[H_{00}^{1/2}(\Gamma_{\mathrm{c}})]^n := \{\underline{v} = \tilde{\underline{v}}|_{\Gamma_{\mathrm{c}}} : \tilde{\underline{v}} \in \left[H^{1/2}(\Gamma)\right]^n, \mathrm{supp}(\tilde{\underline{v}}) \subset \Gamma_{\mathrm{c}}\}$. The Steklov–Poincaré operator is then given by the mapping $S : [H_{00}^{1/2}(\Gamma_{\mathrm{c}})]^n \to [H^{-1/2}(\Gamma_{\mathrm{c}})]^n$.

Here we restrict ourself to these two cases, since we only need those later on. In a similar way we can treat also the cases $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{c}} = \emptyset$, $\overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}} \neq \emptyset$ and $\overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{c}} \neq \emptyset$, $\overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}} \neq \emptyset$. Now we introduce the space Z which is either $[H^{1/2}(\Gamma)]^n$, if we consider case (i), or $[H_{00}^{1/2}(\Gamma_{\mathrm{c}})]^n$, if we consider case (ii). It is important to mention that the dual space $Z^* = [H^{-1/2}(\Gamma_{\mathrm{c}})]^n$ for both cases, see for example [31].

Lemma 1.2. *The Steklov–Poincaré operator $S : Z \to Z^*$ is bounded and self–adjoint.*

Proof. We consider $\underline{v} \in Z$ with an extension $\mathcal{E}\underline{v} \in [H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n$. From the boundary value problem (1.15) and the inverse trace theorem we obtain

$$\begin{aligned} \|S\underline{z}\|_{Z^*} &= \sup_{\underline{0}\neq\underline{v}\in Z} \frac{\langle(\nabla\underline{u})\underline{n}, \underline{v}\rangle_{\Gamma_{\mathrm{c}}}}{\|\underline{v}\|_{\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n}} = \sup_{\underline{0}\neq\underline{v}\in Z} \frac{\langle\nabla\underline{u}, \nabla\mathcal{E}\underline{v}\rangle_{\Omega}}{\|\underline{v}\|_{\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n}} \\ &\leq \sup_{\underline{0}\neq\underline{v}\in Z} \frac{\|\underline{u}\|_{[H^1(\Omega)]^n}\, \|\mathcal{E}\underline{v}\|_{[H^1(\Omega)]^n}}{\|\underline{v}\|_{\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n}} \leq c\, \|\underline{z}\|_{\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n} \end{aligned}$$

for all $\underline{z} \in Z$. Moreover we introduce a solution operator $\widetilde{\mathcal{H}} : Z \to [H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n$. For $\underline{z}, \underline{w} \in Z$ let $\underline{u} = \widetilde{\mathcal{H}}\underline{z}$, $\underline{v} = \widetilde{\mathcal{H}}\underline{w}$ be solutions of the related boundary value problem (1.15). Then there holds

$$\begin{aligned} \langle S\underline{z}, \underline{w}\rangle_{\Gamma_{\mathrm{c}}} &= \langle(\nabla\underline{u})\underline{n}, \underline{w}\rangle_{\Gamma_{\mathrm{c}}} = \int_{\Omega} \sum_{i=1}^{n} \nabla u_i \cdot \nabla v_i \, dx = -\int_{\Omega} \underline{u} \cdot \Delta\underline{v} \, dx + \int_{\Gamma} \underline{u} \cdot (\nabla\underline{v})\underline{n} \, ds_x \\ &= \langle\underline{z}, (\nabla\underline{v})\underline{n}\rangle_{\Gamma_{\mathrm{c}}} = \langle\underline{z}, S\underline{w}\rangle_{\Gamma_{\mathrm{c}}} \end{aligned}$$

for all $\underline{z}, \underline{w} \in Z$. □

In the following we split the solutions of the state equations (1.6) into a homogeneous and an inhomogeneous part. Those are given by the following two problems. We first consider the problem

$$\begin{aligned} -\nu\Delta\underline{u}_f + \nabla p_f &= \underline{f} && \text{in } \Omega,\\ \nabla\cdot\underline{u}_f &= 0 && \text{in } \Omega,\\ \underline{u}_f &= \underline{g} && \text{on } \Gamma_{\mathrm{D}},\\ \nu(\nabla\underline{u}_f)\underline{n} - p_f\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}},\\ \underline{u}_f &= \underline{0} && \text{on } \Gamma_{\mathrm{c}}, \end{aligned}$$

with the corresponding variational formulation: Find $(\underline{u}_f, p_f) \in [H_0^1(\Omega,\Gamma_{\mathrm{c}})]^n \times L_2(\Omega)$ with $\underline{u}_f = \underline{g}$ on Γ_{D}, such that

$$\begin{aligned} &a(\underline{u}_f,\underline{v}) - b(\underline{v},p_f) = \langle \underline{f},\underline{v}\rangle_\Omega,\\ &b(\underline{u}_f,q) \qquad\qquad\;\; = 0 \end{aligned}$$

is satisfied for all test functions $(\underline{v},q) \in [H_0^1(\Omega,\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}})]^n \times L_2(\Omega)$. Furthermore, we consider the problem

$$\begin{aligned} -\nu\Delta\underline{u}_z + \nabla p_z &= \underline{0} && \text{in } \Omega,\\ \nabla\cdot\underline{u}_z &= 0 && \text{in } \Omega,\\ \underline{u}_z &= \underline{0} && \text{on } \Gamma_{\mathrm{D}},\\ \nu(\nabla\underline{u}_z)\underline{n} - p_z\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}},\\ \underline{u}_z &= \underline{z} && \text{on } \Gamma_{\mathrm{c}}. \end{aligned}$$

By the inverse trace theorem, see [31], for $\underline{z} \in Z$ there exists a bounded extension $\mathcal{E}\underline{z} \in [H_0^1(\Omega,\Gamma_{\mathrm{D}})]^n$. We introduce $\underline{u}_z = \underline{u}_0 + \mathcal{E}\underline{z}$, thus the variational formulation for the homogeneous problem reads: Find $(\underline{u}_0, p_z) \in [H_0^1(\Omega,\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}})]^n \times L_2(\Omega)$, such that

$$\begin{aligned} &a(\underline{u}_0,\underline{v}) - b(\underline{v},p_z) = -a(\mathcal{E}\underline{z},\underline{v}),\\ &b(\underline{u}_0,q) \qquad\qquad\;\; = -b(\mathcal{E}\underline{z},q) \end{aligned}$$

is satisfied for all test functions $(\underline{v},q) \in [H_0^1(\Omega,\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}})]^n \times L_2(\Omega)$. Now the unique solution of the state equations (1.6) can be written as $\underline{u} = \underline{u}_z + \underline{u}_f$ and $p = p_z + p_f$.
Due to the compact imbedding $[H^1(\Omega)]^n \subset [L_2(\Omega)]^n$ we introduce the solution operator $\mathcal{H} : Z \to [L_2(\Omega)]^n$ such that $\underline{u}_z = \mathcal{H}\underline{z}$ holds. Now we can introduce the reduced cost functional

$$\begin{aligned} \widetilde{\mathcal{J}}(\underline{z}) :=& \frac{1}{2}\left\|\mathcal{H}\underline{z} + \underline{u}_f - \overline{\underline{u}}\right\|^2_{[L_2(\Omega)]^n} + \frac{1}{2}\varrho\langle S\underline{z},\underline{z}\rangle_{\Gamma_{\mathrm{c}}}\\ =& \frac{1}{2}\left\|\mathcal{H}\underline{z} - \overline{\underline{w}}\right\|^2_{[L_2(\Omega)]^n} + \frac{1}{2}\varrho\langle S\underline{z},\underline{z}\rangle_{\Gamma_{\mathrm{c}}} \end{aligned}$$

where we used $\overline{\underline{w}} := \overline{\underline{u}} - \underline{u}_f$. If we define the set of all admissible controls as

$$Z_{ad} := \{\underline{z} \in Z : z_{a,i} \le z_i \le z_{b,i} \quad \text{a.e. on } \Gamma_{\mathrm{c}},\ i = 1,\dots,n\},$$

the optimal control problem (1.5)–(1.7) can be written as the following reduced minimization problem: Find the optimal control $\hat{\underline{z}} \in Z_{ad}$ such that

$$\widetilde{\mathcal{J}}(\hat{\underline{z}}) = \min_{\underline{z} \in Z_{ad}} \widetilde{\mathcal{J}}(\underline{z}). \tag{1.16}$$

The following theorem shows the existence and uniqueness of an optimal control of the reduced minimization problem (1.16).

Theorem 1.2. [34, Theorem 2.14] *Let $(Z, \langle \cdot, \cdot \rangle_Z)$ and $(X, \langle \cdot, \cdot \rangle_X)$ be real Hilbert spaces, and $Z_{ad} \subseteq Z$ a nonempty, closed, bounded and convex subset. Moreover, let $\varrho \geq 0$, $\underline{\overline{w}} \in X$ and $\mathcal{H} : Z \to X$, $A : Z \to Z^*$ be linear, continuous operators, where A is in addition positive semi–definite. Then for the reduced minimization problem*

$$\min_{\underline{z} \in Z_{ad}} \widetilde{\mathcal{J}}(\underline{z}) = \min_{\underline{z} \in Z_{ad}} \left\{ \frac{1}{2} \|\mathcal{H}\underline{z} - \underline{\overline{w}}\|_X^2 + \frac{1}{2} \varrho \langle A\underline{z}, \underline{z} \rangle_{Z^* \times Z} \right\}$$

there exists an optimal solution. If $\varrho > 0$ or $\mathcal{H}$ is injective, the solution is unique.

For the existence and uniqueness of the solution of the reduced minimization problem (1.16) and accordingly of the optimal control problem (1.5)–(1.7), we have to check the assumptions of Theorem 1.2. The solution operator $\widetilde{\mathcal{H}} : Z \to [H_0^1(\Omega, \Gamma_{\mathrm{D}})]^n$ is linear and bounded, where the latter follows from the stability estimate (1.12). The properties of the Steklov–Poincaré operator follow from Lemma 1.2. It remains to prove the assumptions on the set of admissible controls Z_{ad}, which are given by the following lemma.

Lemma 1.3. [34] *The set of admissible controls Z_{ad} is a nonempty, closed, bounded and convex subset of Z.*

Now all assumptions of Theorem 1.2 are satisfied and we can conclude the unique solvability of the reduced minimization problem (1.16), and of the corresponding optimal control problem (1.5)–(1.7) for a cost coefficient $\varrho > 0$.

1.2.3 Optimality system

In this part we derive the optimality system, which is an equivalent problem to the optimal control problem (1.5)–(1.7). First of all, the following theorem provides an equivalent formulation of the reduced minimization problem.

Theorem 1.3. [34, Theorem 2.22] *Let $(Z, \langle \cdot, \cdot \rangle_Z)$ and $(X, \langle \cdot, \cdot \rangle_X)$ be real Hilbert spaces. Under the assumptions of Theorem 1.2, and if in addition the operator $A : Z \to Z^*$ is self adjoint, the following two statements are equivalent:*

$$(i) \qquad \widetilde{\mathcal{J}}(\hat{\underline{z}}) = \min_{\underline{z} \in Z_{ad}} \widetilde{\mathcal{J}}(\underline{z}) = \min_{\underline{z} \in Z_{ad}} \left\{ \frac{1}{2} \|\mathcal{H}\underline{z} - \underline{\overline{w}}\|_X^2 + \frac{1}{2} \varrho \langle A\underline{z}, \underline{z} \rangle_{Z^* \times Z} \right\},$$

$$(ii) \qquad \langle \mathcal{H}\hat{\underline{z}} - \underline{\overline{w}}, \mathcal{H}(\underline{z} - \hat{\underline{z}}) \rangle_X + \varrho \langle A\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{Z^* \times Z} \geq 0 \quad \textit{for all } \underline{z} \in Z_{ad}.$$

Next we introduce the corresponding adjoint operator $\mathcal{H}^* : [L_2(\Omega)]^n \to Z^*$, which is needed for applying Theorem 1.3. The following theorem details on the adjoint problem of the Stokes equations and the adjoint solution operator $\mathcal{H}^*$.

Theorem 1.4. *Let $\underline{z} \in Z$ and $\underline{\psi} \in [L_2(\Omega)]^n$ be arbitrary but fixed. For $\underline{u}_z = \mathcal{H}\underline{z}$ we consider the unique solution $(\underline{u}_z, p_z) \in [H_0^1(\Omega, \Gamma_\mathrm{D})]^n \times L_2(\Omega)$ of the boundary value problem*

$$\begin{aligned} -\nu\Delta\underline{u}_z + \nabla p_z &= \underline{0} && \text{in } \Omega, \\ \nabla \cdot \underline{u}_z &= 0 && \text{in } \Omega, \\ \underline{u}_z &= \underline{0} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla\underline{u}_z)\underline{n} - p_z\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u}_z &= \underline{z} && \text{on } \Gamma_\mathrm{c}. \end{aligned} \tag{1.17}$$

Furthermore, let $H^*\underline{\psi} = -\nu(\nabla\underline{w})\underline{n} - r\underline{n}$ on Γ_c, where $(\underline{w}, r) \in [H_0^1(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n \times L_2(\Omega)$ is the unique solution of the adjoint boundary value problem

$$\begin{aligned} -\nu\Delta\underline{w} - \nabla r &= \underline{\psi} && \text{in } \Omega, \\ \nabla \cdot \underline{w} &= 0 && \text{in } \Omega, \\ \underline{w} &= \underline{0} && \text{on } \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c}, \\ \nu(\nabla\underline{w})\underline{n} + r\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}. \end{aligned} \tag{1.18}$$

Then $\langle \mathcal{H}\underline{z}, \underline{\psi} \rangle_\Omega = \langle \underline{z}, \mathcal{H}^*\underline{\psi} \rangle_{\Gamma_\mathrm{c}}$ is satisfied.

Proof. For the application of the solution operator we have $\mathcal{H}\underline{z} = \underline{u}_z = \underline{u}_0 + \mathcal{E}\underline{z}$, where $\underline{u}_0 \in [H_0^1(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n$ and $\mathcal{E}\underline{z} \in [H_0^1(\Omega, \Gamma_\mathrm{D})]^n$ denotes the extension of some $\underline{z} \in Z$. For the problem (1.17) we obtain the following variational formulation: Find $(\underline{u}_0, p_z) \in [H_0^1(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n \times L_2(\Omega)$, such that

$$\begin{aligned} &a(\underline{u}_0, \underline{v}) - b(\underline{v}, p_z) = -a(\mathcal{E}\underline{z}, \underline{v}), \\ &b(\underline{u}_0, q) \qquad\qquad\;\; = -b(\mathcal{E}\underline{z}, q) \end{aligned}$$

is satisfied for all $(\underline{v}, q) \in [H_0^1(\Omega, \Gamma_\mathrm{D} \cup \Gamma_\mathrm{c})]^n \times L_2(\Omega)$. By using $\underline{v} = \underline{w}$ in the above variational formulation we get the following variational equations

$$\begin{aligned} &a(\underline{u}_0, \underline{w}) - b(\underline{w}, p_z) = -a(\mathcal{E}\underline{z}, \underline{w}), \\ &b(\underline{u}_0, q) \qquad\qquad\;\;\; = -b(\mathcal{E}\underline{z}, q) \end{aligned}$$

for all $q \in L_2(\Omega)$. Analogously, we obtain for the adjoint problem (1.18), with $\underline{v} = \underline{u}_0$,

$$\begin{aligned} &a(\underline{w}, \underline{u}_0) + b(\underline{u}_0, r) = \langle \underline{\psi}, \underline{u}_0 \rangle_\Omega, \\ &b(\underline{w}, q) \qquad\qquad\;\; = 0 \end{aligned}$$

for all $q \in L_2(\Omega)$. Now we can subtract the respective first equations and apply the symmetry of the bilinear form $a(\cdot, \cdot)$, this yields

$$-b(\mathcal{E}\underline{z}, r) = a(\mathcal{E}\underline{z}, \underline{w}) + \langle \underline{\psi}, \underline{u}_0 \rangle_\Omega.$$

By applying integration by parts we obtain

$$\begin{aligned}\langle \underline{\psi}, \underline{u}_0 \rangle_\Omega &= -a(\mathcal{E}\underline{z}, \underline{w}) - b(\mathcal{E}\underline{z}, r) \\ &= \int_\Omega (\nu \Delta \underline{w} + \nabla r) \cdot \mathcal{E}\underline{z}\, dx - \int_\Gamma (\nu (\nabla \underline{w})\underline{n} + r\underline{n}) \cdot \mathcal{E}\underline{z}\, ds_x \\ &= -\langle \underline{\psi}, \mathcal{E}\underline{z} \rangle_\Omega + \langle -\nu(\nabla \underline{w})\underline{n} - r\underline{n}, \underline{z} \rangle_{\Gamma_c},\end{aligned}$$

which completes the proof by using $\mathcal{H}\underline{z} = \underline{u}_z = \underline{u}_0 + \mathcal{E}\underline{z}$, i.e.

$$\langle \mathcal{H}^* \underline{\psi}, \underline{z} \rangle_{\Gamma_c} = \langle \underline{\psi}, \mathcal{H}\underline{z} \rangle_\Omega = \langle \underline{\psi}, \underline{u}_0 + \mathcal{E}\underline{z} \rangle_\Omega = \langle -\nu(\nabla \underline{w})\underline{n} - r\underline{n}, \underline{z} \rangle_{\Gamma_c}.$$

□

Since the Steklov–Poincaré operator S is self–adjoint, we can apply Theorem 1.3 and obtain

$$\begin{aligned}&\langle \mathcal{H}\hat{\underline{z}} - \overline{\underline{w}}, \mathcal{H}(\underline{z} - \hat{\underline{z}}) \rangle_{[L_2(\Omega)]^n} + \varrho \langle S\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{Z^* \times Z} \\ &= \langle \mathcal{H}^*(\mathcal{H}\hat{\underline{z}} - \overline{\underline{w}}), \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_c} + \varrho \langle S\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_c} \\ &= \langle \mathcal{H}^*(\mathcal{H}\hat{\underline{z}} - \overline{\underline{w}}) + \varrho S\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_c},\end{aligned}$$

accordingly, the reduced minimization problem (1.16) is equivalent to the following variational inequality: Find $\hat{\underline{z}} \in Z_{ad}$, such that

$$\langle \mathcal{H}^*(\mathcal{H}\hat{\underline{z}} - \overline{\underline{w}}) + \varrho S\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_c} \geq 0 \tag{1.19}$$

for all $\underline{z} \in Z_{ad}$. We get by the definition of $\overline{\underline{w}}$ the equation

$$\mathcal{H}\hat{\underline{z}} - \overline{\underline{w}} = \mathcal{H}\hat{\underline{z}} + \underline{u}_f - \overline{\underline{u}} = \hat{\underline{u}} - \overline{\underline{u}}$$

where we used $\hat{\underline{u}} := \mathcal{H}\hat{\underline{z}} + \underline{u}_f$ and thus

$$\mathcal{H}^*(\mathcal{H}\hat{\underline{z}} - \overline{\underline{w}}) = \mathcal{H}^*(\hat{\underline{u}} - \overline{\underline{u}}) = -\nu(\nabla \hat{\underline{w}})\underline{n} - \hat{r}\underline{n}.$$

In particular we get for the variational inequality (1.19)

$$\langle -\nu(\nabla \hat{\underline{w}})\underline{n} - \hat{r}\underline{n} + \varrho S\hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_c} \geq 0$$

for all $\underline{z} \in Z_{ad}$.
Altogether we get, for the optimal control problem (1.5)–(1.7) respectively the reduced minimization problem (1.16), the following equivalent optimality system:

Primal problem

$$\begin{aligned}-\nu \Delta \hat{\underline{u}} + \nabla \hat{p} &= \underline{f} && \text{in } \Omega, \\ \nabla \cdot \hat{\underline{u}} &= 0 && \text{in } \Omega, \\ \hat{\underline{u}} &= \underline{g} && \text{on } \Gamma_D, \\ \nu(\nabla \hat{\underline{u}})\underline{n} - \hat{p}\underline{n} &= \underline{0} && \text{on } \Gamma_N, \\ \hat{\underline{u}} &= \hat{\underline{z}} && \text{on } \Gamma_c,\end{aligned}$$

Adjoint problem

$$\begin{aligned} -\nu\Delta\underline{\hat{w}} - \nabla\hat{r} &= \underline{\hat{u}} - \underline{\overline{u}} && \text{in } \Omega, \\ \nabla\cdot\underline{\hat{w}} &= 0 && \text{in } \Omega, \\ \underline{\hat{w}} &= \underline{0} && \text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}}, \\ \nu(\nabla\underline{\hat{w}})\underline{n} + \hat{r}\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}}, \end{aligned}$$

Optimality condition

$$\langle -\nu(\nabla\underline{\hat{w}})\underline{n} - \hat{r}\underline{n} + \varrho S\underline{\hat{z}}, \underline{z} - \underline{\hat{z}}\rangle_{\Gamma_{\mathrm{c}}} \geq 0$$

for all $\underline{z}\in Z_{ad}$.

Remark 1.6. *If we do not consider box constraints on the control, i.e. $Z_{ad} = Z$, the above optimality condition, a variational inequality, turns into the variational equation*

$$\langle -\nu(\nabla\underline{\hat{w}})\underline{n} - \hat{r}\underline{n} + \varrho S\underline{\hat{z}}, \underline{z}\rangle_{\Gamma_c} = 0$$

for all $\underline{z}\in Z$.

In the following we give another representation of the optimality condition, see also [23]. We define

$$\underline{\lambda} := -\nu(\nabla\underline{\hat{w}})\underline{n} - \hat{r}\underline{n} + \varrho S\underline{\hat{z}} = -\nu(\nabla\underline{\hat{w}})\underline{n} - \hat{r}\underline{n} + \varrho(\nabla\underline{\hat{u}}_z)\underline{n} \qquad \text{on } \Gamma_{\mathrm{c}},$$

where $\underline{\hat{u}}_z \in [H_0^1(\Omega,\Gamma_{\mathrm{D}})]^n$ is the solution of the vector valued Laplace problem (1.15) for some given $\underline{\hat{z}}\in Z$. Now we are able to rewrite the optimality condition in the following form

$$\langle\underline{\lambda}, \underline{z} - \underline{\hat{z}}\rangle_{\Gamma_{\mathrm{c}}} \geq 0$$

for all $\underline{z}\in Z_{ad}$. If we set $\underline{z} = \underline{z}_b \in Z_{ad}$, we obtain

$$\langle\underline{\lambda}, \underline{z}_b - \underline{\hat{z}}\rangle_{\Gamma_{\mathrm{c}}} \geq 0. \tag{1.20}$$

Next we choose, for $\underline{\hat{z}}\in Z_{ad}$, some $\underline{\varphi}\in Z$, such that $\varphi_i \geq 0$ on Γ_{c} $\forall i = 1,\dots,n$ and $\underline{z} = \underline{\hat{z}} - \underline{\varphi}\in Z_{ad}$ hold. It follows

$$\langle\underline{\lambda}, \underline{\varphi}\rangle_{\Gamma_{\mathrm{c}}} \leq 0.$$

This means $\lambda_i \leq 0$ on Γ_{c} $\forall i = 1,\dots,n$ in the sense of Z. Moreover, we can choose $\underline{\varphi} = \underline{z}_b - \underline{\hat{z}}$, and obtain

$$\langle\underline{\lambda}, \underline{z}_b - \underline{\hat{z}}\rangle_{\Gamma_{\mathrm{c}}} \leq 0.$$

Together with (1.20) we conclude

$$\langle \underline{\lambda}, \underline{z}_b - \hat{\underline{z}} \rangle_{\Gamma_\mathrm{c}} = 0.$$

This leads to the following complementary conditions

$$\hat{z}_i \leq z_{b,i}, \quad \lambda_i \leq 0 \quad \text{for } \hat{\underline{z}} = \underline{z}_b, \quad \lambda_i = 0 \quad \text{for } z_{a,i} < \hat{z}_i < z_{b,i} \qquad \text{on } \Gamma_\mathrm{c},$$

and similarly,

$$\hat{z}_i \geq z_{a,i}, \quad \lambda_i \geq 0 \quad \text{for } \hat{\underline{z}} = \underline{z}_a, \quad \lambda_i = 0 \quad \text{for } z_{a,i} < \hat{z}_i < z_{b,i} \qquad \text{on } \Gamma_\mathrm{c}$$

for all $i = 1, \ldots, n$. Due to this reason, $\hat{\underline{u}}_z \in [H_0^1(\Omega, \Gamma_\mathrm{D})]^n$ is the unique solution of the following Signorini boundary value problem

$$\begin{aligned} -\Delta \hat{\underline{u}}_z &= \underline{0} && \text{in } \Omega, \\ \hat{\underline{u}}_z &= \underline{0} && \text{on } \Gamma_\mathrm{D}, \\ (\nabla \hat{\underline{u}}_z)\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \end{aligned}$$

with the bilateral constraints

$$\begin{aligned} &\hat{u}_{z,i} \leq z_{b,i}, \quad \varrho[(\nabla \hat{\underline{u}}_z)\underline{n}]_i \leq \nu[(\nabla \hat{\underline{w}})\underline{n}]_i + \hat{r} n_i \quad \text{for } \hat{\underline{u}}_z = \underline{z}_b \qquad \text{on } \Gamma_\mathrm{c}, \\ &\hat{u}_{z,i} \geq z_{a,i}, \quad \varrho[(\nabla \hat{\underline{u}}_z)\underline{n}]_i \geq \nu[(\nabla \hat{\underline{w}})\underline{n}]_i + \hat{r} n_i \quad \text{for } \hat{\underline{u}}_z = \underline{z}_a \qquad \text{on } \Gamma_\mathrm{c} \end{aligned}$$

for all $i = 1, \ldots, n$, and

$$\left(\varrho[(\nabla \hat{\underline{u}}_z)\underline{n}]_i - \nu[(\nabla \hat{\underline{w}})\underline{n}]_i + \hat{r} n_i\right)(\hat{u}_{z,i} - z_{a,i})(\hat{u}_{z,i} - z_{b,i}) = 0 \qquad \text{on } \Gamma_\mathrm{c}$$

for all $i = 1, \ldots, n$.

Remark 1.7. *Since the optimality condition, i.e. the variational inequality, can be written as a Signorini boundary value problem, we are able to prove some regularity results of the solution, see [23, Proposition 2.2]. It is important to mention that, if the mixed boundary value is considered as in this case, we can only expect some lower regularity, see also [23, Proposition 2.2].*

In this section we have considered the optimal control problem of the Stokes equations (1.5)–(1.7). We proved existence and uniqueness of the solution of the Stokes equations itself, and of the optimal control problem. Moreover, we derived the corresponding optimality system as an equivalent problem. In general it is not so easy to see how the adjoint equation of a partial differential equation looks like, especially for nonlinear equations. Therefore the adjoint equation can be also calculated via the so called formal Lagrange method, see [34].

1.3 Navier–Stokes equations

In this section we consider the optimal control problem for the Navier–Stokes equations. For this case the optimal control problem of the Navier–Stokes equations is given as follows: Minimize the cost functional

$$\mathcal{J}(\underline{u},\underline{z}) := \frac{1}{2}\,\|\underline{u}-\overline{\underline{u}}\|^2_{[L_2(\Omega)]^n} + \frac{1}{2}\varrho\langle S\underline{z},\underline{z}\rangle_{\Gamma_\mathrm{c}} \tag{1.21}$$

under the constraint

$$\begin{aligned} -\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega,\\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega,\\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D},\\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N},\\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}, \end{aligned} \tag{1.22}$$

where the control satisfies the box constraint

$$z_{a,i} \le z_i \le z_{b,i} \qquad \text{a.e. on } \Gamma_\mathrm{c},\; i = 1,\dots,n. \tag{1.23}$$

The aim of this section is to derive the optimality system for the optimal control problem (1.21)–(1.23).

1.3.1 Unique solvability of the state equations

In the following we derive the variational formulation for the Navier–Stokes equations. Moreover we comment on existence and uniqueness results of the solution. We consider the mixed boundary value problem, where $\Gamma = \overline{\Gamma}_\mathrm{D}\cup\overline{\Gamma}_\mathrm{N}$ with $\mathrm{meas}(\Gamma_\mathrm{D}) > 0$ and $\mathrm{meas}(\Gamma_\mathrm{N}) > 0$, which is given by

$$\begin{aligned} -\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega,\\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega,\\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D},\\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}. \end{aligned} \tag{1.24}$$

The related variational formulation is similar to that of the Stokes equations as considered in Section 1.2.1, with an additional term due to the nonlinearity, see for example [15, 26]. The corresponding variational formulation is given as follows: Find $(\underline{u},p) \in [H^1(\Omega)]^n \times L_2(\Omega)$ with $\underline{u} = \underline{g}$ on Γ_D, such that

$$\begin{aligned} &a(\underline{u},\underline{v}) + a_1(\underline{u},\underline{u},\underline{v}) - b(\underline{v},p) &&= \langle\underline{f},\underline{v}\rangle_\Omega,\\ &b(\underline{u},q) &&= 0 \end{aligned} \tag{1.25}$$

is satisfied for all test functions $(\underline{v},q) \in [H^1_0(\Omega,\Gamma_\mathrm{D})]^n \times L_2(\Omega)$, where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are those from (1.10), and the additional trilinear form is given by

$$a_1(\underline{w},\underline{u},\underline{v}) = \int_\Omega [(\underline{w}\cdot\nabla)\underline{u}]\cdot\underline{v}\;dx.$$

Remark 1.8. *If we consider the Dirichlet boundary value problem, i.e.* $\Gamma = \Gamma_D$ *in (1.24), we have to assume the solvability condition*

$$\int_\Gamma \underline{g} \cdot \underline{n} \; ds_x = 0. \tag{1.26}$$

Moreover the pressure p *is only unique up to an additive constant. For this reason we introduce* $L_{2,0}(\Omega) \subset L_2(\Omega)$*, where* $p \in L_{2,0}(\Omega)$ *satisfies the scaling condition* $\int_\Omega p \; dx = 0$*, see [31].*

Next we consider the unique solvability of the variational problem (1.25). The nonlinearity makes the situation more difficult, than before for the Stokes equations. Instead we have to make use of a fixed point theorem, as shown in [15]. The first result is a useful property of $a_1(\cdot,\cdot,\cdot)$.

Lemma 1.4. *The trilinear form* $a_1(\cdot,\cdot,\cdot)$ *is bounded on* $\left([H^1(\Omega)]^n\right)^3$.

Proof. By the definition of $a_1(\cdot,\cdot,\cdot)$ we have

$$a_1(\underline{w},\underline{u},\underline{v}) \leq c_1 \, |\underline{u}|_{[H^1(\Omega)]^n} \, \| \, |\underline{w}||\underline{v}| \, \|_{L_2(\Omega)} .$$

Due to the continuous imbedding of $[H^1(\Omega)]^n$ in $[L_4(\Omega)]^n$, we obtain

$$a_1(\underline{w},\underline{u},\underline{v}) \leq c_1 \, |\underline{u}|_{[H^1(\Omega)]^n} \, \|\underline{w}\|_{[L_4(\Omega)]^n} \, \|\underline{v}\|_{[L_4(\Omega)]^n} \leq c \, \|\underline{u}\|_{[H^1(\Omega)]^n} \, \|\underline{w}\|_{[H^1(\Omega)]^n} \, \|\underline{v}\|_{[H^1(\Omega)]^n}$$

for all $\underline{u},\underline{w},\underline{v} \in [H^1(\Omega)]^n$. □

The following theorem ensures the existence of a solution when we are considering the Dirichlet boundary value problem, i.e. $\Gamma = \Gamma_{\mathrm{D}}$.

Theorem 1.5. [15, Theorem 2.3] *Let* Ω *be a bounded Lipschitz domain,* $\underline{f} \in [H^{-1}(\Omega)]^n$ *and* $\underline{g} \in \left[H^{1/2}(\Gamma)\right]^n$ *satisfying (1.26). Then there exists at least one solution* $(\underline{u}, p) \in [H^1(\Omega)]^n \times L_{2,0}(\Omega)$ *with* $\underline{u} = \underline{g}$ *on* Γ *such that (1.25) is satisfied.*

Now we introduce the space $V_{\mathrm{div}} := \{\underline{v} \in [H^1_0(\Omega)]^n : \nabla \cdot \underline{v} = 0 \quad \text{in } \Omega\}$ and define

$$\eta := \sup_{0 \neq \underline{u},\underline{v},\underline{w} \in V_{\mathrm{div}}} \frac{a_1(\underline{w},\underline{u},\underline{v})}{|\underline{w}|_{[H^1(\Omega)]^n} \, |\underline{u}|_{[H^1(\Omega)]^n} \, |\underline{v}|_{[H^1(\Omega)]^n}} .$$

Moreover we set $V_{\mathrm{div},g} := \{\underline{v} \in [H^1(\Omega)]^n : \nabla \cdot \underline{v} = 0 \quad \text{in } \Omega, \; \underline{v} = \underline{g} \quad \text{on } \Gamma\}$ and

$$\nu_0 := \inf_{0 \neq \underline{v} \in V_{\mathrm{div},g}} \left\{ \sup_{0 \neq \underline{w} \in V_{\mathrm{div}}} \frac{a_1(\underline{w},\underline{v},\underline{w})}{|\underline{w}|^2_{[H^1(\Omega)]^n}} + \left(\eta \sup_{0 \neq \underline{w} \in V_{\mathrm{div}}} \frac{\langle \underline{f}, \underline{w} \rangle_\Omega - a(\underline{v},\underline{w}) - a_1(\underline{v},\underline{v},\underline{w})}{|\underline{w}|_{[H^1(\Omega)]^n}} \right)^{1/2} \right\} .$$

This leads to the following statement on the uniqueness of the solution.

Theorem 1.6. [15, Theorem 2.4] *Under the assumptions of Theorem 1.5 and if in addition* $\nu > \nu_0$ *is satisfied, there exists a unique solution* $(\underline{u}, p) \in [H^1(\Omega)]^n \times L_{2,0}(\Omega)$ *with* $\underline{u} = \underline{g}$ *on* Γ *of the variational formulation (1.25).*

Remark 1.9. *For the existence and uniqueness of the solution of the mixed boundary value problem, as we considered in (1.24), we refer to [2, p. 127–131].*

1.3.2 Optimality system

In the following we derive the optimality system for the optimal control problem of the Navier–Stokes equations, which is given by (1.21)–(1.23). Since we consider a nonlinear problem, it is not so clear how the adjoint equations looks like. That is why we derive the optimality system by the so called formal Lagrange method, as described in [34]. Let $\underline{f} \in \left[\widetilde{H}^{-1}(\Omega)\right]^n$, $\underline{\overline{u}} \in [L_2(\Omega)]^n$ and $\underline{z}_a, \underline{z}_b \in Z$, where Z is either $\left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$ or an appropriate subset, which we described in Section 1.2.2 for the Stokes equations.
In the first step we introduce the following Lagrange functional

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\underline{u}, p, \underline{z}, \underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, v)\\ &= \mathcal{J}(\underline{u}, \underline{z}) + \int_\Omega (-\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p - \underline{f})\cdot \underline{v}_1\, dx + \int_\Omega (\nabla\cdot\underline{u}) v\, dx\\ &\quad + \int_{\Gamma_\mathrm{D}} (\underline{u} - \underline{g})\cdot\underline{v}_2\, ds_x + \int_{\Gamma_\mathrm{N}} (\nu(\nabla\underline{u})\underline{n} - p\underline{n})\cdot\underline{v}_3\, ds_x + \int_{\Gamma_\mathrm{c}} (\underline{u} - \underline{z})\cdot\underline{v}_4\, ds_x.\end{aligned}$$

By applying integration by parts twice, we get the following expression

$$\begin{aligned}\mathcal{L} &= \mathcal{J}(\underline{u}, \underline{z}) - \int_\Omega \nu\Delta\underline{v}_1\cdot\underline{u}\, dx + \int_\Gamma \nu[(\nabla\underline{v}_1)\underline{n}]\cdot\underline{u}\, ds_x - \int_\Gamma \nu[(\nabla\underline{u})\underline{n}]\cdot\underline{v}_1\, ds_x\\ &\quad - \int_\Omega [(\nabla\underline{v}_1)\underline{u}]\cdot\underline{u}\, dx + \int_\Gamma (\underline{u}\cdot\underline{n})(\underline{u}\cdot\underline{v}_1)\, ds_x - \int_\Omega p\ \nabla\cdot\underline{v}_1\, dx\\ &\quad + \int_\Gamma p\ \underline{v}_1\cdot\underline{n}\, ds_x - \int_\Omega \underline{f}\cdot\underline{v}_1\, dx - \int_\Omega \nabla v\cdot\underline{u}\, dx + \int_\Gamma \underline{u}\cdot\underline{n}\ v\, ds_x\\ &\quad + \int_{\Gamma_\mathrm{D}} (\underline{u} - \underline{g})\cdot\underline{v}_2\, ds_x + \int_{\Gamma_\mathrm{N}} (\nu(\nabla\underline{u})\underline{n} - p\underline{n})\cdot\underline{v}_3\, ds_x + \int_{\Gamma_\mathrm{c}} (\underline{u} - \underline{z})\cdot\underline{v}_4\, ds_x.\end{aligned}$$

At a local optimum $(\hat{\underline{u}}, \hat{p}, \hat{\underline{z}}, \underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, v)$ we expect the following conditions to be satisfied: $D_{\underline{u}}\mathcal{L} = 0$, $D_p\mathcal{L} = 0$ and $D_{\underline{z}}\mathcal{L}(\underline{z} - \hat{\underline{z}}) \geq 0$ for all $\underline{z} \in Z_{ad}$. For the derivative with respect to $\underline{u}$ we obtain

$$\begin{aligned}D_{\underline{u}}\mathcal{L}\underline{h} &= \int_\Omega (\hat{\underline{u}} - \underline{\overline{u}})\cdot\underline{h}\, dx - \int_\Omega \nu\Delta\underline{v}_1\cdot\underline{h}\, dx + \int_\Gamma \nu[(\nabla\underline{v}_1)\underline{n}]\cdot\underline{h}\, ds_x - \int_\Gamma \nu[(\nabla\underline{h})\underline{n}]\cdot\underline{v}_1\, ds_x\\ &\quad - \int_\Omega [(\nabla\underline{v}_1)\hat{\underline{u}}]\cdot\underline{h}\, dx - \int_\Omega [(\nabla\underline{v}_1)^\top\hat{\underline{u}}]\cdot\underline{h}\, dx + \int_\Gamma (\underline{h}\cdot\underline{n})(\hat{\underline{u}}\cdot\underline{v}_1)\, ds_x\\ &\quad + \int_\Gamma (\hat{\underline{u}}\cdot\underline{n})(\underline{h}\cdot\underline{v}_1)\, ds_x - \int_\Omega \nabla v\cdot\underline{h}\, dx + \int_\Gamma \underline{h}\cdot\underline{n}\ v\, ds_x + \int_{\Gamma_\mathrm{D}} \underline{h}\cdot\underline{v}_2\, ds_x\\ &\quad + \int_{\Gamma_\mathrm{N}} \nu[(\nabla\underline{h})\underline{n}]\cdot\underline{v}_3\, ds_x + \int_{\Gamma_\mathrm{c}} \underline{h}\cdot\underline{v}_4\, ds_x = 0\end{aligned}$$

for all $\underline{h} \in [C_0^\infty(\Omega)]^n$. For claiming $\underline{h} = (\nabla\underline{h})\underline{n} = \underline{0}$ on Γ we conclude

$$\nu\Delta\underline{v}_1 + (\nabla\underline{v}_1)\hat{\underline{u}} + (\nabla\underline{v}_1)^\top\hat{\underline{u}} + \nabla v = \hat{\underline{u}} - \underline{\overline{u}} \quad \text{in } \Omega.$$

Analogously for $\underline{h} = \underline{0}$ on Γ and $(\nabla \underline{h})\underline{n} = \underline{0}$ on Γ_{N} we get

$$\underline{v_1} = \underline{0} \quad \text{on } \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}}.$$

Now we consider $\underline{h} = \underline{0}$ on $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}}$ and $(\nabla \underline{h})\underline{n} = \underline{0}$ on Γ from which we obtain

$$\nu(\nabla \underline{v}_1)\underline{n} + (\hat{\underline{u}} \cdot \underline{v}_1)\underline{n} + (\hat{\underline{u}} \cdot \underline{n})\underline{v}_1 + v\underline{n} = \underline{0} \quad \text{on } \Gamma_{\mathrm{N}}.$$

Similarly $\underline{h} = \underline{0}$ on $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ and $(\nabla \underline{h})\underline{n} = \underline{0}$ on Γ leads to

$$\nu(\nabla \underline{v}_1)\underline{n} + (\hat{\underline{u}} \cdot \underline{v}_1)\underline{n} + (\hat{\underline{u}} \cdot \underline{n})\underline{v}_1 + v\underline{n} + \underline{v}_2 = \underline{0} \quad \text{on } \Gamma_{\mathrm{c}}.$$

For the second condition we obtain

$$D_p \mathcal{L} h = -\int_{\Omega} h \; \nabla \cdot \underline{v}_1 \; dx + \int_{\Gamma} h \; \underline{v}_1 \cdot \underline{n} \; ds_x - \int_{\Gamma_{\mathrm{N}}} h \; \underline{v}_3 \cdot \underline{n} \; ds_x = 0$$

for all $h \in C_0^\infty(\Omega)$ from which we conclude

$$\nabla \cdot \underline{v}_1 = 0 \quad \text{in } \Omega.$$

For the derivative with respect to the control variable we get

$$D_{\underline{z}} \mathcal{L} \underline{h} = \varrho \int_{\Gamma_{\mathrm{c}}} S \hat{\underline{z}} \cdot \underline{h} \; ds_x - \int_{\Gamma_{\mathrm{c}}} \underline{h} \cdot \underline{v}_2 \; ds_x = 0$$

for all $\underline{h} \in [C_0^\infty(\Omega)]^n$. Since $D_{\underline{z}} \mathcal{L}(\underline{z} - \hat{\underline{z}}) \geq 0$ for all $\underline{z} \in Z_{ad}$, we obtain the variational inequality

$$\langle \underline{v}_2 - \varrho S \hat{\underline{z}}, \underline{z} - \hat{\underline{z}} \rangle_{\Gamma_{\mathrm{c}}} \geq 0$$

for all $\underline{z} \in Z_{ad}$.
Now we set $\hat{\underline{w}} = -\underline{v}_1$ and $\hat{r} = -v$ and obtain the optimality system as an equivalent problem to the optimal control problem (1.21)–(1.22). This consists of the following equations:

Primal problem

$$\begin{aligned}
-\nu \Delta \hat{\underline{u}} + (\hat{\underline{u}} \cdot \nabla)\hat{\underline{u}} + \nabla \hat{p} &= \underline{f} && \text{in } \Omega,\\
\nabla \cdot \hat{\underline{u}} &= 0 && \text{in } \Omega,\\
\hat{\underline{u}} &= \underline{g} && \text{on } \Gamma_{\mathrm{D}},\\
\nu(\nabla \hat{\underline{u}})\underline{n} - \hat{p}\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}},\\
\hat{\underline{u}} &= \hat{\underline{z}} && \text{on } \Gamma_{\mathrm{c}},
\end{aligned}$$

Adjoint problem

$$\begin{aligned} -\nu\Delta\underline{\hat{w}} - (\nabla\underline{\hat{w}})\underline{\hat{u}} - (\nabla\underline{\hat{w}})^\top\underline{\hat{u}} - \nabla\hat{r} = \underline{\hat{u}} - \underline{\overline{u}} \qquad & \text{in } \Omega, \\ \nabla\cdot\underline{\hat{w}} = 0 \qquad & \text{in } \Omega, \\ \underline{\hat{w}} = \underline{0} \qquad & \text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}}, \\ \nu(\nabla\underline{\hat{w}})\underline{n} + (\underline{\hat{u}}\cdot\underline{\hat{w}})\underline{n} + (\underline{\hat{u}}\cdot\underline{n})\underline{\hat{w}} + \hat{r}\underline{n} = \underline{0} \qquad & \text{on } \Gamma_{\mathrm{N}}, \end{aligned}$$

Optimality condition

$$\langle -\nu(\nabla\underline{\hat{w}})\underline{n} - (\underline{\hat{u}}\cdot\underline{\hat{w}})\underline{n} - (\underline{\hat{u}}\cdot\underline{n})\underline{\hat{w}} - \hat{r}\underline{n} + \rho S\underline{\hat{z}}, \underline{z} - \underline{\hat{z}}\rangle_{\Gamma_{\mathrm{c}}} \geq 0$$

for all $\underline{z} \in Z_{ad}$.

Remark 1.10. *If we do not consider any box constraints, i.e. $Z = Z_{ad}$, the above optimality condition turns into the variational equation*

$$\langle -\nu(\nabla\underline{\hat{w}})\underline{n} - (\underline{\hat{u}}\cdot\underline{\hat{w}})\underline{n} - (\underline{\hat{u}}\cdot\underline{n})\underline{\hat{w}} - \hat{r}\underline{n} + \rho S\underline{\hat{z}}, \underline{z}\rangle_{\Gamma_c} = 0$$

for all $\underline{z} \in Z$.

In this chapter we have considered optimal control problems for the Stokes and Navier–Stokes equations. In the Stokes case we proved the unique solvability of the optimal control problem. For both equations we derived the optimality system as an equivalent formulation of the optimal control problem. In the following we want to discretize these problems by using stabilized finite element methods. In chapter 2 we will give an overview on stabilized finite element methods for Stokes and Navier–Stokes equations. In chapter 3 we apply these methods to the optimal control problems and address especially the issue of the difference between a control in $[L_2(\Gamma_{\mathrm{c}})]^n$ and $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$.

2 Stabilized finite element methods

In this chapter we introduce stabilized finite element methods for the Stokes and Navier–Stokes equations. There are several different methods around, see [3, 6, 8, 9, 14, 21, 22, 26, 28, 32, 33], but we will just focus on a few of them.
We start with the idea of stabilized finite element methods for the Stokes equations (1.8), with homogeneous Dirichlet boundary conditions for simplicity. When we consider the variational formulation of this problem, in the continuous setting we have to satisfy the inf–sup condition (1.11) for existence and uniqueness of the solution. This follows in this case from Nečas inequality, see Lemma 1.1. However, when we discretize the Stokes equations we have to fulfill a discrete inf–sup condition, which we cannot conclude from the continuous setting in general. This condition depends on the ansatz spaces $V_h \subset [H_0^1(\Omega)]^n$ and $Q_h \subset L_{2,0}(\Omega)$, and is given as follows: There exists an h independent constant $\tilde{c}_s > 0$, such that

$$\tilde{c}_s \left\| q_h \right\|_{L_2(\Omega)} \leq \sup_{0 \neq \underline{v}_h \in V_h} \frac{b(\underline{v}_h, q_h)}{\left\| \underline{v}_h \right\|_{[H^1(\Omega)]^n}} \tag{2.1}$$

is satisfied for all $q_h \in Q_h$. To satisfy the discrete inf–sup condition one possibility is to chose special finite elements. The most common are the Taylor–Hood–, Crouzeix–Raviart– and the Mini–element. For details we refer, for example, to [4, 6, 15, 26]. Here we present an idea how to ensure stability independent of the choice of the finite elements. We do so by a "consistent modification" of the variational formulation such that the discrete inf–sup condition (2.1) is satisfied. This means that we add so called stabilization terms in the variational formulation. For different choices of these stabilization terms we get different methods. The advantage of the most stabilization methods is, that existence and uniqueness can be shown for arbitrary finite elements. Moreover it is important to mention that for the most methods constant shape functions are excluded, because of the derivatives in the stabilization terms we just get the standard variational formulation again, which leads to instabilities. An example for this case will be presented. A good overview on stabilized finite element methods for the Stokes equations can be found, for example, in [3, 6, 9, 14, 21, 22, 28]. The idea for the Navier–Stokes equations is the same, but here we have to treat the nonlinearity. There are several papers on stabilized finite element methods for this problem, see for example [8, 32, 33].
As we already mentioned, we have the advantage of using arbitrary finite elements, except the constant ones, when we make use of stabilization methods. In particular we are able to use low order elements, as the equal order 1 elements. This leads to less degrees of freedoms, which is not possible for a standard discretization. Another advantage of stabilization, which is especially usefull for the Navier–Stokes equations, is the problem of high Reynolds

numbers, respectively low viscosity. For this case the convective part is dominant. There are possibilities to treat this problem, we will present an idea.
This chapter is organized as follows: We start with an overview on different stabilizations for the Stokes equations, where we consider the so called Galerkin Least–Square method in more detail. The stability of this method will be proven and some error estimates will be given afterwards. This section ends by giving some numerical results which confirm the theoretical estimates. In the second section a stabilized finite element method for the Navier–Stokes equations will be presented. Moreover we will discuss how to treat the nonlinearity and give some numerical results.

2.1 Stokes equations

In this section we give an overview of the most popular stabilized finite element methods for the Stokes equations (1.8). The idea is to add stabilization terms in the variational formulation (1.9). A first possibility is to add a term $\Phi_h(\underline{u}_h, p_h; q_h)$ in the second equation, this leads to the following problem: Find $(\underline{u}_h, p_h) \in V_h \times Q_h$, such that

$$\begin{aligned} a(\underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) \qquad\qquad\qquad\qquad &= \langle \underline{f}, \underline{v}_h \rangle_\Omega, \\ b(\underline{u}_h, q_h) \qquad\qquad\qquad + \Phi_h(\underline{u}_h, p_h; q_h) &= 0 \end{aligned}$$

is satisfied for all $(\underline{v}_h, q_h) \in V_h \times Q_h$. A first example for this stabilization term was given by Brezzi and Pitkäranta (1984) with

$$\Phi_h(\underline{u}_h, p_h; q_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \nabla p_h \cdot \nabla q_h \; dx. \tag{2.2}$$

Further examples where given by Hughes, Franca and Balestra (1986) with

$$\Phi_h(\underline{u}_h, p_h; q_h) = \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu \Delta \underline{u}_h + \nabla p_h - \underline{f}) \cdot \nabla q_h \; dx \tag{2.3}$$

with some suitable chosen $\delta > 0$, as well as by Brezzi and Douglas (1988) with

$$\Phi_h(\underline{u}_h, p_h; q_h) = \delta \sum_{T \in \mathcal{T}_h} h_T^2 \left[\int_T (\nabla p_h - \underline{f}) \cdot \nabla q_h \; dx - \int_{\partial T \cap \Gamma} (\nu \Delta \underline{u}_h \cdot \underline{n}) q_h \; ds_x \right].$$

Now we consider a generalization, where we also allow a stabilization term $\Psi_h(\underline{u}_h, p_h; \underline{v}_h)$ in the first equation. Therefore we get the following variational formulation: Find $(\underline{u}_h, p_h) \in V_h \times Q_h$, such that

$$\begin{aligned} a(\underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) + \Psi_h(\underline{u}_h, p_h; \underline{v}_h) &= \langle \underline{f}, \underline{v}_h \rangle_\Omega, \\ b(\underline{u}_h, q_h) \qquad\qquad\qquad + \Phi_h(\underline{u}_h, p_h; q_h) &= 0 \end{aligned} \tag{2.4}$$

is satisfied for all $(\underline{v}_h, q_h) \in V_h \times Q_h$. Therefore we chose $\Phi_h(\underline{u}_h, p_h; q_h)$ as in (2.3) and

$$\Psi_h(\underline{u}_h, p_h; \underline{v}_h) = \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu \Delta \underline{u}_h + \nabla p_h - \underline{f}) \cdot (-\rho \nu \Delta \underline{v}_h)\, dx \tag{2.5}$$

with some constants $\rho \in \{-1, 0, 1\}$ and $\delta > 0$. Depending on the constant ρ we get the following methods, see also [6, 9]:

$\rho = -1$: *Galerkin Least–Square method* (GLS)
$\rho = 0$: *Pressure Stabilization Petrov–Galerkin method* (PSPG)
$\rho = 1$: *Unusual Galerkin Least–Square method* (UGLS)

Remark 2.1. *Let linear shape functions for $\underline{u}_h$ be given, then there holds $\Delta \underline{u}_h|_T = 0$ for all $T \in \mathcal{T}_h$. So the stabilization term (2.5) vanishes and the variational formulation is independent of the constant ρ. Hence all methods, except (2.2) coincide.*

It remains to answer the questions which method for piecewise linear and constant shape functions for $\underline{u}_h$ and p_h, respectively, makes sense. The problem, considering this paring, is that we receive the standard variational formulation for all methods up to now and so obtain an unstable method. An example which is anticipating this is given by

$$\Psi_h(\underline{u}_h, p_h; \underline{v}_h) = 0 \quad \text{and} \quad \Phi_h(\underline{u}_h, p_h; q_h) = \delta \sum_{e \in \mathcal{E}_h} h_e \int_e [p_h]_e [q_h]_e \, ds_x$$

with a given constant $\delta > 0$, where $\mathcal{E}_h$ denotes the set of all inner edges e of the triangulation and $[q_h]_e$ the jump on e for $q_h \in Q_h$.
Another stabilization method, a so called minimal stabilization, is the *Local Projection stabilization* (LPS), which is based on a local L_2–projection P_h. The corresponding stabilization terms are given by

$$\Psi_h(\underline{u}_h, p_h; \underline{v}_h) = 0 \quad \text{and} \quad \Phi_h(\underline{u}_h, p_h; q_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (P_h \nabla p_h) \cdot (P_h \nabla q_h)\, dx.$$

Details for this method can be found for example in [3, 14, 21, 27].
In the following we will consider the Galerkin Least–Square method in detail, applied to the Stokes equations. We will prove stability of this method, from which we can conclude the unique solvability of the discrete problem. Moreover we will prove error estimates. For the other methods, as the Pressure Stabilization Petrov–Galerkin and the Unusual Galerkin Least–Square method, the proofs are similar, see for example [6, 9]. We end by giving some numerical results, which confirm the error estimates.

2.1.1 Stability

We consider the Galerkin Least–Square method for the Stokes equations, which is given by the equations (2.4) for $\rho = -1$. Since the equations must be valid for all test functions, we are also able to subtract them. Let us consider the trial space $S_h^k(\Omega)$ of piecewise polynomials of degree $k \in \mathbb{N}_0$, which are globally continuous. This definition leads for the ansatz spaces $V_h = \left[S_h^k(\Omega) \cap H_0^1(\Omega)\right]^n$ and $Q_h = S_h^l(\Omega) \cap L_{2,0}(\Omega)$, for some $k, l \in \mathbb{N}$, to the following variational formulation: Find $(\underline{u}_h, p_h) \in V_h \times Q_h$, such that

$$A(\underline{u}_h, p_h; \underline{v}_h, q_h) = F(\underline{v}_h, q_h) \tag{2.6}$$

is valid for all $(\underline{v}_h, q_h) \in V_h \times Q_h$, where the linear forms are given by

$$\begin{aligned} A(\underline{u}_h, p_h; \underline{v}_h, q_h) &= a(\underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) - b(\underline{u}_h, q_h) \\ &\quad - \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu \Delta \underline{u}_h + \nabla p_h) \cdot (-\nu \Delta \underline{v}_h + \nabla q_h) \, dx \end{aligned} \tag{2.7}$$

and

$$F(\underline{v}_h, q_h) = \langle \underline{f}, \underline{v}_h \rangle_\Omega - \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \underline{f} \cdot (-\nu \Delta \underline{v}_h + \nabla q_h) \, dx. \tag{2.8}$$

Our aim is to prove stability and unique solvability of the Galerkin Least–Square method. For the proof itself we need the following lemma first.

Lemma 2.1. *There exist positive and h independent constants $\tilde{c}_{s,1}$ and $\tilde{c}_{s,2}$, such that*

$$\tilde{c}_{s,1} \left\| q_h \right\|_{L_2(\Omega)} - \tilde{c}_{s,2} \left(\sum_{T \in \mathcal{T}_h} h_T^2 \left| q_h \right|_{H^1(T)}^2 \right)^{\frac{1}{2}} \le \sup_{\underline{0} \neq \underline{v}_h \in V_h} \frac{b(\underline{v}_h, q_h)}{\left\| \underline{v}_h \right\|_{[H^1(\Omega)]^n}} \tag{2.9}$$

is satisfied for all $q_h \in Q_h$.

Proof. Let $q_h \in Q_h \subset L_{2,0}(\Omega)$ be arbitrary but fixed. From the inf–sup condition in the continuous setting we know that there exists a $\underline{v} \in [H_0^1(\Omega)]^n$ with $\underline{v} \neq 0$, such that

$$c \left\| q_h \right\|_{L_2(\Omega)} \left\| \underline{v} \right\|_{[H^1(\Omega)]^n} \le b(\underline{v}, q_h).$$

For this $\underline{v} \in [H_0^1(\Omega)]^n$ there exists an interpolation $R_h : [H_0^1(\Omega)]^n \to V_h$, see [15], with the following properties:

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T^{-1} \left\| \underline{v} - R_h \underline{v} \right\|_{[L_2(T)]^n} &\le c_{R,1} \left\| \underline{v} \right\|_{[H^1(\Omega)]^n}, \\ \left\| \underline{v} - R_h \underline{v} \right\|_{[H^1(\Omega)]^n} &\le c_{R,2} \left\| \underline{v} \right\|_{[H^1(\Omega)]^n}. \end{aligned}$$

Hence we get the following two estimates

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T^{-2} \left\| \underline{v} - R_h \underline{v} \right\|_{[L_2(T)]^n}^2 &\le (c_{R,1})^2 \left\| \underline{v} \right\|_{[H^1(\Omega)]^n}^2, \\ \left\| R_h \underline{v} \right\|_{[H^1(\Omega)]^n} \le \left\| \underline{v} - R_h \underline{v} \right\|_{[H^1(\Omega)]^n} + \left\| \underline{v} \right\|_{[H^1(\Omega)]^n} &\le (c_{R,2} + 1) \left\| \underline{v} \right\|_{[H^1(\Omega)]^n}. \end{aligned}$$

Applying these results to the bilinear form $b(\cdot,\cdot)$ we obtain the following inequality:

$$
\begin{aligned}
b(R_h\underline{v}, q_h) &= b(R_h\underline{v} - \underline{v}, q_h) + b(\underline{v}, q_h) \\
&\geq \int_\Omega \nabla\cdot(R_h\underline{v} - \underline{v}) q_h \, dx + c\,\|q_h\|_{L_2(\Omega)}\,\|\underline{v}\|_{[H^1(\Omega)]^n} \\
&= -\sum_{T\in\mathcal{T}_h}\int_T (R_h\underline{v} - \underline{v})\cdot\nabla q_h\, dx + c\,\|q_h\|_{L_2(\Omega)}\,\|\underline{v}\|_{[H^1(\Omega)]^n} \\
&\geq -\left(\sum_{T\in\mathcal{T}_h} h_T^{-2}\,\|R_h\underline{v} - \underline{v}\|^2_{[L_2(T)]^n}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^2\,|q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \\
&\quad + c\,\|q_h\|_{L_2(\Omega)}\,\|\underline{v}\|_{[H^1(\Omega)]^n} \\
&\geq \left[-c_{R,1}\left(\sum_{T\in\mathcal{T}_h} h_T^2\,|q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} + c\,\|q_h\|_{L_2(\Omega)}\right]\|\underline{v}\|_{[H^1(\Omega)]^n} \\
&\geq \left[-c_{R,1}\left(\sum_{T\in\mathcal{T}_h} h_T^2\,|q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} + c\,\|q_h\|_{L_2(\Omega)}\right]\frac{1}{c_{R,2}+1}\,\|R_h\underline{v}\|_{[H^1(\Omega)]^n}.
\end{aligned}
$$

If we now set $\underline{v}_h = R_h\underline{v}$ and divide by its norm, we can do so, because its different from zero, there holds

$$
\tilde{c}_{s,1}\,\|q_h\|_{L_2(\Omega)} - \tilde{c}_{s,2}\left(\sum_{T\in\mathcal{T}_h} h_T^2\,|q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \leq \frac{b(\underline{v}_h, q_h)}{\|\underline{v}_h\|_{[H^1(\Omega)]^n}},
$$

which remains valid for the supremum over all $\underline{v}_h \in V_h$. □

This is now an important result for proving the following theorem, which shows the unique solvability of the Galerkin Least–Square method (2.6). Most interesting is, that uniqueness is guaranteed for arbitrary shape functions with degree $k, l \geq 1$. It is important to mention that we handle from now on with the constant c_I from the local inverse inequality, which can be found for example in [31].

Theorem 2.1. *Let $0 < \delta_0 \leq \delta < \nu^{-1}c_I^{-2}$, then for the Galerkin Least–Square method (2.6) there exists a h and δ independent constant $\bar{c}_s > 0$, such that*

$$
\bar{c}_s\left(\|\underline{v}_h\|_{[H^1(\Omega)]^n} + \|q_h\|_{L_2(\Omega)}\right) \leq \sup_{(\underline{0},0)\neq(\underline{w}_h,r_h)\in V_h\times Q_h} \frac{A(\underline{v}_h, q_h; \underline{w}_h, r_h)}{\|\underline{w}_h\|_{[H^1(\Omega)]^n} + \|r_h\|_{L_2(\Omega)}} \tag{2.10}
$$

is satisfied for all $(\underline{v}_h, q_h) \in V_h \times Q_h$.

Proof. Let $\underline{v}_h \in V_h$ and $q_h \in Q_h$ be given. We have to show that there exist $\underline{w}_h \in V_h$ and $r_h \in Q_h$, such that the above inequality is satisfied. First we consider the interpolation

$R_h\underline{v} \in V_h$ as used in the proof of Lemma 2.1. We scale it, such that $\|R_h\underline{v}\|_{[H^1(\Omega)]^n} = \|q_h\|_{L_2(\Omega)}$ is valid. Moreover the inequality

$$ab < \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2, \qquad \forall\; a, b, \alpha > 0$$

is satisfied. With it, Lemma 2.1 and the local inverse inequality we get the following estimate:

$$\begin{aligned}
&A(\underline{v}_h, q_h; -R_h\underline{v}, 0)\\
&= -\, a(\underline{v}_h, R_h\underline{v}) + b(R_h\underline{v}, q_h) - \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{v}_h + \nabla q_h)\cdot(\nu\Delta R_h\underline{v})\; dx\\
&\geq -\, c_2^A\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, |R_h\underline{v}|_{[H^1(\Omega)]^n} + \tilde{c}_{s,1}\, \|q_h\|^2_{L_2(\Omega)} - \tilde{c}_{s,2}\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}\\
&\quad + \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (\nu\Delta\underline{v}_h)\cdot(\nu\Delta R_h\underline{v})\; dx - \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T \nabla q_h\cdot(\nu\Delta R_h\underline{v})\; dx\\
&\geq -\, c_2^A\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, |R_h\underline{v}|_{[H^1(\Omega)]^n} + \tilde{c}_{s,1}\, \|q_h\|^2_{L_2(\Omega)} - \tilde{c}_{s,2}\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}\\
&\quad - \delta\nu^2 \left(\sum_{T\in\mathcal{T}_h} h_T^2\, \|\Delta\underline{v}_h\|^2_{[L_2(T)]^n}\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^2\, \|\Delta R_h\underline{v}\|^2_{[L_2(T)]^n}\right)^{\frac{1}{2}}\\
&\quad - \delta\nu \left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^2\, \|\Delta R_h\underline{v}\|^2_{[L_2(T)]^n}\right)^{\frac{1}{2}}\\
&\geq -\, c_2^A\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, |R_h\underline{v}|_{[H^1(\Omega)]^n} + \tilde{c}_{s,1}\, \|q_h\|^2_{L_2(\Omega)} - \tilde{c}_{s,2}\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}\\
&\quad - \delta\nu^2 c_I^2\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, |R_h\underline{v}|_{[H^1(\Omega)]^n} - \delta\nu \left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} c_I\, |R_h\underline{v}|_{[H^1(\Omega)]^n}\\
&\geq -\, c_2^A\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, \|q_h\|_{L_2(\Omega)} + \tilde{c}_{s,1}\, \|q_h\|^2_{L_2(\Omega)} - \tilde{c}_{s,2}\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}\\
&\quad - \nu\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, \|q_h\|_{L_2(\Omega)} - \frac{1}{c_I}\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}\\
&= -\, c_1\, |\underline{v}_h|_{[H^1(\Omega)]^n}\, \|q_h\|_{L_2(\Omega)} + \tilde{c}_{s,1}\, \|q_h\|^2_{L_2(\Omega)} - c_2\left(\sum_{T\in\mathcal{T}_h} h_T^2\, |q_h|^2_{H^1(T)}\right)^{\frac{1}{2}} \|q_h\|_{L_2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{c_1}{2\alpha}\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} - \frac{c_1\alpha}{2}\left\|q_h\right\|^2_{L_2(\Omega)} + \tilde{c}_{s,1}\left\|q_h\right\|^2_{L_2(\Omega)} - \frac{c_2}{2\alpha}\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)} \\
&\quad - \frac{c_2\alpha}{2}\left\|q_h\right\|^2_{L_2(\Omega)} \\
&= -\frac{c_1}{2\alpha}\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + \left(\tilde{c}_{s,1} - \frac{\alpha}{2}(c_1+c_2)\right)\left\|q_h\right\|^2_{L_2(\Omega)} - \frac{c_2}{2\alpha}\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)} \\
&\geq -c_3\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + c_4\left\|q_h\right\|^2_{L_2(\Omega)} - c_5\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)},
\end{aligned}
$$

where we have chosen $0 < \alpha < (2\tilde{c}_{s,1})(c_1+c_2)^{-1}$. It follows that

$$
\begin{aligned}
&A(\underline{v}_h, q_h; \underline{v}_h, -q_h) \\
&= a(\underline{v}_h, \underline{v}_h) - \delta\sum_{T\in\mathcal{T}_h} h_T^2\int_T(-\nu\Delta\underline{v}_h + \nabla q_h)\cdot(-\nu\Delta\underline{v}_h - \nabla q_h)\; dx \\
&\geq \nu\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} - \delta\sum_{T\in\mathcal{T}_h} h_T^2\left(\nu^2\left\|\Delta\underline{v}_h\right\|^2_{[L_2(T)]^n} - \left|q_h\right|^2_{H^1(T)}\right) \\
&\geq \nu\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} - \delta\nu^2 c_I^2\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + \delta\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)} \\
&\geq c_6\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + \delta\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)}.
\end{aligned}
$$

Now we set

$$
\underline{w}_h = \underline{v}_h - \varepsilon R_h\underline{v} \quad \text{and} \quad r_h = -q_h
$$

for some ε which satisfies $0 < \varepsilon < \min\{\delta c_5^{-1}, c_6 c_3^{-1}\}$. It follows that

$$
\begin{aligned}
&A(\underline{v}_h, q_h; \underline{w}_h, r_h) \\
&= A(\underline{v}_h, q_h; \underline{v}_h, -q_h) + \varepsilon A(\underline{v}_h, q_h; -R_h\underline{v}, 0) \\
&\geq (c_6 - \varepsilon c_3)\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + \varepsilon c_4\left\|q_h\right\|^2_{L_2(\Omega)} + (\delta - \varepsilon c_5)\sum_{T\in\mathcal{T}_h} h_T^2\left|q_h\right|^2_{H^1(T)} \\
&\geq c_7\left(\left|\underline{v}_h\right|^2_{[H^1(\Omega)]^n} + \left\|q_h\right\|^2_{L_2(\Omega)}\right).
\end{aligned}
$$

From above we obtain

$$
\begin{aligned}
\left|\underline{w}_h\right|_{[H^1(\Omega)]^n} &\leq \left|\underline{v}_h\right|_{[H^1(\Omega)]^n} + \varepsilon\left|R_h\underline{v}\right|_{[H^1(\Omega)]^n} \leq c_8\left(\left|\underline{v}_h\right|_{[H^1(\Omega)]^n} + \left\|q_h\right\|_{L_2(\Omega)}\right), \\
\left\|r_h\right\|_{L_2(\Omega)} &= \left\|q_h\right\|_{L_2(\Omega)}
\end{aligned}
$$

and hence the following inequality

$$
\begin{aligned}
&\left(\|\underline{w}_h\|_{[H^1(\Omega)]^n} + \|r_h\|_{L_2(\Omega)}\right)\left(\|\underline{v}_h\|_{[H^1(\Omega)]^n} + \|q_h\|_{L_2(\Omega)}\right) \\
&\le \left((1+c_p)\,|\underline{w}_h|_{[H^1(\Omega)]^n} + \|r_h\|_{L_2(\Omega)}\right)\left((1+c_p)\,|\underline{v}_h|_{[H^1(\Omega)]^n} + \|q_h\|_{L_2(\Omega)}\right) \\
&\le c_9\left(|\underline{v}_h|_{[H^1(\Omega)]^n} + \|q_h\|_{L_2(\Omega)}\right)^2 \le c_{10}\left(|\underline{v}_h|^2_{[H^1(\Omega)]^n} + \|q_h\|^2_{L_2(\Omega)}\right) \\
&\le \frac{c_{10}}{c_7} A(\underline{v}_h, q_h; \underline{w}_h, r_h),
\end{aligned}
$$

is satisfied, which remains valid for the supremum. □

Remark 2.2. *It is important to remark, that for piecewise linear shape functions for $\underline{u}_h$, the linear forms (2.7)–(2.8) reduce to*

$$
\begin{aligned}
A(\underline{u}_h, p_h; \underline{v}_h, q_h) &= a(\underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) - b(\underline{u}_h, q_h) - \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \nabla p_h \cdot \nabla q_h \; dx, \\
F(\underline{v}_h, q_h) &= \langle \underline{f}, \underline{v}_h \rangle_\Omega - \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \underline{f} \cdot \nabla q_h \; dx.
\end{aligned}
$$

For this case it can be shown that there is no upper bound for the stabilization parameter δ necessary. If we additionally set $F(\underline{v}_h, q_h) = \langle \underline{f}, \underline{v}_h \rangle_\Omega$ we get the method of Brezzi and Pitkäranta, as given in (2.2).

2.1.2 Error estimates

In the following we will derive some error estimates for the Galerkin Least–Square method. For piecewise continuous shape functions we get the following theorem on the error.

Theorem 2.2. *Let $V_h = \left[S_h^k(\Omega) \cap H_0^1(\Omega)\right]^n$ and $Q_h = S_h^l(\Omega) \cap L_{2,0}(\Omega)$, with degree $k, l \ge 1$ be given ansatz spaces. If $0 < \delta_0 \le \delta < \nu^{-1} c_I^{-2}$ is valid, then for the Galerkin Least–Square method the error estimate*

$$
\|\underline{u} - \underline{u}_h\|_{[H^1(\Omega)]^n} + \|p - p_h\|_{L_2(\Omega)} \le c\left(h^k\,|\underline{u}|_{\left[H^{k+1}(\Omega)\right]^n} + h^{l+1}\,|p|_{H^{l+1}(\Omega)}\right) \tag{2.11}
$$

is valid, where $(\underline{u}, p) \in \left[H^{k+1}(\Omega)\right]^n \times H^{l+1}(\Omega)$ is the exact solution.

Proof. Let $I_h\underline{u} \in V_h$ be the interpolation of $\underline{u}$ and $J_h p \in Q_h$ the corresponding one of p. From Theorem 2.1 it follows, that there exist $\underline{v}_h \in V_h$ and $q_h \in Q_h$, such that

$$
\|\underline{v}_h\|_{[H^1(\Omega)]^n} + \|q_h\|_{L_2(\Omega)} \le c \tag{2.12}
$$

and

$$
\|I_h\underline{u} - \underline{u}_h\|_{[H^1(\Omega)]^n} + \|J_h p - p_h\|_{L_2(\Omega)} \le A(I_h\underline{u} - \underline{u}_h, J_h p - p_h; \underline{v}_h, q_h). \tag{2.13}
$$

From the variational formulation (2.6) the Galerkin orthogonality

$$A(\underline{u}-\underline{u}_h, p-p_h; \underline{v}_h, q_h) = 0$$

follows from which we obtain

$$A(I_h\underline{u}-\underline{u}_h, J_hp-p_h; \underline{v}_h, q_h) = A(I_h\underline{u}-\underline{u}, J_hp-p; \underline{v}_h, q_h).$$

Considering the local inverse inequality and the properties (2.12)–(2.13) we get the following estimate

$$\begin{aligned}
&\|I_h\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n} + \|J_hp-p_h\|_{L_2(\Omega)} \le A(I_h\underline{u}-\underline{u}, J_hp-p; \underline{v}_h, q_h)\\
&= a(I_h\underline{u}-\underline{u}, \underline{v}_h) - b(\underline{v}_h, J_hp-p) - b(I_h\underline{u}-\underline{u}, q_h)\\
&\quad - \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta(I_h\underline{u}-\underline{u}) + \nabla(J_hp-p)) \cdot (-\nu\Delta\underline{v}_h + \nabla q_h)\ dx\\
&\le c_2^A \left|\underline{u}-I_h\underline{u}\right|_{[H^1(\Omega)]^n} \left|\underline{v}_h\right|_{[H^1(\Omega)]^n} + \left|\underline{v}_h\right|_{[H^1(\Omega)]^n} \|p-J_hp\|_{L_2(\Omega)} + \left|\underline{u}-I_h\underline{u}\right|_{[H^1(\Omega)]^n} \|q_h\|_{L_2(\Omega)}\\
&\quad + \delta \sum_{T\in\mathcal{T}_h} h_T^2 \left(\nu \|\Delta(\underline{u}-I_h\underline{u})\|_{[L_2(T)]^n} + |p-J_hp|_{H^1(T)}\right) \left(\nu \|\Delta\underline{v}_h\|_{[L_2(T)]^n} + |q_h|_{H^1(T)}\right)\\
&\le c_1 \Big[\left|\underline{u}-I_h\underline{u}\right|_{[H^1(\Omega)]^n} \left|\underline{v}_h\right|_{[H^1(\Omega)]^n} + \|p-J_hp\|_{L_2(\Omega)} \left|\underline{v}_h\right|_{[H^1(\Omega)]^n} + \left|\underline{u}-I_h\underline{u}\right|_{[H^1(\Omega)]^n} \|q_h\|_{L_2(\Omega)}\\
&\quad + \|p-J_hp\|_{L_2(\Omega)} \|q_h\|_{L_2(\Omega)}\\
&\quad + \sum_{T\in\mathcal{T}_h} h_T \left(\|\Delta(\underline{u}-I_h\underline{u})\|_{[L_2(T)]^n} + |p-J_hp|_{H^1(T)}\right) \left(|\underline{v}_h|_{[H^1(T)]^n} + \|q_h\|_{L_2(T)}\right)\Big]\\
&\le c_2 \Big[\left|\underline{u}-I_h\underline{u}\right|_{[H^1(\Omega)]^n} + \|p-J_hp\|_{L_2(\Omega)}\\
&\quad + \sum_{T\in\mathcal{T}_h} h_T \left(\|\Delta(\underline{u}-I_h\underline{u})\|_{[L_2(T)]^n} + |p-J_hp|_{H^1(T)}\right)\Big].
\end{aligned}$$

By applying standard approximation theorems, see for example [4], we have for piecewise polynomial shape functions of the degree k for $\underline{u}$ and degree l for p with $k, l \ge 1$ the following statements:

$$\|\underline{u}-I_h\underline{u}\|_{[H^1(T)]^n} \le c h_T^k \left|\underline{u}\right|_{\left[H^{k+1}(T)\right]^n} \quad \text{and} \quad \|p-J_hp\|_{L_2(T)} \le c h_T^{l+1} |p|_{H^{l+1}(T)}.$$

Applying now the triangle inequality on the error we get the estimate

$$\begin{aligned}
&\|\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n} + \|p-p_h\|_{L_2(\Omega)}\\
&\le \|\underline{u}-I_h\underline{u}\|_{[H^1(\Omega)]^n} + \|I_h\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n} + \|p-J_hp\|_{L_2(\Omega)} + \|J_hp-p_h\|_{L_2(\Omega)}\\
&\le c\left(h^k \left|\underline{u}\right|_{\left[H^{k+1}(\Omega)\right]^n} + h^{l+1} |p|_{H^{l+1}(\Omega)}\right).
\end{aligned}$$

□

For an error estimate for the velocity $\underline{u}_h$ in the $[L_2(\Omega)]^n$–norm we prove next the following Aubin–Nitsche trick.

Lemma 2.2. *Let the assumptions of Theorem 2.2 be satisfied. If there exists a constant* $c_a > 0$ *such that* $\|\underline{u}\|_{[H^2(\Omega)]^n} + \|p\|_{H^1(\Omega)} \leq c_a \left\|\underline{f}\right\|_{[L_2(\Omega)]^n}$ *is satisfied, the error estimate*

$$\|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n} \leq c \left(h^{k+1} |\underline{u}|_{\left[H^{k+1}(\Omega)\right]^n} + h^{l+2} |p|_{H^{l+1}(\Omega)} \right) \tag{2.14}$$

follows.

Proof. Let $(\underline{w}, r) \in \left[H_0^1(\Omega)\right]^n \times L_{2,0}(\Omega)$ be the solution to the problem

$$\begin{aligned} -\nu\Delta\underline{w} + \nabla r &= \underline{u} - \underline{u}_h && \text{in } \Omega, \\ \nabla \cdot \underline{w} &= 0 && \text{in } \Omega, \\ \underline{w} &= \underline{0} && \text{on } \Gamma. \end{aligned}$$

Then, by assumption, $\|\underline{w}\|_{[H^2(\Omega)]^n} + \|r\|_{H^1(\Omega)} \leq c_a \|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n}$ holds. Let $I_h\underline{w} \in V_h$ and $J_h r \in Q_h$ be the interpolands of $\underline{w}$ and r respectively. By using the Galerkin orthogonality we get

$$\begin{aligned} & A(\underline{u} - \underline{u}_h, p - p_h; \underline{w} - I_h\underline{w}, r - J_h r) \\ &= a(\underline{u} - \underline{u}_h, \underline{w}) - b(\underline{w}, p - p_h) - b(\underline{u} - \underline{u}_h, r) \\ &\quad - \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu\Delta(\underline{u} - \underline{u}_h) + \nabla(p - p_h)) \cdot (\underline{u} - \underline{u}_h) \, dx. \end{aligned}$$

Moreover we obtain, by integration by parts,

$$\begin{aligned} & a(\underline{u} - \underline{u}_h, \underline{w}) - b(\underline{w}, p - p_h) - b(\underline{u} - \underline{u}_h, r) \\ &= \int_\Omega (-\nu\Delta\underline{w} + \nabla r) \cdot (\underline{u} - \underline{u}_h) \, dx = \|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n}^2 . \end{aligned}$$

Using the properties of the interpolations we get the estimate

$$\begin{aligned} & \|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n}^2 \\ &= A(\underline{u} - \underline{u}_h, p - p_h; \underline{w} - I_h\underline{w}, r - J_h r) \\ &\quad + \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu\Delta(\underline{u} - \underline{u}_h) + \nabla(p - p_h)) \cdot (\underline{u} - \underline{u}_h) \, dx \\ &\leq c_1 h \left[\|\underline{u} - \underline{u}_h\|_{[H^1(\Omega)]^n} + \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\Delta(\underline{u} - \underline{u}_h)\|_{[L_2(\Omega)]^n}^2 \right)^{\frac{1}{2}} + \|p - p_h\|_{L_2(\Omega)} \right. \\ &\quad \left. + \left(\sum_{T \in \mathcal{T}_h} h_T^2 |p - p_h|_{H^2(\Omega)}^2 \right)^{\frac{1}{2}} \right] \left(\|\underline{w}\|_{[H^2(\Omega)]^n} + \|r\|_{H^1(\Omega)} + \|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n} \right) \end{aligned}$$

and by using the assumption it follows that

$$\|\underline{u}-\underline{u}_h\|_{[L_2(\Omega)]^n} \leq c_2 h \left[\|\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n} + \left(\sum_{T\in\mathcal{T}_h} h_T^2 \left\|\Delta(\underline{u}-\underline{u}_h)\right\|^2_{[L_2(\Omega)]^n} \right)^{\frac{1}{2}} \right.$$
$$\left. + \|p-p_h\|_{L_2(\Omega)} + \left(\sum_{T\in\mathcal{T}_h} h_T^2 \left|p-p_h\right|^2_{H^2(\Omega)} \right)^{\frac{1}{2}} \right].$$

Finally we obtain the result by using the error estimate from Theorem 2.2, the triangle inequality and the standard approximation theorem. □

We have seen different error estimates for the Galerkin Least–Square method. In the following we present some numerical examples which confirm these estimates.

2.1.3 Numerical results

In the following we present some numerical examples for stabilized finite element methods for the Stokes equations. For this we consider the Galerkin Least–Square method (2.6). As a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ we consider for simplicity the unit square, where we take a symmetric uniform triangulation into consideration. We start with four elements and apply our refinement strategy by decomposing each triangle into four congruent ones, by taking the midpoints of each edge. Furthermore we consider $\nu = 1$ as viscosity constant and the following exact solution

$$\underline{u} = \begin{pmatrix} \cos(2\pi x_1)\sin(2\pi x_2) - \sin(2\pi x_2) \\ -\cos(2\pi x_2)\sin(2\pi x_1) + \sin(2\pi x_1) \end{pmatrix} \quad \text{and} \quad p = \sin(2\pi x_1)\cos(2\pi x_2) \tag{2.15}$$

which fulfills a homogeneous boundary condition. Moreover, we are using equal order 1 elements. As we have already mentioned in Remark 2.1, for this kind of shape functions all methods except (2.2) coincide. The resulting linear system of equations is solved by the direct solver PARDISO, see [29, 30].
Since the stabilization parameter $\delta \in (\delta_0, \nu^{-1}c_I^{-2})$ is still to choose, we will present two examples, where we do once the right choice of this constant and in the other case not. We will see which consequences such a failure has on the solution and the convergence of the error. In the following we will denote by L the level of refinement and by N the number of elements. At level $L = 1$ we have $N = 16$ elements with 23 degrees of freedom. Computations will be done up to level $L = 9$ with $N = 1048576$ elements and 1571843 degrees of freedom. The theory on stabilized finite element methods will be confirmed by getting the expected order of convergence in corresponding norms.

Example 1

It is obvious that we get for $\delta = 0$ the standard variational formulation, which is for equal order 1 elements an unstable method. That is why we want to start in this example with a rather small stabilization parameter, and choose for this reason $\delta = 1.0$ e-04.

L	$\|\underline{u}-\underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n}$	eoc	$\|p-p_h\|_{L_2(\Omega)}$	eoc
0	1.23675 e+00	–	8.30020 e+00	–	7.29825 e-01	–
1	5.38226 e-01	1.20	6.51614 e+00	0.35	5.15092 e-01	0.50
2	1.44395 e-01	1.90	3.02764 e+00	1.11	5.33179 e-01	–
3	4.16439 e-02	1.79	1.57269 e+00	0.94	1.04934 e+00	–
4	1.07568 e-02	1.95	7.92627 e-01	0.99	8.50726 e-01	0.30
5	2.69292 e-03	2.00	3.96132 e-01	1.00	6.66923 e-01	0.35
6	6.67586 e-04	2.01	1.97270 e-01	1.01	4.33483 e-01	0.62
7	1.65177 e-04	2.01	9.82703 e-02	1.01	2.07571 e-01	1.06
theory:		2		1		1

Table 2.1: Errors and order of convergence for $\delta = 1.0$ e-04.

In Table 2.1 we present errors for the velocity $\underline{u}$ and pressure p in corresponding norms. Since the exact solution fulfills the assumptions of Theorem 2.2 we can compare for $k = l = 1$ the rate of convergence with the error estimates (2.11) and (2.14). We expect linear convergence for $\underline{u}$ in the $[H^1(\Omega)]^n$–norm and for p in the $L_2(\Omega)$–norm. For this rather small stabilization parameter δ, the velocity converges as expected but for the pressure we do not get this result. Actually there is no convergence of the pressure on the first levels up to 1024 elements for this example, which is very important to be aware of. In Figure 2.1 we can see that this causes oscillations in the pressure p, but on the other hand this phenomena can not be seen in the velocity $\underline{u}$, which coincides with the convergence of the error.

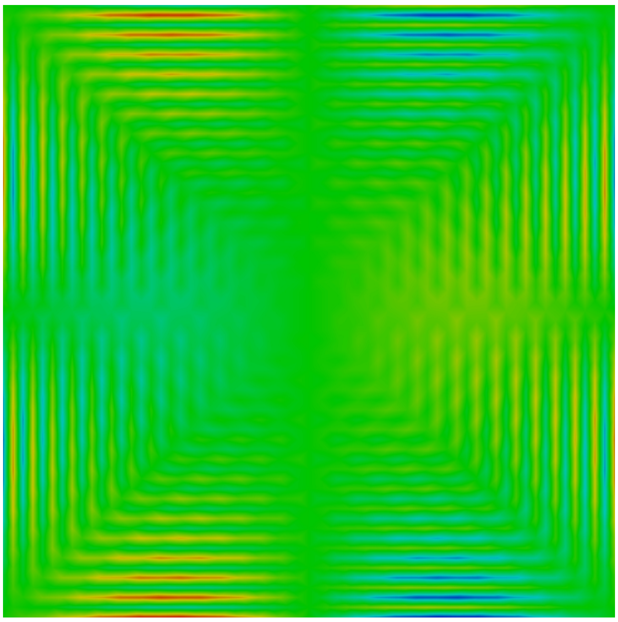

Figure 2.1: Oscillations of p with $N = 4096$ and $\delta = 1.0$ e-04.

Example 2

In this example we present results for a stabilization parameter $\delta = 0.08323$, see Table 2.2. As we will see later, this choice is nearly the optimal for the equal order 1 elements.

L	$\|\underline{u} - \underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|\underline{u} - \underline{u}_h\|_{[H^1(\Omega)]^n}$	eoc	$\|p - p_h\|_{L_2(\Omega)}$	eoc
0	1.23685 e+00	–	8.30057 e+00	–	6.72323 e-01	–
1	5.38555 e-01	1.20	6.51873 e+00	0.35	3.44121 e-01	0.97
2	1.37752 e-01	1.97	3.00964 e+00	1.11	2.52441 e-01	0.45
3	3.84498 e-02	1.84	1.54627 e+00	0.96	7.55617 e-02	1.74
4	9.93251 e-03	1.95	7.80300 e-01	0.99	2.39254 e-02	1.66
5	2.50092 e-03	1.99	3.91284 e-01	1.00	6.55964 e-03	1.87
6	6.26251 e-04	2.00	1.95796 e-01	1.00	1.69328 e-03	1.95
7	1.56638 e-04	2.00	9.79167 e-02	1.00	4.28586 e-04	1.98
8	3.91637 e-05	2.00	4.89606 e-02	1.00	1.07769 e-04	1.99
9	9.79119 e-06	2.00	2.44806 e-02	1.00	2.76877 e-05	1.96
theory:		2		1		1

Table 2.2: Errors and order of convergence for $\delta = 0.08323$.

For this example we get the expected linear convergence for $\underline{u}$ in the $[H^1(\Omega)]^n$–norm and quadratic in the $[L_2(\Omega)]^n$–norm. This coincides with the error estimates (2.11) and (2.14) of the Galerkin Least–Square method. It is important to mention that we also expect linear convergence for p, by the error estimate (2.14), but get nearly second order for this example.

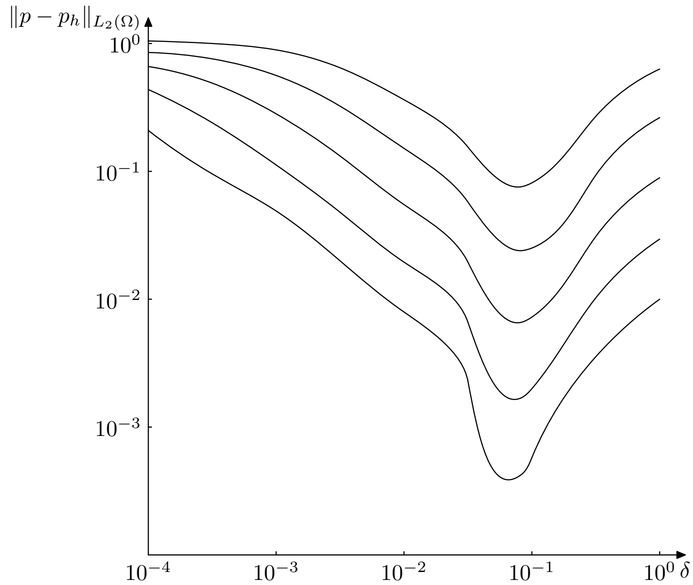

Figure 2.2: Errors for p with $N = 256, \ldots, 65536$.

Up to now we considered two examples for the stabilization parameter δ, one rather small value and at $\delta = 0.08323$. Now we want to study the behavior of the error for values of δ through a complete interval, which here will be $(0, 1]$. As a first point we want to mention that there is nearly no change of the error of $\underline{u}$ on each of the levels for different stabilization parameters δ. The attitude of the error of p is indeed much more interesting. We start with $\delta = 1.0$ e-04 from the first example. In Figure 2.2 we see, that for the $L_2(\Omega)$–error of the pressure p the value decreases up to the optimal stabilization parameter around $\delta = 0.08323$ and afterwards increases again. The higher the level, the faster the error decreases and increases around the optimum. For this figure we did computations up to $\delta = 1$.

2.2 Navier–Stokes equations

Similar to the previous section we apply the idea of stabilized finite elements to the Navier–Stokes equations (1.24) with homogeneous Dirichlet boundary conditions now. We consider the variational formulation of the Navier–Stokes equations (1.25) and add stabilization terms in both equations. This leads to the following problem: Find $(\underline{u}_h, p_h) \in V_h \times Q_h$, such that

$$\begin{aligned} &a(\underline{u}_h, \underline{v}_h) + a_1(\underline{u}_h, \underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) + \Psi_h(\underline{u}_h, p_h; \underline{v}_h) = \langle \underline{f}, \underline{v}_h \rangle_\Omega, \\ &b(\underline{u}_h, q_h) \qquad\qquad\qquad\qquad\qquad\qquad + \Phi_h(\underline{u}_h, p_h; q_h) = 0 \end{aligned} \tag{2.16}$$

is satisfied for all $(\underline{v}_h, q_h) \in V_h \times Q_h$. The corresponding stabilization terms are given by

$$\begin{aligned} \Psi_h(\underline{u}_h, p_h; \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{SUPG}} (-\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}) \cdot ((\underline{u}_h \cdot \nabla) \underline{v}_h - \rho \nu \Delta \underline{v}_h) \, dx \\ &\quad + \sum_{T \in \mathcal{T}_h} \int_T \xi \, (\nabla \cdot \underline{u}_h)(\nabla \cdot \underline{v}_h) \, dx, \\ \Phi_h(\underline{u}_h, p_h; q_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{PSPG}} (-\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}) \cdot \nabla q_h \, dx. \end{aligned}$$

The situation here is more difficult than for the Stokes equations before, since we have to deal with the nonlinearity. We see that the use of stabilization terms has the consequence of additional nonlinearities. An overview on stabilized finite element methods for the Navier–Stokes equations can be found for example in [8, 10, 24, 32, 33].
Another difference we are now dealing with, is that there are local stabilization parameters δ^{SUPG} (*Streamline upwinding Petrov–Galerkin*) and δ^{PSPG} (*Pressure Stabilization Petrov–Galerkin*). Moreover we are using an additional so called *grad–div stabilization* (GDS) with local parameters ξ. For this method we take for simplicity $\delta^{\text{SUPG}} = \delta^{\text{PSPG}}$ into consideration.

The stabilization parameters will be calculated as follows:

$$\delta^{\text{SUPG}} = \delta^{\text{PSPG}} = \begin{cases} \dfrac{h_T}{2\|\underline{u}_h\|_2}\,\theta\left(\dfrac{\lambda_1\|\underline{u}_h\|_2 h_T}{4\nu}\right), & \text{if } \|\underline{u}_h\|_2 \neq 0, \\ 0, & \text{else,} \end{cases}$$
$$\xi = \lambda_2\|\underline{u}_h\|_2 h_T\,\theta\left(\frac{\lambda_1\|\underline{u}_h\|_2 h_T}{4\nu}\right) \tag{2.17}$$

where

$$\theta(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \quad \text{and} \quad \lambda_1, \lambda_2 \in \mathbb{R}_+.$$

Remark 2.3. *We consider $\lambda_1 \in \mathbb{R}_+$. In [8], for example, the constant is considered to be a local one, and is so depending on the element. In this paper the constant is chosen as $\lambda_1 = \min\{1/3, 2c_I\}$, where c_I is a constant from a local inverse inequality. A similar approach can be found in [10].*
Other ways, how to choose the stabilization parameters can be found for example in [32, 33]. In the latter the two stabilization parameters δ^{SUPG} and δ^{PSPG} are not chosen to be equal, they are calculated by local element matrices.
The choice of the constant λ_2, which is acting as a scaling factor for the stabilization parameter ξ from the grad–div stabilization, is important for small viscosity constants ν respectively high Reynolds numbers. It has an huge influence on the number of Newton iterations. More details can be found in [24, 25].

Remark 2.4. *A proof of the stability and an error estimate for linear shape functions can be found for example in [17]. There the stabilization parameters differ from our chosen ones.*

Now we have to discuss how to treat the nonlinearity. Since we consider a stabilized finite element method we have to deal with more nonlinear terms, than in a standard approach. We using Newtons method and make two additional simplifications. In the SUPG stabilization term $(\underline{u}_h \cdot \nabla)\underline{v}_h$ is considered, which is depending on the solution $\underline{u}_h$. Instead we take $(\underline{u}_h^k \cdot \nabla)\underline{v}_h$ into consideration, where k denotes the previous Newton step. The same will be done for the calculation of the stabilization parameters, since they also depend on the solution $\underline{u}_h$.
Now we apply Newtons method to the variational formulation (2.16) and get for the $k+1$

Newton step the following formulation: Find $(\underline{u}_h^{k+1}, p_h^{k+1}) \in V_h \times Q_h$, such that

$$
\begin{aligned}
& a(\underline{u}_h^{k+1}, \underline{v}_h) + a_1(\underline{u}_h^{k+1}, \underline{u}_h^k, \underline{v}_h) + a_1(\underline{u}_h^k, \underline{u}_h^{k+1}, \underline{v}_h) \\
& \quad + \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{SUPG}} \left(-\nu \Delta \underline{u}_h^{k+1} + (\underline{u}_h^{k+1} \cdot \nabla) \underline{u}_h^k + (\underline{u}_h^k \cdot \nabla) \underline{u}_h^{k+1} \right) \cdot \left((\underline{u}_h^k \cdot \nabla) \underline{v}_h - \rho \nu \Delta \underline{v}_h \right) \, dx \\
& \quad - b(\underline{v}_h, p_h^{k+1}) + \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{SUPG}} \nabla p_h^{k+1} \cdot \left((\underline{u}_h^k \cdot \nabla) \underline{v}_h - \rho \nu \Delta \underline{v}_h \right) \, dx \\
& = \langle \underline{f}, \underline{v}_h \rangle_\Omega + a_1(\underline{u}_h^k, \underline{u}_h^k, \underline{v}_h) + \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{SUPG}} \left((\underline{u}_h^k \cdot \nabla) \underline{u}_h^k + \underline{f} \right) \cdot \left((\underline{u}_h^k \cdot \nabla) \underline{v}_h - \rho \nu \Delta \underline{v}_h \right) \, dx, \\
& b(\underline{u}_h^{k+1}, q_h) + \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{PSPG}} \left(-\nu \Delta \underline{u}_h^{k+1} + (\underline{u}_h^{k+1} \cdot \nabla) \underline{u}_h^k + (\underline{u}_h^k \cdot \nabla) \underline{u}_h^{k+1} \right) \cdot \nabla q_h \, dx \\
& \quad + \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{PSPG}} \nabla p_h^{k+1} \cdot \nabla q_h \, dx \\
& = \sum_{T \in \mathcal{T}_h} \int_T \delta^{\text{PSPG}} \left((\underline{u}_h^k \cdot \nabla) \underline{u}_h^k + \underline{f} \right) \cdot \nabla q_h \, dx
\end{aligned}
$$

is satisfied for all $(\underline{v}_h, q_h) \in V_h \times Q_h$.

Remark 2.5. *We see from the above linearized variational formulation that some terms do not depend on the solution of the previous Newton step k. Due to this reason we do not need to assemble all terms in each Newton step.*

2.2.1 Numerical results

In the following we present some numerical examples for the stabilized finite element method (2.16) of the Navier–Stokes equations. As a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ we consider for simplicity the unit square, where we take a symmetric uniform triangulation into consideration, as in the Stokes case, see Section 2.1.3. Furthermore we consider $\nu = 1$ as viscosity constant and the exact solution (2.15). Moreover we are using equal order 1 elements. As initial solution $(\underline{u}_h^0, p_h^0)$ for Newtons method we will use the solution of the Stokes equations. The Newton method will be stopped if $\|\underline{u}_h^{k+1} - \underline{u}_h^k\|_2 < \varepsilon$ is satisfied, with the accuracy $\varepsilon = 1.0$ e-08. In every Newton step the resulting linear system is solved by the direct solver PARDISO, see [29, 30].
Since the parameters $\lambda_1, \lambda_2 \in \mathbb{R}_+$ are still to choose, we will present two examples, with different constants. We will see which consequences the choice has on the solution and on the order of convergence of the error. In the following N_{iter} denotes the number of Newton iterations.

Example 1

In this first example we choose $\lambda_1 = 1/3$ and $\lambda_2 = 1$. In Table 2.3, we see the corresponding results, where we get second order of convergence for the velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm and first order in the $[H^1(\Omega)]^n$–norm. However, for the pressure p, we only get 3/2 as an order of convergence in the $L_2(\Omega)$–norm. This causes the choice of the constant λ_1 in the stabilization parameters.

L	N_{iter}	$\|\underline{u}-\underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n}$	eoc	$\|p-p_h\|_{L_2(\Omega)}$	eoc
0	6	1.23679 e+00	–	8.30035 e+00	–	9.85133 e-01	–
1	5	5.39185 e-01	1.20	6.51715 e+00	0.35	8.61788 e-01	0.19
2	5	1.40191 e-01	1.94	3.01200 e+00	1.11	3.89441 e-01	1.15
3	4	3.92741 e-02	1.84	1.54730 e+00	0.96	1.42218 e-01	1.45
4	4	1.01411 e-02	1.95	7.80595 e-01	0.99	4.79510 e-02	1.57
5	3	2.55176 e-03	1.99	3.91354 e-01	1.00	1.51560 e-02	1.66
6	3	6.38722 e-04	2.00	1.95813 e-01	1.00	4.90187 e-03	1.63
7	3	1.59721 e-04	2.00	9.79208 e-02	1.00	1.64091 e-03	1.58
8	3	3.99300 e-05	2.00	4.89617 e-02	1.00	5.62596 e-04	1.54
9	3	9.98217 e-06	2.00	2.44809 e-02	1.00	1.96986 e-04	1.51

Table 2.3: Errors and order of convergence for $\lambda_1 = 1/3$, $\lambda_2 = 1$.

Example 2

In this example we choose the following constants in the stabilization parameters, $\lambda_1 = 2/3$ and $\lambda_2 = 1$. In Table 2.4 we obtain second order of convergence of the error for the velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm and for the pressure p in the $L_2(\Omega)$–norm. Moreover we obtain first order of convergence for the velocity in the $[H^1(\Omega)]^n$–norm.

L	N_{iter}	$\|\underline{u}-\underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|\underline{u}-\underline{u}_h\|_{[H^1(\Omega)]^n}$	eoc	$\|p-p_h\|_{L_2(\Omega)}$	eoc
0	8	1.23685 e+00	–	8.30057 e+00	–	9.72977 e-01	–
1	5	5.39789 e-01	1.20	6.51850 e+00	0.35	7.88429 e-01	0.30
2	5	1.38582 e-01	1.96	3.01043 e+00	1.11	3.47962 e-01	1.18
3	5	3.87741 e-02	1.84	1.54648 e+00	0.96	1.03879 e-01	1.74
4	4	1.00218 e-02	1.95	7.80331 e-01	0.99	3.03018 e-02	1.78
5	4	2.52375 e-03	1.99	3.91288 e-01	1.00	8.06883 e-03	1.91
6	3	6.31992 e-04	2.00	1.95796 e-01	1.00	2.06264 e-03	1.97
7	3	1.58075 e-04	2.00	9.79168 e-02	1.00	5.20104 e-04	1.99
8	3	3.95232 e-05	2.00	4.89606 e-02	1.00	1.30546 e-04	1.99
9	3	9.88105 e-06	2.00	2.44806 e-02	1.00	3.32619 e-05	1.97

Table 2.4: Errors and order of convergence for $\lambda_1 = 2/3$, $\lambda_2 = 1$.

Remark 2.6. *In the numerical examples for the Stokes and Navier–Stokes equations, Section 2.1.3 and Section 2.2.1, we took for both the same exact solution (2.15). If we compare* *the errors for both equations, see Table 2.2 and Table 2.4, we see that the values of the errors nearly match. In particular we see this behavior for the velocity.*

We considered in this chapter, different stabilized finite element methods for Stokes and Navier–Stokes equations. For the Stokes equation we proved stability of the Galerkin Least–Square method, error estimates and presented related numerical results. In the second part we considered stabilizations for the Navier–Stokes equations and some related numerical examples. In the following chapter we want to apply these methods to the solution of the optimal control problems for the Stokes and Navier–Stokes equations, from chapter 1.

3 Stabilized finite element methods for optimal control problems

In this chapter we consider stabilized finite element methods for optimal control problems. In particular we will focus on optimal control problems for the Stokes and the Navier–Stokes equations. In chapter 1 we have seen those optimal control problems, where we focused on the analysis and derived the optimality system as an equivalent problem. Now we want to discretize these problems, by the use of stabilized finite element methods. An overview on stabilized finite element methods, for the Stokes and Navier–Stokes equations, was given in chapter 2. In this chapter we put the ideas of chapter 1 and 2 together for a discretization of the optimal control problem.
This chapter will be organized as follows: In the first section we will derive the discrete variational formulation for the optimal control problem of the Stokes equations, where we focus especially on low order elements. Moreover, we will describe the resulting linear system and treat the realization of the Steklov–Poincaré operator. The section will end by giving some related numerical results, where we especially address the issue of the difference between a control in $[L_2(\Gamma_c)]^n$ and $\left[H^{1/2}(\Gamma_c)\right]^n$.
In the second section we consider the optimal control problem of the Navier–Stokes equations. We introduce a discretization by using stabilized finite elements. Moreover we give numerical results where we also consider the difference of a control in $[L_2(\Gamma_c)]^n$ and $\left[H^{1/2}(\Gamma_c)\right]^n$ and their consequences. Finally, we consider a more realistic example, where the flow around an airfoil is considered. Here we also comment on the different control spaces and their consequences.

3.1 Stokes equations

In this section we consider the mixed optimal control problem for the Stokes equations (1.5)–(1.7), without box constraints. We consider the boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_c$ with a Dirichlet boundary Γ_D, a Neumann boundary Γ_N, and a control boundary Γ_c. Again we denote by S the Steklov–Poincaré operator applied to a control in $\left[H^{1/2}(\Gamma_c)\right]^n$. In this case we have to use the correct control space, which is due to the different boundary conditions, see Section 1.2.2.
In Section 1.2.3 we derived the corresponding optimality system, which is given by the following equations:

Primal problem

$$\begin{aligned} -\nu\Delta\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}, \end{aligned}$$

Adjoint problem

$$\begin{aligned} -\nu\Delta\underline{w} - \nabla r &= \underline{u} - \overline{\underline{u}} && \text{in } \Omega, \\ \nabla\cdot\underline{w} &= 0 && \text{in } \Omega, \\ \underline{w} &= \underline{0} && \text{on } \Gamma_\mathrm{D}\cup\Gamma_\mathrm{c}, \\ \nu(\nabla\underline{w})\underline{n} + r\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \end{aligned}$$

Optimality condition

$$-\nu(\nabla\underline{w})\underline{n} - r\underline{n} + \varrho S\underline{z} = 0 \qquad \text{on } \Gamma_\mathrm{c}.$$

In the following we derive the variational formulation of the optimality system in the continuous setting first. Afterwards we discretize the problem and introduce a corresponding stabilized finite element formulation. This stabilization will be reduced to the special case of equal order 1 elements. Afterwards we discuss the resulting linear system for a control in $[L_2(\Gamma_\mathrm{c})]^n$ and $\left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$, in particular we comment on the realization of the Steklov–Poincaré operator. In the last part of this section we present different numerical examples for the optimal control problems and discuss the difference of a control in $[L_2(\Gamma_\mathrm{c})]^n$ and $\left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$.

3.1.1 Variational formulation

In the following we derive the variational formulation for the optimality system. We start with the derivation of the variational formulation for the optimality condition. Let $\mathcal{E}\underline{\varphi} \in [H_0^1(\Omega,\Gamma_\mathrm{D})]^n$ denote an extension of some test function $\underline{\varphi} \in \left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$, or an appropriate subspace. Taking the adjoint equations into account and use integration by parts, we get

$$\int_{\Gamma_\mathrm{c}} (\nu(\nabla\underline{w})\underline{n} + r\underline{n})\cdot\underline{\varphi}\, ds_x = a(\underline{w},\mathcal{E}\underline{\varphi}) + b(\mathcal{E}\underline{\varphi}, r) - \int_\Omega (\underline{u} - \overline{\underline{u}})\cdot\mathcal{E}\underline{\varphi}\, dx.$$

This leads to the following form of the optimality condition

$$\varrho\int_{\Gamma_\mathrm{c}} S\underline{z}\cdot\underline{\varphi}\, ds_x = a(\underline{w},\mathcal{E}\underline{\varphi}) + b(\mathcal{E}\underline{\varphi}, r) - \int_\Omega (\underline{u} - \overline{\underline{u}})\cdot\mathcal{E}\underline{\varphi}\, dx.$$

Let Z be either $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$ or an appropriate subspace, see Section 1.2.2. With the standard variational formulation for the Stokes equations, see (1.9), we get the following variational formulation for the optimality system: Find $(\underline{u}, p, \underline{w}, r, \underline{z}) \in [H^1(\Omega)]^n \times L_2(\Omega) \times [H_0^1(\Omega, \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}})]^n \times L_2(\Omega) \times Z$ with $\underline{u} = \underline{g}$ on Γ_{D} and $\underline{u} = \underline{z}$ on Γ_{c} such that

$$\begin{aligned}
-\langle \underline{u}, \underline{\sigma}\rangle_\Omega \quad & + a(\underline{w}, \underline{\sigma}) + b(\underline{\sigma}, r) && = -\langle \overline{\underline{u}}, \underline{\sigma}\rangle_\Omega, \\
& b(\underline{w}, s) && = 0, \\
a(\underline{u}, \underline{v}) - b(\underline{v}, p) & && = \langle \underline{f}, \underline{v}\rangle_\Omega, \\
b(\underline{u}, q) & && = 0, \\
\langle \underline{u}, \mathcal{E}\underline{\varphi}\rangle_\Omega \quad & - a(\underline{w}, \mathcal{E}\underline{\varphi}) - b(\mathcal{E}\underline{\varphi}, r) + \varrho\langle S\underline{z}, \underline{\varphi}\rangle_{\Gamma_{\mathrm{c}}} && = \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}\rangle_\Omega
\end{aligned} \tag{3.1}$$

is satisfied for all $(\underline{v}, q, \underline{\sigma}, s, \underline{\varphi}) \in [H_0^1(\Omega, \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}})]^n \times L_2(\Omega) \times [H_0^1(\Omega, \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}})]^n \times L_2(\Omega) \times Z$, where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are those from (1.10).

Remark 3.1. *If there is no Neumann boundary Γ_{N} considered, we have to introduce the space $L_{2,0}(\Omega) \subset L_2(\Omega)$ for the primal and adjoint pressure. Then we are able to guarantee unique solvability.*

Remark 3.2. *For the above variational formulation (3.1), of the optimal control problem for the Stokes equations, there exists a unique determined solution, see Section 1.2.2.*

Next we derive the discrete variational formulation for the problem (3.1), where we are using stabilized finite elements. Therefore we introduce the following ansatz spaces:

$$\begin{aligned}
\widetilde{V}_h &= \left[S_h^k(\Omega)\right]^n \subset \left[H^1(\Omega)\right]^n, & V_h &= \left[S_h^k(\Omega) \cap H_0^1(\Omega, \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{c}})\right]^n, \\
Q_h &= S_h^l(\Omega) \subset L_2(\Omega), & Z_h &= \left[S_h^k(\Gamma_{\mathrm{c}})\right]^n \subset Z
\end{aligned} \tag{3.2}$$

of some polynomial order $k, l \in \mathbb{N}$. For the stabilized finite element method we add stabilization terms in the primal and adjoint equations, similar to Section 2.1. In the following we denote by I_h an appropriate interpolation. This leads to the following variational formulation: Find $(\underline{u}_h, p_h, \underline{w}_h, r_h, \underline{z}_h) \in \widetilde{V}_h \times Q_h \times V_h \times Q_h \times Z_h$ with $\underline{u}_h = I_h\underline{g}$ on Γ_{D} and $\underline{u}_h = \underline{z}_h$ on Γ_{c}, such that

$$\begin{aligned}
-\langle \underline{u}_h, \underline{\sigma}_h\rangle_\Omega \quad & + a(\underline{w}_h, \underline{\sigma}_h) + b(\underline{\sigma}_h, r_h) + \Psi_h^*(\underline{u}_h, \underline{w}_h, r_h; \underline{\sigma}_h) && = -\langle \overline{\underline{u}}, \underline{\sigma}_h\rangle_\Omega, \\
& b(\underline{w}_h, s_h) + \Phi_h^*(\underline{u}_h, \underline{w}_h, r_h; s_h) && = 0, \\
a(\underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) & + \Psi_h(\underline{u}_h, p_h; \underline{v}_h) && = \langle \underline{f}, \underline{v}_h\rangle_\Omega, \\
b(\underline{u}_h, q_h) & + \Phi_h(\underline{u}_h, p_h; q_h) && = 0, \\
\langle \underline{u}_h, \mathcal{E}\underline{\varphi}_h\rangle_\Omega \quad & - a(\underline{w}_h, \mathcal{E}\underline{\varphi}_h) - b(\mathcal{E}\underline{\varphi}_h, r_h) + \varrho\langle A\underline{z}_h, \underline{\varphi}_h\rangle_{\Gamma_{\mathrm{c}}} && = \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}_h\rangle_\Omega
\end{aligned} \tag{3.3}$$

is satisfied for all $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$, where the corresponding stabilization terms are given by

$$
\begin{aligned}
\Psi_h(\underline{u}_h, p_h; \underline{v}_h) &= \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{u}_h + \nabla p_h - \underline{f}) \cdot (-\rho\nu\Delta\underline{v}_h)\; dx, \\
\Phi_h(\underline{u}_h, p_h; q_h) &= \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{u}_h + \nabla p_h - \underline{f}) \cdot \nabla q_h\; dx, \\
\Psi_h^*(\underline{u}_h, \underline{w}_h, r_h; \underline{\sigma}_h) &= \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{w}_h - \nabla r_h - (\underline{u}_h - \underline{\overline{u}})) \cdot (-\rho\nu\Delta\underline{\sigma}_h)\; dx, \\
\Phi_h^*(\underline{u}_h, \underline{w}_h, r_h; s_h) &= \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{w}_h - \nabla r_h - (\underline{u}_h - \underline{\overline{u}})) \cdot \nabla s_h\; dx.
\end{aligned}
\tag{3.4}
$$

Now we describe the Galerkin Least–Square method, where $\rho = -1$ in (3.3). Since the equations must be valid for all test functions, we are able to subtract them. This leads to the following variational formulation: Find $(\underline{u}_h, p_h, \underline{w}_h, r_h, \underline{z}_h) \in \widetilde{V}_h \times Q_h \times V_h \times Q_h \times Z_h$ with $\underline{u}_h = I_h\underline{g}$ on Γ_{D} and $\underline{u}_h = \underline{z}_h$ on Γ_{c}, such that

$$
\begin{aligned}
A^*(\underline{u}_h, \underline{w}, r_h; \underline{\sigma}_h, s_h) &= F^*(\underline{\sigma}_h, s_h), \\
A(\underline{u}_h, p_h; \underline{v}_h, q_h) &= F(\underline{v}_h, q_h), \\
A^c(\underline{u}_h, \underline{w}, r_h, \underline{z}_h; \underline{\varphi}_h) &= F^c(\underline{\varphi}_h)
\end{aligned}
\tag{3.5}
$$

is satisfied for all $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$. The forms $A(\cdot,\cdot;\cdot,\cdot)$ and $F(\cdot,\cdot)$ are those from (2.7)–(2.8). The additional forms are given by

$$
\begin{aligned}
A^*(\underline{u}_h, \underline{w}, r_h; \underline{\sigma}_h, s_h) &= -\langle \underline{u}_h, \underline{\sigma}_h\rangle_\Omega + a(\underline{w}_h, \underline{\sigma}_h) + b(\underline{\sigma}_h, r_h) - b(\underline{w}_h, s_h) \\
&\quad - \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T (-\nu\Delta\underline{w}_h - \nabla r_h - \underline{u}_h) \cdot (-\nu\Delta\underline{\sigma}_h + \nabla s_h)\; dx, \\
A^c(\underline{u}_h, \underline{w}, r_h, \underline{z}_h; \underline{\varphi}_h) &= \langle \underline{u}_h, \mathcal{E}\underline{\varphi}_h\rangle_\Omega - a(\underline{w}_h, \mathcal{E}\underline{\varphi}_h) - b(\mathcal{E}\underline{\varphi}_h, r_h) + \varrho\langle A\underline{z}_h, \underline{\varphi}_h\rangle_{\Gamma_{\mathrm{c}}}
\end{aligned}
$$

and

$$
\begin{aligned}
F^*(\underline{\sigma}_h, s_h) &= \langle \underline{f}, \underline{v}_h\rangle_\Omega + \delta \sum_{T\in\mathcal{T}_h} h_T^2 \int_T \underline{\overline{u}} \cdot (-\nu\Delta\underline{\sigma}_h + \nabla s_h)\; dx, \\
F^c(\underline{\varphi}_h) &= \langle \underline{\overline{u}}, \mathcal{E}\underline{\varphi}_h\rangle_\Omega.
\end{aligned}
$$

The aim is to discretize the problem (3.5) with low order elements. In the following we present a simplification of the above problem, where we use equal order 1 elements. This has the consequence that all second order derivatives in the stabilization terms in (3.4) vanish. For the Dirichlet datum we consider an extension $\underline{u}_g \in [H_0^1(\Omega,\Gamma_{\mathrm{c}})]^n$, and for the boundary control $\underline{z}_h \in Z_h$ we introduce an extension $\underline{u}_{z,h} \in \widetilde{V}_h \cap [H_0^1(\Omega,\Gamma_{\mathrm{D}})]^n$. If we consider equal order 1 elements, we obtain the following variational formulation: Find

$(\underline{u}_{0,h}, p_h, \underline{w}_h, r_h, \underline{z}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$, such that

$$\begin{aligned}
&-\langle \underline{u}_{0,h}, \underline{\sigma}_h \rangle_\Omega && && + a(\underline{w}_h, \underline{\sigma}_h) && + b(\underline{\sigma}_h, r_h) && - \langle \underline{u}_{z,h}, \underline{\sigma}_h \rangle_\Omega && = -\langle \overline{\underline{u}}, \underline{\sigma}_h \rangle_\Omega \\
& && && && && && + \langle \underline{u}_g, \underline{\sigma}_h \rangle_\Omega, \\
&-d_h(\underline{u}_{0,h}, s_h) && && + b(\underline{w}_h, s_h) && - c_h(s_h, r_h) && - d_h(\underline{u}_{z,h}, s_h) && = -d_h(\overline{\underline{u}}, s_h) \\
& && && && && && + d_h(\underline{u}_g, s_h), \\
&a(\underline{u}_{0,h}, \underline{v}_h) && - b(\underline{v}_h, p_h) && && && + a(\underline{u}_{z,h}, \underline{v}_h) && = \langle \underline{f}, \underline{v}_h \rangle_\Omega \\
& && && && && && - a(\underline{u}_g, \underline{v}_h), \\
&b(\underline{u}_{0,h}, q_h) && + c_h(q_h, p_h) && && && + b(\underline{u}_{z,h}, q_h) && = d_h(\underline{f}, q_h) \\
& && && && && && - b(\underline{u}_g, q_h), \\
&\langle \underline{u}_{0,h}, \mathcal{E}\underline{\varphi}_h \rangle_\Omega && && - a(\underline{w}_h, \mathcal{E}\underline{\varphi}_h) && - b(\mathcal{E}\underline{\varphi}_h, r_h) && + \langle \underline{u}_{z,h}, \mathcal{E}\underline{\varphi}_h \rangle_\Omega && \\
& && && && && + \varrho \langle A\underline{z}_h, \underline{\varphi}_h \rangle_{\Gamma_c} && = \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}_h \rangle_\Omega \\
& && && && && && - \langle \underline{u}_g, \mathcal{E}\underline{\varphi}_h \rangle_\Omega
\end{aligned} \tag{3.6}$$

is satisfied for all test function $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$, where the additional bilinear forms are given by

$$\begin{aligned}
c_h(q_h, p_h) &= \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \nabla p_h \cdot \nabla q_h \, dx, \\
d_h(\underline{v}_h, q_h) &= \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T \underline{v}_h \cdot \nabla q_h \, dx.
\end{aligned} \tag{3.7}$$

3.1.2 Linear system

In this section we consider the variational formulation (3.6) of the optimal control problem for the Stokes equations with equal order 1 elements. This problem is equivalent to a linear system, which we describe in this section. In particular, we consider the realization of the Steklov–Poincaré operator for a control in $\left[H^{1/2}(\Gamma)\right]^n$. Let us consider the discrete ansatz spaces (3.2), where we use polynomials of order 1 for the velocity as well as for the pressure, i.e. $k = l = 1$. Those spaces are spanned by the following basis:

$$V_h = \operatorname{span}\{\underline{\varphi}_i\}_{i=1}^m \quad \text{and} \quad Q_h = \operatorname{span}\{\psi_i\}_{i=1}^m.$$

The bilinear forms of the variational formulation (3.6) induce the following matrices with entries

$$\begin{aligned}
A_h[i,j] &= a(\underline{\varphi}_j, \underline{\varphi}_i), & M_h[i,j] &= \langle \underline{\varphi}_j, \underline{\varphi}_i \rangle_\Omega, \\
B_h[i,j] &= b(\underline{\varphi}_j, \psi_i), & D_h[i,j] &= d_h(\underline{\varphi}_j, \psi_i), \\
C_h[i,j] &= c_h(\psi_j, \psi_i)
\end{aligned} \tag{3.8}$$

for all $i, j = 1, \ldots, m$. We write the assembled matrices in the following form

$$\begin{aligned} A_h &= \begin{pmatrix} A_{II} & A_{IC} \\ A_{CI} & A_{CC} \end{pmatrix}, & M_h &= \begin{pmatrix} M_{II} & M_{IC} \\ M_{CI} & M_{CC} \end{pmatrix}, \\ B_h &= \begin{pmatrix} B_I & B_C \end{pmatrix}, & D_h &= \begin{pmatrix} D_I & D_C \end{pmatrix}, \end{aligned}$$

where the index I denotes the interior and Neumann degrees of freedom. The index C denotes the degrees of freedom of the control.
Note that we obtain for the primal and adjoint Stokes equations the following block structure

$$\begin{pmatrix} A_{II} & -B_I^\top \\ B_I & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{II} & B_I^\top \\ B_I & \end{pmatrix}.$$

Since we consider equal order 1 elements, no stabilization occurs in the respective first equations of the primal and adjoint equations. Due to this reason the blocks from the linear system of the primal and the adjoint problem occur in the global linear system without additional terms. In the following, we derive the global linear system for a control in $[L_2(\Gamma_c)]^n$ as well as for $\left[H^{1/2}(\Gamma_c)\right]^n$, or in an appropriate subspace.

$L_2(\Gamma)$–control

In the first case we consider the control in $[L_2(\Gamma_c)]^n$, which leads to a linear system of the following form

$$\begin{pmatrix} -M_{II} & & A_{II} & B_I^\top & -M_{IC} \\ -D_I & & B_I & -C & -D_C \\ A_{II} & -B_I^\top & & & A_{IC} \\ B_I & C & & & B_C \\ M_{CI} & & -A_{CI} & -B_C^\top & M_{CC} + \varrho \widetilde{M}_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{w}_I \\ \underline{r} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} \underline{g}_I \\ \underline{g}_D \\ \underline{f}_I \\ \underline{f}_D \\ \underline{g}_C \end{pmatrix}, \tag{3.9}$$

where the mass matrix on the boundary is given by

$$\widetilde{M}_h[i, j] = \langle \underline{\varphi}_j, \underline{\varphi}_i \rangle_{\Gamma_c}$$

for all $i, j = \widetilde{m}, \ldots, m$. The blocks C, D_I and D_C are those which appear from the stabilization. If we consider, for example, a discretization with Taylor–Hood elements without any stabilization, these parts are not necessary.

$H^{1/2}(\Gamma)$–control

Next we consider the control in $\left[H^{1/2}(\Gamma_c)\right]^n$, or in an appropriate subspace, see Section 1.2.2. The Steklov–Poincaré operator induces the matrix $\widetilde{S}_h$. We already mentioned in Section 1.2.2, how to realize the Steklov–Poincaré operator via the solution of a boundary value problem of the vector valued Laplacian. Due to this reason the matrix $\widetilde{S}_h$ can be represented by the Schur complement, i.e.

$$\widetilde{S}_h = A_{CC} - A_{CI} A_{II}^{-1} A_{IC}.$$

This approach induces an inverse matrix A_{II}^{-1}. A possibility to handle this, is to introduce a new variable $\underline{x} := A_{II}^{-1} A_{IC} \underline{z}$. The resulting linear system is then given by

$$\begin{pmatrix} -M_{II} & & A_{II} & B_I^\top & -M_{IC} & \\ -D_I & & B_I & -C & -D_C & \\ A_{II} & -B_I^\top & & & A_{IC} & \\ B_I & C & & & B_C & \\ M_{CI} & & -A_{CI} & -B_C^\top & M_{CC} + \varrho A_{CC} & -\varrho A_{CI} \\ & & & & \varrho A_{IC} & -\varrho A_{II} \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{w}_I \\ \underline{r} \\ \underline{z} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \underline{g}_I \\ \underline{g}_D \\ \underline{f}_I \\ \underline{f}_D \\ \underline{g}_C \\ \underline{0} \end{pmatrix}. \tag{3.10}$$

Consequently, for the realization of the control in $\left[H^{1/2}(\Gamma_c)\right]^n$, respective the Steklov–Poincaré operator we do not need any additional terms. All this can be done by considering the matrix A_h, which will be needed in any case.

Remark 3.3. *Here we will solve the linear system by a direct solver, as explained in the next section. Especially for 3D computations and for time dependent problems preconditioned iterative solvers may be used instead.*

3.1.3 Numerical results

In the following we present some numerical examples for the stabilized finite element method for the optimal control problem of the Stokes equations (1.5)–(1.6). The control is either considered in $[L_2(\Gamma_c)]^n$ or in $\left[H^{1/2}(\Gamma_c)\right]^n$. In particular, we will discuss the difference of these two choices. As a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ we consider for simplicity the unit square, where we take a symmetric uniform triangulation into consideration, as in Section 2.1.3. The complete boundary is considered as a control boundary, i.e. $\Gamma = \Gamma_c$. Furthermore, we consider the viscosity constant $\nu = 1$ and the following tracking function and right hand side, which are given by

$$\overline{\underline{u}} = \begin{pmatrix} x_1(x_1 - 1) + 1 \\ x_2(x_2 - 1) + 1 \end{pmatrix} \quad \text{and} \quad \underline{f} = \underline{1}. \tag{3.11}$$

Moreover, we are using equal order 1 elements, which leads to the linear system (3.9) for a $[L_2(\Gamma)]^n$ control and to (3.10) for a $\left[H^{1/2}(\Gamma)\right]^n$ control, respectively. We already mentioned in Remark 2.1, that for this kind of shape functions all stabilized finite element methods except (2.2) coincide, for the primal as well as for the adjoint equations. For the stabilization parameter we choose $\delta = 0.08323$, which we also considered for the Stokes equations in Section 2.1.3. Since we do not know the exact solution of the optimal control problem for the given data (3.11), we take the solution from level $L = 9$ with $N = 1048576$

elements, which we denote by $(\underline{u}_{h_9}, p_{h_9}, \underline{w}_{h_9}, r_{h_9}, \underline{z}_{h_9})$, as reference solution. Thus, we will get slightly better results than for a standard error calculation, in particular we will see this in the estimated order of convergence of the error. The resulting linear system of equations is solved by the direct solver PARDISO, see [29, 30]. In the following we present numerical results for a $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$ control, where we especially focus on the difference of these two control spaces. We consider different cost coefficients $\varrho > 0$.

$L_2(\Gamma)$–control

In the first numerical example we consider the control in $[L_2(\Gamma)]^n$. The corresponding linear system is given by (3.9).

L	$\left\|\underline{u}_{h_9} - \underline{u}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{[L_2(\Gamma)]^n}$	eoc
0	2.04163 e-01	–	7.28408 e+00	–	3.70473 e-01	–
1	1.20184 e-01	0.76	6.81423 e+00	0.10	2.28836 e-01	0.70
2	7.06360 e-02	0.77	6.29465 e+00	0.11	1.62837 e-01	0.49
3	3.86655 e-02	0.87	5.71430 e+00	0.14	1.14279 e-01	0.51
4	2.06899 e-02	0.90	5.05445 e+00	0.18	7.98782 e-02	0.52
5	1.07972 e-02	0.94	4.30991 e+00	0.23	5.54419 e-02	0.53
6	5.42025 e-03	0.99	3.46626 e+00	0.31	3.78323 e-02	0.55
7	2.51623 e-03	1.11	2.50409 e+00	0.47	2.47553 e-02	0.61
8	9.43471 e-04	1.42	1.39883 e+00	0.84	1.42914 e-02	0.79
		1		0.5		0.5

Table 3.1: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

L	$\left\|\underline{w}_{h_9} - \underline{w}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	1.17440 e-03	–	7.74222 e-02	–
1	7.94678 e-04	0.56	3.19051 e-02	1.28
2	2.78431 e-04	1.51	1.21914 e-02	1.39
3	8.96704 e-05	1.63	5.06974 e-03	1.27
4	2.73687 e-05	1.71	2.22068 e-03	1.19
5	8.91721 e-06	1.62	1.00021 e-03	1.15
6	3.17598 e-06	1.49	4.48316 e-04	1.16
7	1.16110 e-06	1.45	1.88095 e-04	1.25
8	3.55650 e-07	1.71	6.20343 e-05	1.60
		1.5		1.2

Table 3.2: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

In Table 3.1 we give the errors for the primal velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm, for the primal pressure p in the $L_2(\Omega)$–norm and for the control $\underline{z}$ in the $[L_2(\Gamma)]^n$–norm, for the cost coefficient $\varrho = 1$. We get nearly first order of convergence for the primal velocity $\underline{u}$. For the primal pressure p and for the control $\underline{z}$ we obtain an order of convergence close to 0.5. In Table 3.2 we present the results for the adjoint velocity $\underline{w}$ in the $[L_2(\Omega)]^n$–norm and for the adjoint pressure r in the $L_2(\Omega)$–norm. We get an order of convergence near to 1.5 for the adjoint velocity and approximately 1.2 for the adjoint pressure.

L	$\|\underline{u}_{h_9} - \underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\|\underline{z}_{h_9} - \underline{z}_h\|_{[L_2(\Gamma)]^n}$	eoc
0	7.45066 e-02	–	1.02944 e+01	–	4.62326 e-01	–
1	5.03206 e-02	0.57	1.02007 e+01	0.01	3.89629 e-01	0.25
2	3.55392 e-02	0.50	9.99927 e+00	0.03	3.59060 e-01	0.12
3	2.53705 e-02	0.49	9.63007 e+00	0.05	2.90842 e-01	0.30
4	1.68865 e-02	0.59	8.94089 e+00	0.11	2.09139 e-01	0.48
5	1.04347 e-02	0.69	7.89583 e+00	0.18	1.38816 e-01	0.59
6	6.03102 e-03	0.79	6.51738 e+00	0.28	8.85768 e-02	0.65
7	3.20276 e-03	0.91	4.81769 e+00	0.44	5.52332 e-02	0.68
8	1.39943 e-03	1.19	2.76369 e+00	0.80	3.13394 e-02	0.82
		1		0.5		0.6

Table 3.3: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1.0$ e-02.

L	$\|\underline{w}_{h_9} - \underline{w}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	6.83056 e-04	–	1.17363 e-02	–
1	2.30556 e-04	1.57	2.32271 e-03	2.34
2	7.05226 e-05	1.71	1.54781 e-03	0.59
3	2.34120 e-05	1.59	6.68842 e-04	1.21
4	6.21951 e-06	1.91	2.78011 e-04	1.27
5	1.59974 e-06	1.96	1.07399 e-04	1.37
6	4.02552 e-07	1.99	3.85900 e-05	1.48
7	9.73901 e-08	2.05	1.28824 e-05	1.58
8	2.04042 e-08	2.25	3.53038 e-06	1.87
		2		1.5

Table 3.4: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1.0$ e-02.

Similarly we present in Table 3.3 and Table 3.4 numerical results for the cost coefficient $\varrho = 1.0$ e-02. For the primal velocity $\underline{u}$ we obtain first order convergence in the $[L_2(\Omega)]^n$–norm and for the primal pressure p we obtain close to 0.5 as order of convergence in the $L_2(\Omega)$–norm. The order of convergence for the control $\underline{z}$ in the $[L_2(\Gamma)]^n$–norm is close to 0.6. Moreover, we obtain second order of convergence for the adjoint velocity $\underline{w}$ in the $[L_2(\Omega)]^n$–norm and around 1.5 as order of convergence for the adjoint pressure r in the $L_2(\Omega)$–norm.

	$\varrho = 1$	$\varrho = 1.0$ e-02
L	$\Vert \underline{u}_h - \overline{\underline{u}} \Vert^2_{[L_2(\Omega)]^n}$	$\Vert \underline{u}_h - \overline{\underline{u}} \Vert^2_{[L_2(\Omega)]^n}$
0	8.55461 e-01	1.38664 e-02
1	6.98711 e-01	1.31973 e-02
2	6.54036 e-01	1.36744 e-02
3	6.30896 e-01	1.38943 e-02
4	6.20928 e-01	1.38696 e-02
5	6.16574 e-01	1.37247 e-02
6	6.14614 e-01	1.36239 e-02
7	6.13702 e-01	1.35807 e-02
8	6.13267 e-01	1.35655 e-02

Table 3.5: Tracking functional for $[L_2(\Gamma)]^n$–control.

In Table 3.5 we present the numerical results of the tracking functional $\Vert \underline{u}_h - \overline{\underline{u}} \Vert^2_{[L_2(\Omega)]^n}$ for the cost coefficients $\varrho = 1$ and $\varrho = 1.0$ e-02. We remark that the values of the tracking functional are higher for the cost coefficient $\varrho = 1$. This is due to the fact that in this case we consider a higher cost coefficient and hence the control has more influence on the solution. From this point of view, we are also able to consider the cost coefficient ϱ as a regularization parameter.

$H^{1/2}(\Gamma)$–control

In the second numerical example we consider the control in $\left[H^{1/2}(\Gamma)\right]^n$, which is realized by the Steklov–Poincaré operator S. The given data are the same as in the previous case for the control in $[L_2(\Gamma)]^n$, to compare the solutions later on. The corresponding linear system is given by (3.10).

L	$\Vert \underline{u}_{h_9} - \underline{u}_h \Vert_{[L_2(\Omega)]^n}$	eoc	$\Vert p_{h_9} - p_h \Vert_{L_2(\Omega)}$	eoc	$\Vert \underline{z}_{h_9} - \underline{z}_h \Vert_{[L_2(\Gamma)]^n}$	eoc
0	2.12995 e-03	–	2.22242 e-02	–	5.58722 e-03	–
1	1.51862 e-03	0.49	1.77507 e-02	0.32	3.45150 e-03	0.69
2	3.96755 e-04	1.94	2.94866 e-03	2.59	7.67450 e-04	2.17
3	1.08915 e-04	1.87	1.15375 e-03	1.35	2.26969 e-04	1.76
4	2.88255 e-05	1.92	6.40047 e-04	0.85	6.47265 e-05	1.81
5	7.39090 e-06	1.96	2.85279 e-04	1.17	1.73715 e-05	1.90
6	1.85191 e-06	2.00	1.10594 e-04	1.37	4.46905 e-06	1.96
7	4.45214 e-07	2.06	3.85642 e-05	1.52	1.09238 e-06	2.03
8	9.08554 e-08	2.29	1.09900 e-02	1.81	2.27189 e-07	2.27
		2		1.5		2

Table 3.6: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

L	$\left\|\underline{w}_{h_9} - \underline{w}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	6.88230 e-04	–	1.03416 e-02	–
1	1.41745 e-04	2.28	4.41739 e-03	1.23
2	2.07336 e-05	2.77	9.66792 e-04	2.19
3	2.17883 e-06	3.25	2.25512 e-04	2.10
4	3.23744 e-07	2.75	5.36731 e-05	2.07
5	7.04321 e-08	2.20	1.30379 e-05	2.04
6	1.70513 e-08	2.05	3.20983 e-06	2.02
7	4.06891 e-09	2.07	7.94715 e-07	2.01
8	8.35861 e-10	2.28	1.91924 e-07	2.05
		2		2

Table 3.7: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

In Table 3.6 we present the errors, for the cost coefficient $\varrho = 1$, for the primal velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm, for the primal pressure p in the $L_2(\Omega)$–norm and for the control $\underline{z}$ in the $[L_2(\Gamma)]^n$–norm. For the velocity $\underline{u}$ we obtain second order of convergence, for the pressure p we obtain 1.5 as order of convergence and for the control $\underline{z}$ also second order. The errors for the adjoint variables are given in Table 3.8. Here we present the errors of the adjoint velocity $\underline{w}$ in the $[L_2(\Omega)]^n$–norm and the adjoint pressure r in the $L_2(\Omega)$–norm. For both we obtain second order of convergence in the corresponding norms.

L	$\left\|\underline{u}_{h_9} - \underline{u}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{[L_2(\Gamma)]^n}$	eoc
0	2.87115 e-02	–	2.62239 e-01	–	9.93803 e-02	–
1	1.83660 e-02	0.64	2.48259 e-01	0.08	5.88196 e-02	0.76
2	5.13242 e-03	1.84	1.93535 e-01	0.36	1.30867 e-02	2.17
3	2.09703 e-03	1.29	1.43905 e-01	0.43	5.68438 e-03	1.20
4	7.61095 e-04	1.46	8.38048 e-02	0.78	2.15886 e-03	1.40
5	2.39335 e-04	1.67	4.18159 e-02	1.00	6.64430 e-04	1.70
6	6.90416 e-05	1.79	1.92336 e-02	1.12	1.83137 e-04	1.86
7	1.84545 e-05	1.90	8.25194 e-03	1.22	4.65457 e-05	1.98
8	4.16817 e-06	2.15	2.89102 e-03	1.51	1.00445 e-05	2.21
		2		1.2		2

Table 3.8: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1.0$ e-02.

For the cost coefficient $\varrho = 1.0$ e-02 we present the numerical results in Table 3.8 and Table 3.9. We obtain second order convergence for the primal velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm as well as for the control $\underline{z}$ in the $[L_2(\Gamma)]^n$–norm. For the primal pressure p we get around 1.2 as order of convergence in the $L_2(\Omega)$–norm. For the adjoint velocity $\underline{w}$ as well as for the adjoint pressure r we obtain second order convergence in the $[L_2(\Omega)]^n$–norm and $L_2(\Omega)$–norm, respectively.

L	$\left\|\underline{w}_{h_9} - \underline{w}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\left\|r_{h_9} - r_h\right\|_{L_2(\Omega)}$	eoc
0	6.55258 e-04	–	7.92555 e-03	–
1	1.34114 e-04	2.29	2.09168 e-03	1.92
2	1.82894 e-05	2.87	8.09764 e-04	1.37
3	3.89455 e-06	2.23	2.18467 e-04	1.89
4	9.94573 e-07	1.97	5.65031 e-05	1.95
5	2.51936 e-07	1.98	1.43478 e-05	1.98
6	6.26223 e-08	2.01	3.61496 e-06	1.99
7	1.50811 e-08	2.05	8.96047 e-07	2.01
8	3.18304 e-09	2.24	2.05164 e-07	2.13
		2		2

Table 3.9: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1.0$ e-02.

	$\varrho = 1$	$\varrho = 1.0$ e-02
L	$\|\underline{u}_h - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n}$	$\|\underline{u}_h - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n}$
0	1.10734 e-02	1.06883 e-02
1	1.09908 e-02	9.11345 e-03
2	1.09058 e-02	8.52571 e-03
3	1.08806 e-02	8.34883 e-03
4	1.08737 e-02	8.30174 e-03
5	1.08719 e-02	8.28773 e-03
6	1.08714 e-02	8.28381 e-03
7	1.08712 e-02	8.28277 e-03
8	1.08712 e-02	8.28250 e-03

Table 3.10: Tracking functional for $\left[H^{1/2}(\Gamma)\right]^n$–control.

In Table 3.5 we present the numerical results of the tracking functional $\|\underline{u}_h - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n}$ for the cost coefficients $\varrho = 1.0$ and $\varrho = 1.0$ e-02. Again we obtain smaller values for smaller cost coefficient ϱ.

Comparison

In the following we compare the control in $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$. We have seen in the last two sections numerical examples for these two types of control spaces and we want to compare their solutions now. In particular, we focus on the consequences. First of all, let us consider the optimality condition for a control in $[L_2(\Gamma)]^n$, which is given by

$$-\nu(\nabla \underline{w})\underline{n} - r\underline{n} + \varrho\underline{z} = \underline{0} \qquad \text{on } \Gamma.$$

In the former numerical examples we have choose as computational domain $\Omega = (0,1)^2$. Since we have to consider homogeneous Dirichlet boundary conditions on the whole boundary Γ for the adjoint velocity $\underline{w}$ we can conclude

$$(\nabla \underline{w})\underline{n} = \underline{0}$$

in each corner point of the domain Ω. Due to this reason the optimality condition reads

$$\varrho \underline{z} = r\underline{n} \tag{3.12}$$

in each corner point. Since the control was assumed to be continuous, and since the normal vector is discontinuous in each corner point, we conclude

$$r = 0 \quad \text{and} \quad \underline{z} = \underline{0}$$

in each corner point of the domain Ω. This is now an necessary condition, which the control $\underline{z}$ and the adjoint pressure r have to satisfy. On the other hand, if we consider the optimal control in $\left[H^{1/2}(\Gamma)\right]^n$, we get

$$\varrho S\underline{z} = r\underline{n}$$

in each corner point. In this case, the Steklov–Poincaré operator S realizes the proper mapping.

Remark 3.4. *More generally, we get for an arbitrary corner point of a domain, by multiplying (3.12) by the outer normal vector, $\varrho \underline{z} \cdot \underline{n} = r$. From this we conclude*

$$\varrho \underline{z} \cdot (\underline{n}^1 - \underline{n}^2) = 0$$

in each corner point, where $\underline{n}^1$, $\underline{n}^2$ denote the two outer normals. This is now a necessary condition on the control $\underline{z}$ in $[L_2(\Gamma)]^n$ which has to be satisfied. However we are not having such an additional condition on the control in $\left[H^{1/2}(\Gamma)\right]^n$, as mentioned above. Similarly we can derive a condition for the three dimensional case.

Let us consider the numerical results for the $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$ control for the cost coefficient $\varrho = 1$, see Table 3.1 – Table 3.2 and Table 3.6 – Table 3.7. We see that the errors for the primal variables as well as for the adjoint variables are much smaller in the case when considering the control in $\left[H^{1/2}(\Gamma)\right]^n$. In particular, we observe this behavior for the error of the control $\underline{z}$. Here the difference is around a factor of e-02 at level $L = 0$ up to a factor of e-04 at level $L = 7$.
Moreover we see, that we obtain for the control in $\left[H^{1/2}(\Gamma)\right]^n$ a considerably better order of convergence. In particular, we obtain for the control $\underline{z}$ an order of convergence around 0.5 in the $[L_2(\Gamma)]^n$ case and 2.0 in the $\left[H^{1/2}(\Gamma)\right]^n$ case.
In Table 3.3 – Table 3.4 and Table 3.8 – Table 3.9, we find the results for the cost coefficient $\varrho = 1.0$ e-02. In this case the results for the $\left[H^{1/2}(\Gamma)\right]^n$ control are still better, even though the cost coefficient is $\varrho = 1.0$ e-02 and so the control has less influence on the solution.
In Table 3.5 and Table 3.10 we present the numerical results for the tracking functional $\|\underline{u}_h - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n}$. Considering the results for the cost coefficient $\varrho = 1$, we see that the

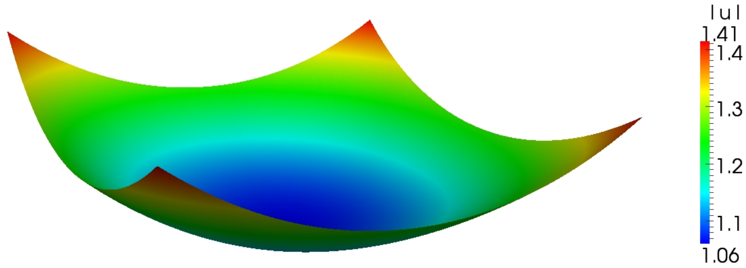

Figure 3.1: Absolute value of the tracking function $\underline{\overline{u}}$.

values are around a factor of e-01 better in the $\left[H^{1/2}(\Gamma)\right]^n$ case. Moreover we see that the difference is smaller for smaller cost coefficients.
In Figure 3.1, we plot the absolute value of the tracking function $\underline{\overline{u}}$ as given in (3.11). In Figure 3.2 – Figure 3.7 we present the absolute values of different states for the $[L_2(\Gamma)]^n$–control (left) and for the $\left[H^{1/2}(\Gamma)\right]^n$ (right), for different cost coefficients ϱ, at level $L = 7$ with $N = 65536$ elements. As we concluded above, the control is tending to zero in each corner point for the $[L_2(\Gamma)]^n$–control, see Figure 3.2 (This effect would be even more visible if we would use an adaptive mesh, where we refine at the corner points.). On the other hand, we do not have this behavior for the $\left[H^{1/2}(\Gamma)\right]^n$ case, see Figure 3.3. Moreover we observe, the smaller the cost coefficient ϱ, the smaller is the difference of the solutions, which is clear by the cost functional, see Figure 3.6 and Figure 3.7.

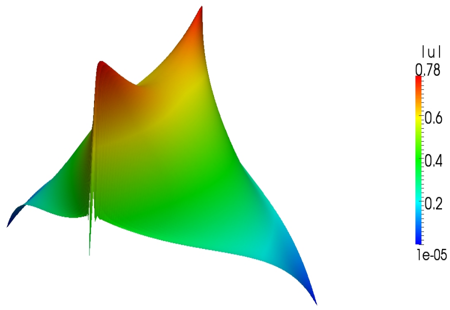

Figure 3.2: $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

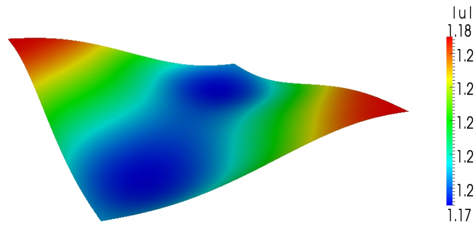

Figure 3.3: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

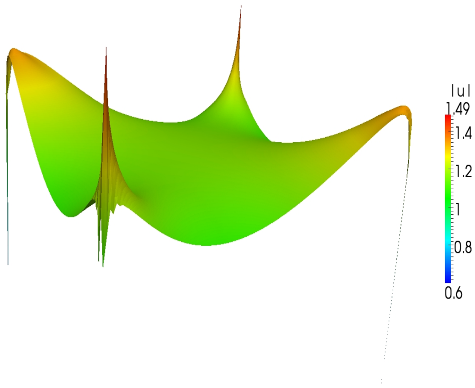

Figure 3.4: $[L_2(\Gamma)]^n$–control, $\varrho = 10^{-3}$.

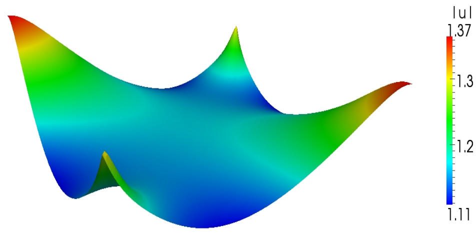

Figure 3.5: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 10^{-3}$.

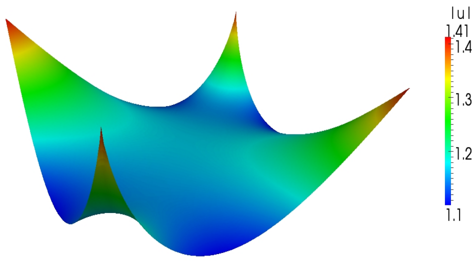

Figure 3.6: $[L_2(\Gamma)]^n$–control, $\varrho = 10^{-6}$.

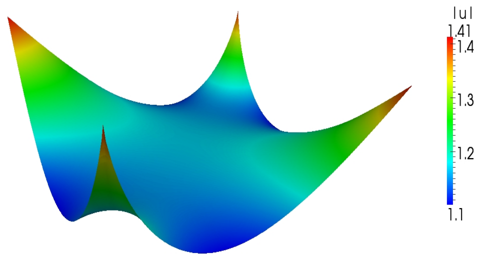

Figure 3.7: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 10^{-6}$.

Let us consider the previous examples with a cost coefficient $\varrho = 10^{-6}$, with an adaptive and an coarse mesh. In the numerical examples before we have seen that there is nearly no difference of a control in $[L_2(\Gamma)]^n$ and in $\left[H^{1/2}(\Gamma)\right]^n$. We refine for the adaptive mesh to the corner points with a certain factor, with $N = 64642$ elements. For the coarse mesh we consider a mesh with $N = 65536$ elements. Considering the control in $[L_2(\Gamma)]^n$, see Figure 3.8 and Figure 3.10, we observe non zero values in the corner points for the control for the coarse mesh, but for the adaptive mesh this effect is still visible. On the other hand, we do not have this behavior for the control in $\left[H^{1/2}(\Gamma)\right]^n$, see Figure 3.9 and Figure 3.11. This example shows, that adaptivity is an important aspect for optimal control problems.

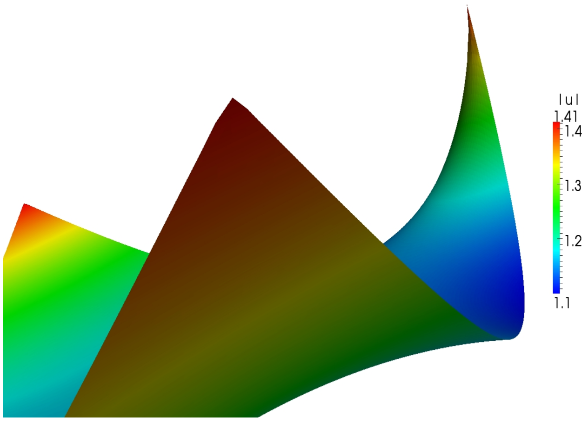

Figure 3.8: $[L_2(\Gamma)]^n$, coarse mesh.

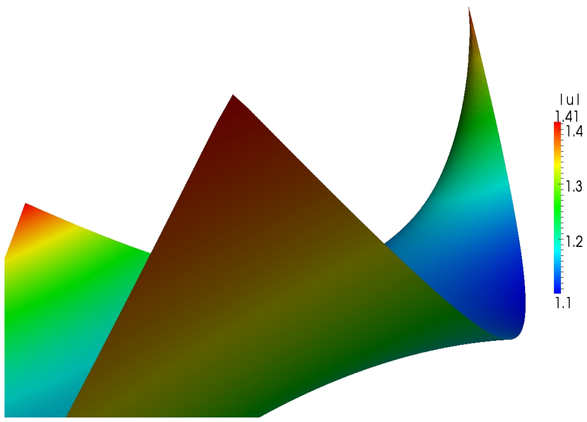

Figure 3.9: $\left[H^{1/2}(\Gamma)\right]^n$, coarse mesh.

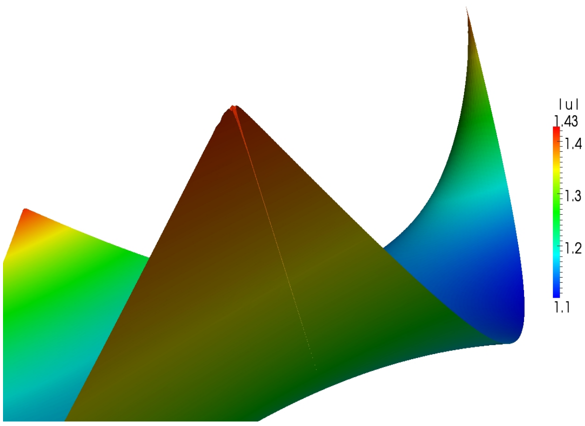

Figure 3.10: $[L_2(\Gamma)]^n$, adaptive mesh.

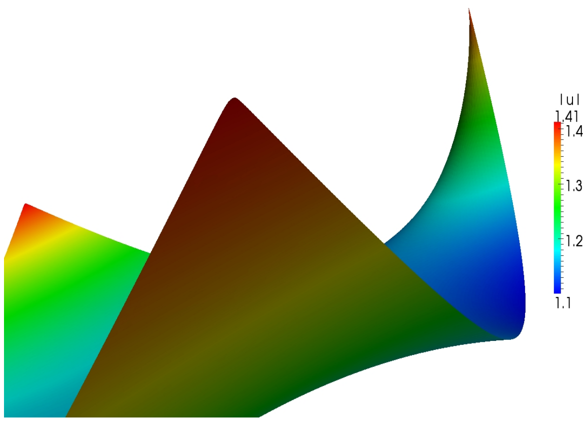

Figure 3.11: $\left[H^{1/2}(\Gamma)\right]^n$, adaptive mesh.

In this section we considered the optimality system of the Stokes equations, which was discretized by stabilized finite element methods. Moreover we treated the difference of a $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$ control with related numerical examples. In the next section we want to apply the same ideas to the optimal control problem of the Navier–Stokes equations.

3.2 Navier–Stokes equations

In this section we consider the optimal control problem for the Navier–Stokes equations (1.21)–(1.23), without box constraints. For the mixed boundary conditions we consider the boundary $\Gamma = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{c}}$ with a Dirichlet boundary Γ_{D}, a Neumann boundary Γ_{N}, and a control boundary Γ_{c}. In Section 1.3.2 we derived the corresponding optimality system as an equivalent problem to the optimal control problem, which is given by the following equations:

Primal problem

$$
\begin{aligned}
-\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega,\\
\nabla\cdot\underline{u} &= 0 && \text{in } \Omega,\\
\underline{u} &= \underline{g} && \text{on } \Gamma_{\mathrm{D}},\\
\nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}},\\
\underline{u} &= \underline{z} && \text{on } \Gamma_{\mathrm{c}},
\end{aligned}
$$

Adjoint problem

$$
\begin{aligned}
-\nu\Delta\underline{w} - (\nabla\underline{w})\underline{u} - (\nabla\underline{w})^{\top}\underline{u} - \nabla r &= \underline{u} - \overline{\underline{u}} && \text{in } \Omega,\\
\nabla\cdot\underline{w} &= 0 && \text{in } \Omega,\\
\underline{w} &= \underline{0} && \text{on } \Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{c}},\\
\nu(\nabla\underline{w})\underline{n} + (\underline{u}\cdot\underline{w})\underline{n} + (\underline{u}\cdot\underline{n})\underline{w} + r\underline{n} &= \underline{0} && \text{on } \Gamma_{\mathrm{N}},
\end{aligned}
$$

Optimality condition

$$
-\nu(\nabla\underline{w})\underline{n} - (\underline{u}\cdot\underline{w})\underline{n} - (\underline{u}\cdot\underline{n})\underline{w} - r\underline{n} + \rho S\underline{z} = \underline{0} \qquad \text{on } \Gamma_{\mathrm{c}}.
$$

The aim of this section is to derive a stabilized finite element formulation for the above optimality system of the Navier–Stokes equations. We start with the variational formulation in the continuous setting. Afterwards we explain the discretization, where we introduce stabilization terms for the primal and adjoint equations. Moreover, we consider a discretization with equal order 1 elements and explain how to apply Newtons method. Afterwards, we consider the resulting linear system and their properties.
We give some related numerical examples for a control in $[L_2(\Gamma_{\mathrm{c}})]^n$ and in $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$. In particular, we focus on the difference of these two control spaces. Finally we present a more realistic example, where the flow around an airfoil is considered.

3.2.1 Variational formulation

In the following we derive the variational formulation for the above optimality system. We start with the derivation for the optimality condition. Let $\mathcal{E}\underline{\varphi} \in [H_0^1(\Omega,\Gamma_{\mathrm{D}})]^n$ denote an extension of some test function $\underline{\varphi} \in \left[H^{1/2}(\Gamma_c)\right]^n$, or in an appropriate subspace. Taking the adjoint equations into account and use integration by parts, we obtain

$$\begin{aligned}\int_\Omega (\underline{u} - \overline{\underline{u}}) \cdot \mathcal{E}\underline{\varphi}\, dx = a(\underline{w}, \mathcal{E}\underline{\varphi}) - \nu \int_{\Gamma_c} [(\nabla \underline{w})\underline{n}] \cdot \underline{\varphi}\, ds_x \\ + b(\underline{u}, \underline{w} \cdot \mathcal{E}\underline{\varphi}) + \int_\Omega [(\nabla \mathcal{E}\underline{\varphi})\underline{u}] \cdot \underline{w}\, dx - \int_{\Gamma_c} (\underline{u} \cdot \underline{n})(\underline{w} \cdot \underline{\varphi})\, ds_x \\ + b(\mathcal{E}\underline{\varphi}, \underline{w} \cdot \underline{u}) + \int_\Omega [(\nabla \underline{u})\mathcal{E}\underline{\varphi}] \cdot \underline{w}\, dx - \int_{\Gamma_c} (\underline{w} \cdot \underline{u})(\underline{n} \cdot \underline{\varphi})\, ds_x \\ + b(\mathcal{E}\underline{\varphi}, r) - \int_{\Gamma_c} r\underline{n} \cdot \underline{\varphi}\, ds_x.\end{aligned}$$

This leads to the following representation of the optimality condition

$$\varrho \int_{\Gamma_c} S\underline{z} \cdot \underline{\varphi}\, ds_x = a(\underline{w}, \mathcal{E}\underline{\varphi}) + a_2(\underline{u}, \underline{w}, \mathcal{E}\underline{\varphi}) + b(\mathcal{E}\underline{\varphi}, r) - \int_\Omega (\underline{u} - \overline{\underline{u}}) \cdot \mathcal{E}\underline{\varphi}\, dx,$$

where S denotes the Steklov–Poincaré operator and

$$a_2(\underline{u}, \underline{w}, \underline{v}) = \int_\Omega [(\nabla \underline{v})\underline{u}] \cdot \underline{w}\, dx + \int_\Omega [(\nabla \underline{u})\underline{v}] \cdot \underline{w}\, dx + b(\underline{u}, \underline{w} \cdot \underline{v}) + b(\underline{v}, \underline{w} \cdot \underline{u}).$$

Furthermore, we introduce the trilinear form

$$a_1^*(\underline{w}, \underline{u}, \underline{v}) = \int_\Omega [(\nabla \underline{w})\underline{u} + (\nabla \underline{w})^\top \underline{u}] \cdot \underline{v}\, dx.$$

Let Z be either $\left[H^{1/2}(\Gamma_c)\right]^n$ or an appropriate subspace, see Section 1.2.2. With the standard variational formulation (1.25) for the Navier–Stokes equations, we obtain the following variational formulation for the optimality system: Find $(\underline{u}, p, \underline{w}, r, \underline{z}) \in [H^1(\Omega)]^n \times L_2(\Omega) \times [H_0^1(\Omega, \Gamma_D \cup \Gamma_c)]^n \times L_2(\Omega) \times Z$ with $\underline{u} = \underline{g}$ on Γ_D and $\underline{u} = \underline{z}$ on Γ_c, such that

$$\begin{aligned}
-\langle \underline{u}, \underline{\sigma} \rangle_\Omega \quad & + a(\underline{w}, \underline{\sigma}) - a_1^*(\underline{w}, \underline{u}, \underline{\sigma}) + b(\underline{\sigma}, r) &&= -\langle \overline{\underline{u}}, \underline{\sigma} \rangle_\Omega, \\
& b(\underline{w}, s) &&= 0, \\
a(\underline{u}, \underline{v}) + a_1(\underline{u}, \underline{u}, \underline{v}) - b(\underline{v}, p) & &&= \langle \underline{f}, \underline{v} \rangle_\Omega, \\
b(\underline{u}, q) & &&= 0, \\
\langle \underline{u}, \mathcal{E}\underline{\varphi} \rangle_\Omega + a_2(\underline{u}, \underline{w}, \mathcal{E}\underline{\varphi}) \quad & - a(\underline{w}, \mathcal{E}\underline{\varphi}) \quad - b(\mathcal{E}\underline{\varphi}, r) && \\
& + \varrho \langle S\underline{z}, \underline{\varphi} \rangle_\Gamma &&= \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi} \rangle_\Omega
\end{aligned} \tag{3.13}$$

is satisfied for all $(\underline{v}, q, \underline{\sigma}, s, \underline{\varphi}) \in [H_0^1(\Omega, \Gamma_D \cup \Gamma_c)]^n \times L_2(\Omega) \times [H_0^1(\Omega, \Gamma_D \cup \Gamma_c)]^n \times L_2(\Omega) \times Z$.

Remark 3.5. *If there is no Neumann boundary Γ_N considered, we have to take the space $L_{2,0}(\Omega) \subset L_2(\Omega)$ for the primal and adjoint pressures, p and r, to guarantee the uniqueness of the solution.*

Next we discretize the variational formulation (3.13). We do so by using stabilized finite element methods, which we already considered for the Navier–Stokes equations in Section 2.2. In the following, we denote by I_h an appropriate interpolation. Adding stabilization terms in the variational formulation and considering the same ansatz spaces as for the optimal control problem of the Stokes equations as given by (3.2), we obtain the following variational problem: Find $(\underline{u}_h, p_h, \underline{w}_h, r_h, \underline{z}_h) \in \widetilde{V}_h \times Q_h \times V_h \times Q_h \times Z_h$ with $\underline{u}_h = I_h \underline{g}$ on Γ_{D} and $\underline{u}_h = \underline{z}_h$ on Γ_{c}, such that

$$
\begin{aligned}
-\langle \underline{u}_h, \underline{\sigma}_h \rangle_\Omega + a(\underline{w}_h, \underline{\sigma}_h) - a_1^*(\underline{w}_h, \underline{u}_h, \underline{\sigma}_h) + b(\underline{\sigma}_h, r_h) \qquad & \\
+ \Psi_h^*(\underline{u}_h, \underline{w}_h, r_h; \underline{\sigma}_h) &= -\langle \overline{\underline{u}}, \underline{\sigma}_h \rangle_\Omega, \\
b(\underline{w}_h, s_h) + \Phi_h^*(\underline{u}_h, \underline{w}_h, r_h; s_h) &= 0, \\
a(\underline{u}_h, \underline{v}_h) + a_1(\underline{u}_h, \underline{u}_h, \underline{v}_h) - b(\underline{v}_h, p_h) + \Psi_h(\underline{u}_h, p_h; \underline{v}_h) &= \langle \underline{f}, \underline{v}_h \rangle_\Omega, \\
b(\underline{u}_h, q_h) + \Phi_h(\underline{u}_h, p_h; q_h) &= 0, \\
\langle \underline{u}_h, \mathcal{E}\underline{\varphi}_h \rangle_\Omega + a_2(\underline{u}_h, \underline{w}_h, \mathcal{E}\underline{\varphi}_h) - a(\underline{w}_h, \mathcal{E}\underline{\varphi}_h) - b(\mathcal{E}\underline{\varphi}_h, r_h) \qquad & \\
+ \varrho \langle S\underline{z}_h, \underline{\varphi}_h \rangle_\Gamma &= \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}_h \rangle_\Omega
\end{aligned}
\tag{3.14}
$$

is satisfied for all $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$. The corresponding stabilization terms for the primal equations are given by

$$
\begin{aligned}
\Psi_h(\underline{u}_h, p_h; \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\mathrm{SUPG}} (-\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}) \\
&\qquad \cdot ((\underline{u}_h \cdot \nabla) \underline{v}_h - \rho \nu \Delta \underline{v}_h) \, dx \\
&\quad + \sum_{T \in \mathcal{T}_h} \int_T \xi \, (\nabla \cdot \underline{u}_h)(\nabla \cdot \underline{v}_h) \, dx, \\
\Phi_h(\underline{u}_h, p_h; q_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\mathrm{PSPG}} (-\nu \Delta \underline{u}_h + (\underline{u}_h \cdot \nabla) \underline{u}_h + \nabla p_h - \underline{f}) \cdot \nabla q_h \, dx,
\end{aligned}
\tag{3.15}
$$

and for the adjoint equations by

$$
\begin{aligned}
\Psi_h^*(\underline{u}_h, \underline{w}_h, r_h; \underline{\sigma}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\mathrm{SUPG}} (-\nu \Delta \underline{w}_h - (\nabla \underline{w}_h) \underline{u}_h - (\nabla \underline{w}_h)^\top \underline{u}_h - \nabla r_h \\
&\qquad - (\underline{u}_h - \overline{\underline{u}})) \cdot (-\rho \nu \Delta \underline{\sigma}_h - (\nabla \underline{\sigma}_h) \underline{u}_h - (\nabla \underline{\sigma}_h)^\top \underline{u}_h) \, dx \\
&\quad + \sum_{T \in \mathcal{T}_h} \int_T \xi \, (\nabla \cdot \underline{w}_h)(\nabla \cdot \underline{\sigma}_h) \, dx, \\
\Phi_h^*(\underline{u}_h, \underline{w}_h, r_h; s_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta^{\mathrm{PSPG}} (-\nu \Delta \underline{w}_h - (\nabla \underline{w}_h) \underline{u}_h - (\nabla \underline{w}_h)^\top \underline{u}_h - \nabla r_h \\
&\qquad - (\underline{u}_h - \overline{\underline{u}})) \cdot \nabla s_h \, dx.
\end{aligned}
\tag{3.16}
$$

The stabilization parameters δ^{SUPG}, δ^{PSPG} and ξ are calculated in the same way as described in (2.17). In particular, we set $\delta = \delta^{\mathrm{SUPG}} = \delta^{\mathrm{PSPG}}$.

Since we want to discretize the optimal control problem of the Navier–Stokes equations with low order elements, we present in the following a simplification of the above problem, where we use equal order 1 elements, i.e. $k = l = 1$ in (3.2). This has the consequence that all second order derivatives in the stabilization terms in (3.15)–(3.16) vanish. For this particular case we obtain the following variational formulation: Find $(\underline{u}_h, p_h, \underline{w}_h, r_h, \underline{z}_h) \in \widetilde{V}_h \times Q_h \times V_h \times Q_h \times Z_h$ with $\underline{u}_h = I_h \underline{g}$ on Γ_{D} and $\underline{u}_h = \underline{z}_h$ on Γ_{c}, such that

$$
\begin{aligned}
&- \langle \underline{u}_h, \underline{\sigma}_h \rangle_\Omega + a^*_{1,h}(\underline{u}_h, \underline{\sigma}_h, \underline{u}_h) + a(\underline{w}_h, \underline{\sigma}_h) - a^*_1(\underline{w}_h, \underline{u}_h, \underline{\sigma}_h) \\
&\quad + a^*_{1,h}(\underline{u}_h, \underline{w}_h, (\nabla \underline{\sigma}_h)\underline{u}_h) + a^*_{1,h}(\underline{u}_h, \underline{w}_h, (\nabla \underline{\sigma}_h)^\top \underline{u}_h) + \widetilde{a}_h(\underline{w}_h, \underline{\sigma}_h) \\
&\quad\; + a^*_{1,h}(\underline{u}_h, \underline{\sigma}_h, \nabla r_h) + b(\underline{\sigma}_h, r_h) \\
&= -\langle \overline{\underline{u}}, \underline{\sigma}_h \rangle_\Omega + a^*_{1,h}(\underline{u}_h, \underline{\sigma}_h, \overline{\underline{u}}), \\[2ex]
&- d_h(\underline{u}_h, s_h) - a^*_{1,h}(\underline{u}_h, \underline{w}_h, \nabla s_h) + b(\underline{w}_h, s_h) - c_h(s_h, r_h) \\
&= -d_h(\overline{\underline{u}}, s_h), \\[2ex]
&a(\underline{u}_h, \underline{v}_h) + a_1(\underline{u}_h, \underline{u}_h, \underline{v}_h) + a_{1,h}(\underline{u}_h, \underline{u}_h, (\underline{u}_h \cdot \nabla)\underline{v}_h) + \widetilde{a}_h(\underline{u}_h, \underline{v}_h) \\
&\quad - b(\underline{v}_h, p_h) + a_{1,h}(\underline{u}_h, \underline{v}_h, \nabla p_h) \\
&= \langle \underline{f}, \underline{v}_h \rangle_\Omega + a_{1,h}(\underline{u}_h, \underline{v}_h, \underline{f}), \\[2ex]
&b(\underline{u}_h, q_h) + a_{1,h}(\underline{u}_h, \underline{u}_h, \nabla q_h) + c_h(q_h, p_h) \\
&= d_h(\underline{f}, q_h), \\[2ex]
&\langle \underline{u}_h, \mathcal{E}\underline{\varphi}_h \rangle_\Omega + a_2(\underline{u}_h, \underline{w}_h, \mathcal{E}\underline{\varphi}_h) - a(\underline{w}_h, \mathcal{E}\underline{\varphi}_h) - b(\mathcal{E}\underline{\varphi}_h, r_h) + \varrho \langle S\underline{z}_h, \underline{\varphi}_h \rangle_\Gamma \\
&= \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}_h \rangle_\Omega
\end{aligned}
\tag{3.17}
$$

is satisfied for all $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$. We used therefore the additional bilinear forms $c_h(\cdot,\cdot)$ and $d_h(\cdot,\cdot)$ from (3.7). Moreover, we took

$$
\begin{aligned}
a_{1,h}(\underline{u}_h, \underline{w}_h, \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta \left((\underline{u}_h \cdot \nabla)\underline{w}_h \right) \cdot \underline{v}_h \, dx, \\
a^*_{1,h}(\underline{u}_h, \underline{w}_h, \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \delta \left((\nabla \underline{w}_h)\underline{u}_h + (\nabla \underline{w}_h)^\top \underline{u}_h \right) \cdot \underline{v}_h \, dx, \\
\widetilde{a}_h(\underline{u}_h, \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T \xi \, (\nabla \cdot \underline{u}_h)(\nabla \cdot \underline{v}_h) \, dx
\end{aligned}
$$

into consideration. Now it is possible to apply Newtons method, where we assume the following simplifications. For the term $(\underline{u}_h \cdot \nabla)\underline{v}_h$, which is multiplied by the primal residual in the stabilization term $\Psi_h(\underline{u}_h, p_h; \underline{v}_h)$, we take $(\underline{u}^k_h \cdot \nabla)\underline{v}_h$ into account, where k denotes the previous Newton step. In this way we are able to avoid additional nonlinearities. Nevertheless we have to treat the nonlinearity, which occurs in the residual. Similarly we take for $(\nabla\underline{\sigma}_h)\underline{u}_h + (\nabla\underline{\sigma}_h)^\top\underline{u}_h$ the term $(\nabla\underline{\sigma}_h)\underline{u}_h^k + (\nabla\underline{\sigma}_h)^\top\underline{u}_h^k$. This leads to the following variational formulation for the $k+1$ Newton step: Find $(\underline{u}_h^{k+1}, p_h^{k+1}, \underline{w}_h^{k+1}, r_h^{k+1}, \underline{z}_h^{k+1}) \in \widetilde{V}_h \times Q_h \times V_h \times Q_h \times Z_h$ with $\underline{u}_h^{k+1} = I_h\underline{g}$ on Γ_D and $\underline{u}_h^{k+1} = \underline{z}_h^{k+1}$ on Γ_c, such that

$$\begin{aligned}
&-\langle \underline{u}_h^{k+1}, \underline{\sigma}_h\rangle_\Omega + a_{1,h}^*(\underline{u}_h^{k+1}, \underline{\sigma}_h, \underline{u}_h^k) + a_{1,h}^*(\underline{u}_h^k, \underline{\sigma}_h, \underline{u}_h^{k+1}) + a(\underline{w}_h^{k+1}, \underline{\sigma}_h)\\
&\quad - a_1^*(\underline{w}_h^{k+1}, \underline{u}_h^k, \underline{\sigma}_h) - a_1^*(\underline{w}_h^k, \underline{u}_h^{k+1}, \underline{\sigma}_h) + a_{1,h}^*(\underline{u}_h^{k+1}, \underline{w}_h^k, (\nabla\underline{\sigma}_h)\underline{u}_h^k)\\
&\qquad + a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^{k+1}, (\nabla\underline{\sigma}_h)\underline{u}_h^k) + a_{1,h}^*(\underline{u}_h^{k+1}, \underline{w}_h^k, (\nabla\underline{\sigma}_h)^\top\underline{u}_h^k)\\
&\qquad\quad + a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^{k+1}, (\nabla\underline{\sigma}_h)^\top\underline{u}_h^k) + \widetilde{a}_h(\underline{w}_h^{k+1}, \underline{\sigma}_h) + a_{1,h}^*(\underline{u}_h^k, \underline{\sigma}_h, \nabla r_h^{k+1})\\
&\qquad\qquad + b(\underline{\sigma}_h, r_h^{k+1})\\
&= -\langle \overline{\underline{u}}, \underline{\sigma}_h\rangle_\Omega + a_{1,h}^*(\underline{u}_h^k, \underline{\sigma}_h, \overline{\underline{u}}) + a_{1,h}^*(\underline{u}_h^k, \underline{\sigma}_h, \underline{u}_h^k) - a_1^*(\underline{w}_h^k, \underline{u}_h^k, \underline{\sigma}_h)\\
&\quad + a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^k, (\nabla\underline{\sigma}_h)\underline{u}_h^k) + a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^k, (\nabla\underline{\sigma}_h)^\top\underline{u}_h^k),\\[1ex]
&-d_h(\underline{u}_h^{k+1}, s_h) - a_{1,h}^*(\underline{u}_h^{k+1}, \underline{w}_h^k, \nabla s_h) - a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^{k+1}, \nabla s_h) + b(\underline{w}_h^{k+1}, s_h)\\
&\quad - c_h(s_h, r_h^{k+1})\\
&= -d_h(\overline{\underline{u}}, s_h) - a_{1,h}^*(\underline{u}_h^k, \underline{w}_h^k, \nabla s_h),
\end{aligned}\tag{3.18}$$

$$\begin{aligned}
&a(\underline{u}_h^{k+1}, \underline{v}_h) + a_1(\underline{u}_h^{k+1}, \underline{u}_h^k, \underline{v}_h) + a_1(\underline{u}_h^k, \underline{u}_h^{k+1}, \underline{v}_h) + a_{1,h}(\underline{u}_h^{k+1}, \underline{u}_h^k, (\underline{u}_h^k\cdot\nabla)\underline{v}_h)\\
&\quad + a_{1,h}(\underline{u}_h^k, \underline{u}_h^{k+1}, (\underline{u}_h^k\cdot\nabla)\underline{v}_h) + \widetilde{a}_h(\underline{u}_h^{k+1}, \underline{v}_h) - b(\underline{v}_h, p_h^{k+1}) + a_{1,h}(\underline{u}_h^k, \underline{v}_h, \nabla p_h^{k+1})\\
&= \langle \underline{f}, \underline{v}_h\rangle_\Omega + a_{1,h}(\underline{u}_h^k, \underline{v}_h, \underline{f}) + a_1(\underline{u}_h^k, \underline{u}_h^k, \underline{v}_h) + a_{1,h}(\underline{u}_h^k, \underline{u}_h^k, (\underline{u}_h^k\cdot\nabla)\underline{v}_h),
\end{aligned}$$

$$\begin{aligned}
&b(\underline{u}_h^{k+1}, q_h) + a_{1,h}(\underline{u}_h^{k+1}, \underline{u}_h^k, \nabla q_h) + a_{1,h}(\underline{u}_h^k, \underline{u}_h^{k+1}, \nabla q_h) + c_h(q_h, p_h^{k+1})\\
&= d_h(\underline{f}, q_h) + a_{1,h}(\underline{u}_h^k, \underline{u}_h^k, \nabla q_h),
\end{aligned}$$

$$\begin{aligned}
&\langle \underline{u}_h^{k+1}, \mathcal{E}\underline{\varphi}_h\rangle_\Omega + a_2(\underline{u}_h^{k+1}, \underline{w}_h^k, \mathcal{E}\underline{\varphi}_h) + a_2(\underline{u}_h^k, \underline{w}_h^{k+1}, \mathcal{E}\underline{\varphi}_h) - a(\underline{w}_h^{k+1}, \mathcal{E}\underline{\varphi}_h)\\
&\quad - b(\mathcal{E}\underline{\varphi}_h, r_h^{k+1}) + \varrho\langle S\underline{z}_h^{k+1}, \underline{\varphi}_h\rangle_\Gamma\\
&= \langle \overline{\underline{u}}, \mathcal{E}\underline{\varphi}_h\rangle_\Omega + a_2(\underline{u}_h^k, \underline{w}_h^k, \mathcal{E}\underline{\varphi}_h)
\end{aligned}$$

is satisfied for all $(\underline{v}_h, q_h, \underline{\sigma}_h, s_h, \underline{\varphi}_h) \in V_h \times Q_h \times V_h \times Q_h \times Z_h$. The above variational formulation (3.18) is now no longer nonlinear. Due to this reason we are now able to introduce an extension $\underline{u}_{z,h} \in \widetilde{V}_h$ of the control $\underline{z}_h \in Z_h$ such that $\underline{u} = \underline{u}_{0,h} + \underline{u}_{z,h}$ with $\underline{u}_{0,h} \in V_h$ is satisfied.
In the following we consider the corresponding linear system of the variational formulation (3.18). Further we present some related numerical results for a control in $[L_2(\Gamma_\mathrm{c})]^n$ as well as in $\left[H^{1/2}(\Gamma_\mathrm{c})\right]^n$ and discuss their difference.

3.2.2 Linear system

In this section we consider the variational formulation (3.18) of the optimal control problem for the Navier–Stokes equations with equal order 1 elements. The problem is equivalent to a linear system, which we describe in this section. In particular, we consider the realization of the Steklov–Poincaré operator for a control in $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$, or in an appropriate subspace, see Section 1.2.2. Let us consider the discrete ansatz spaces (3.2) with $k = l = 1$, where we use piecewise polynomials of order 1 for the velocity as well as for the pressure.
For the linear system we consider the following matrices with entries

$$
\begin{aligned}
K_h[i,j] &= a(\underline{\varphi}_j, \underline{\varphi}_i) + \widetilde{a}_h(\underline{\varphi}_j, \underline{\varphi}_i),\\
A_h^1[i,j] &= a_1(\underline{\varphi}_j, \underline{u}_h^k, \underline{\varphi}_i) + a_1(\underline{u}_h^k, \underline{\varphi}_j, \underline{\varphi}_i) + a_{1,h}(\underline{\varphi}_j, \underline{u}_h^k, (\underline{u}_h^k \cdot \nabla)\underline{\varphi}_i)\\
&\quad + a_{1,h}(\underline{u}_h^k, \underline{\varphi}_j, (\underline{u}_h^k \cdot \nabla)\underline{\varphi}_i),\\
A_h^2[i,j] &= a_2(\underline{u}_h^k, \underline{\varphi}_j, \underline{\varphi}_i),\\
A_h^{*,1}[i,j] &= a_{1,h}^*(\underline{\varphi}_j, \underline{\varphi}_i, \underline{u}_h^k) + a_{1,h}^*(\underline{u}_h^k, \underline{\varphi}_i, \underline{\varphi}_j) - a_1^*(\underline{w}_h^k, \underline{\varphi}_j, \underline{\varphi}_i)\\
&\quad + a_{1,h}^*(\underline{\varphi}_j, \underline{w}_h^k, (\nabla\underline{\varphi}_i)\underline{u}_h^k) + a_{1,h}^*(\underline{\varphi}_j, \underline{w}_h^k, (\nabla\underline{\varphi}_i)^\top\underline{u}_h^k),\\
A_h^{*,2}[i,j] &= a_1^*(\underline{\varphi}_j, \underline{u}_h^k, \underline{\varphi}_i) + a_{1,h}^*(\underline{u}_h^k, \underline{\varphi}_j, (\nabla\underline{\varphi}_i)\underline{u}_h^k)\\
&\quad + a_{1,h}^*(\underline{u}_h^k, \underline{\varphi}_j, (\nabla\underline{\varphi}_i)^\top\underline{u}_h^k),\\
A_h^{*,3}[i,j] &= a_2(\underline{\varphi}_j, \underline{w}_h^k, \underline{\varphi}_i),\\
B_h^1[i,j] &= a_{1,h}(\underline{u}_h^k, \underline{\varphi}_i, \nabla\psi_j),\\
B_h^2[i,j] &= a_{1,h}(\underline{\varphi}_j, \underline{u}_h^k, \nabla\psi_i) + a_{1,h}(\underline{u}_h^k, \underline{\varphi}_j, \nabla\psi_i),\\
B_h^{*,1}[i,j] &= a_{1,h}^*(\underline{u}_h^k, \underline{\varphi}_j, \nabla\psi_i),\\
B_h^{*,2}[i,j] &= a_{1,h}^*(\underline{\varphi}_j, \underline{w}_h^k, \nabla\psi_i)
\end{aligned}
\tag{3.19}
$$

for all $i, j = 1, \ldots, m$. In addition, we consider those of (3.8). In the following we derive the linear system for a control in $[L_2(\Gamma_{\mathrm{c}})]^n$ and $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$.

$L_2(\Gamma)$–control

For a control in $[L_2(\Gamma_{\mathrm{c}})]^n$ the global matrix for the optimal control problem of the Navier–Stokes equations is of the form

$$
\begin{pmatrix}
-M_{II} + A_{II}^{*,2} & & K_{II} + A_{II}^{*,1} & (B_I + B_I^{*,1})^\top & -M_{IC} + A_{IC}^{*,2}\\
-D_I - B_I^{*,2} & & B_I - B_I^{*,1} & -C & -D_C - B_C^{*,2}\\
K_{II} + A_{II}^1 & (-B_I + B_I^1)^\top & & & K_{IC} + A_{IC}^1\\
B_I + B_I^2 & C & & & B_C + B_C^2\\
M_{CI} + A_{CI}^{*,3} & & A_{CI}^2 - A_{CI} & -B_C^\top & M_{CC} + A_{CI}^{*,3} + \varrho\widetilde{M}_h
\end{pmatrix}
$$

where $\widetilde{M}_h$ denotes the mass matrix on the control boundary Γ_{c}.

$H^{1/2}(\Gamma)$–**control**

Let us consider the control in $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$ next. We obtain a similar block structure for the linear system as in the $[L_2(\Gamma_{\mathrm{c}})]^n$ case. Now we consider $\widetilde{S}_h$ which is the Galerkin matrix of the Steklov–Poincaré operator. This will be realized by the Schur complement system, see Section 3.1.2 and leads to the following linear system

$$\begin{pmatrix} -M_{II}+A_{II}^{*,2} & & K_{II}+A_{II}^{*,1} & (B_I+B_I^{*,1})^\top & -M_{IC}+A_{IC}^{*,2} & \\ -D_I-B_I^{*,2} & & B_I-B_I^{*,1} & -C & -D_C-B_C^{*,2} & \\ K_{II}+A_{II}^1 & (-B_I+B_I^1)^\top & & & K_{IC}+A_{IC}^1 & \\ B_I+B_I^2 & C & & & B_C+B_C^2 & \\ M_{CI}+A_{CI}^{*,3} & & A_{CI}^2-A_{CI} & -B_C^\top & M_{CC}+A_{CI}^{*,3}+\varrho A_{CC} & -\varrho A_{CI} \\ & & & & \varrho A_{IC} & -\varrho A_{II} \end{pmatrix}.$$

Remark 3.6. *We already mentioned in Section 2.2, that it is not necessary to assemble all blocks in every Newton step. For the optimal control problem of the Navier–Stokes equations, this idea can be applied too, see (3.19).*

3.2.3 Numerical results

In the following we present some numerical examples for stabilized finite element methods for the optimal control problem of the Navier–Stokes equations (1.21)–(1.22). The control is either considered in $[L_2(\Gamma_{\mathrm{c}})]^n$ or in $\left[H^{1/2}(\Gamma_{\mathrm{c}})\right]^n$. In particular we will discuss the difference of these two choices. For a first numerical example we take, similar to the optimal control problem of the Stokes equations, see Section 3.1, as a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ the unit square, with a symmetric uniform triangulation into consideration, as in Section 2.1.3. The complete boundary is considered as a control boundary, i.e. $\Gamma = \Gamma_{\mathrm{c}}$. The viscosity constant $\nu = 1$ is considered and the following tracking function and right hand side, which are given by

$$\underline{\overline{u}} = \begin{pmatrix} x_2(x_2-1)+1 \\ x_1(x_1-1)+1 \end{pmatrix} \quad \text{and} \quad \underline{f} = \underline{1}. \tag{3.20}$$

Moreover, we are using equal order 1 elements, which leads to the linear systems as given in Section 3.2.2, for a $[L_2(\Gamma)]^n$ control and for a $\left[H^{1/2}(\Gamma)\right]^n$ control, respectively. The stabilization parameter are calculated as in (2.17). Since we do not know the exact solution of the optimal control problem for the given data (3.20), we take the solution from level $L = 9$ with $N = 1048576$ elements, which we denote by $(\underline{u}_{h_9}, p_{h_9}, \underline{w}_{h_9}, r_{h_9}, \underline{z}_{h_9})$, as reference solution. The resulting linear system of equations is solved by the direct solver PARDISO, see [29, 30]. In the following, we present numerical results for a $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$ control, where we especially focus on the difference of these two control spaces. We consider different cost coefficients $\varrho > 0$.

$L_2(\Gamma)$–control

First, we consider the control in $[L_2(\Gamma)]^n$. In the following we present numerical results for different cost coefficients ϱ.

L	$\left\|\underline{u}_{h_9} - \underline{u}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{[L_2(\Gamma)]^n}$	eoc
0	1.98177 e-01	–	7.56383 e+00	–	3.55102 e-01	–
1	1.28031 e-01	0.63	7.12123 e+00	0.09	2.43439 e-01	0.54
2	7.44316 e-02	0.78	6.54632 e+00	0.12	1.70947 e-01	0.51
3	4.03145 e-02	0.88	5.93605 e+00	0.14	1.19022 e-01	0.52
4	2.15093 e-02	0.91	5.24858 e+00	0.18	8.29968 e-02	0.52
5	1.12160 e-02	0.94	4.47486 e+00	0.23	5.75723 e-02	0.53
6	5.62909 e-03	0.99	3.59872 e+00	0.31	3.92801 e-02	0.55
7	2.61296 e-03	1.11	2.59969 e+00	0.47	2.57017 e-02	0.61
8	9.79682 e-04	1.42	1.45215 e+00	0.84	1.48376 e-02	0.79
		1.1		0.5		0.6

Table 3.11: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

L	$\left\|\underline{w}_{h_9} - \underline{w}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	1.59452 e-03	–	7.07079 e-02	–
1	1.21603 e-03	0.39	3.14117 e-02	1.17
2	4.08984 e-04	1.57	1.27162 e-02	1.30
3	1.25544 e-04	1.70	5.28880 e-03	1.27
4	3.65198 e-05	1.78	2.31058 e-03	1.19
5	1.12082 e-05	1.70	1.03950 e-03	1.15
6	3.75218 e-06	1.58	4.65699 e-04	1.16
7	1.30756 e-06	1.52	1.95347 e-04	1.25
8	3.89019 e-07	1.75	6.44209 e-05	1.60
		1.5		1.2

Table 3.12: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

In Table 3.11 and Table 3.12 we present numerical results for the cost coefficient $\varrho = 1$. In the former table we find the errors of the primal variables, i.e. of the primal velocity in the $[L_2(\Omega)]^n$–norm, of the primal pressure in the $L_2(\Omega)$–norm, and of the control in the $[L_2(\Gamma)]^n$–norm. For the primal velocity we obtain close to 1.1 as order of convergence and for the primal pressure around 0.5. For the control, in this case considered in $[L_2(\Gamma)]^n$, we obtain around 0.6 as order of convergence. In Table 3.12 we present the results for the adjoint variables, where the errors of the adjoint velocity $\underline{w}$, in the $[L_2(\Omega)]^n$–norm, and of the adjoint pressure r, in the $L_2(\Omega)$–norm, are given. We obtain around 1.5 for the adjoint velocity and 1.2 for the adjoint pressure as order of convergence.

L	$\left\Vert \underline{u}_{h_9} - \underline{u}_h \right\Vert_{[L_2(\Omega)]^n}$	eoc	$\Vert p_{h_9} - p_h \Vert_{L_2(\Omega)}$	eoc	$\left\Vert \underline{z}_{h_9} - \underline{z}_h \right\Vert_{[L_2(\Gamma)]^n}$	eoc
0	7.58012 e-02	–	1.05656 e+01	–	4.05960 e-01	–
1	4.95944 e-02	0.61	1.05706 e+01	–	4.43468 e-01	–
2	3.83087 e-02	0.37	1.03517 e+01	0.03	3.96832 e-01	0.16
3	2.69095 e-02	0.51	9.94359 e+00	0.06	3.09012 e-01	0.36
4	1.75789 e-02	0.61	9.20326 e+00	0.11	2.17423 e-01	0.51
5	1.07613 e-02	0.71	8.10982 e+00	0.18	1.42965 e-01	0.60
6	6.19581 e-03	0.80	6.68613 e+00	0.28	9.09168 e-02	0.65
7	3.28570 e-03	0.92	4.93966 e+00	0.44	5.66363 e-02	0.68
8	1.43491 e-03	1.20	2.83284 e+00	0.80	3.21268 e-02	0.82
		1		0.5		0.7

Table 3.13: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1.0$ e-02.

L	$\left\Vert \underline{w}_{h_9} - \underline{w}_h \right\Vert_{[L_2(\Omega)]^n}$	eoc	$\Vert r_{h_9} - r_h \Vert_{L_2(\Omega)}$	eoc
0	6.54373 e-04	–	1.53352 e-03	–
1	2.34628 e-04	1.48	1.83445 e-03	–
2	8.01604 e-05	1.55	1.56344 e-03	0.23
3	2.65148 e-05	1.60	6.93531 e-04	1.17
4	7.08067 e-06	1.90	2.89684 e-04	1.26
5	1.81706 e-06	1.96	1.12844 e-04	1.36
6	4.55788 e-07	2.00	4.13400 e-05	1.45
7	1.10039 e-07	2.05	1.42321 e-05	1.54
8	2.30713 e-08	2.25	4.04007 e-06	1.82
		2		1.5

Table 3.14: Errors for $[L_2(\Gamma)]^n$–control, $\varrho = 1.0$ e-02.

	$\varrho = 1$	$\varrho = 1.0$ e-02
L	$\Vert \underline{u}_h - \overline{\underline{u}} \Vert^2_{[L_2(\Omega)]^n}$	$\Vert \underline{u}_h - \overline{\underline{u}} \Vert^2_{[L_2(\Omega)]^n}$
0	8.55401 e-01	1.35292 e-02
1	7.35134 e-01	6.35149 e-03
2	6.79265 e-01	7.06321 e-03
3	6.51704 e-01	7.53401 e-03
4	6.40394 e-01	7.54312 e-03
5	6.35549 e-01	7.39589 e-03
6	6.33384 e-01	7.29334 e-03
7	6.32381 e-01	7.25001 e-03
8	6.31903 e-01	7.23501 e-03

Table 3.15: Tracking functional for $[L_2(\Gamma)]^n$–control.

The numerical results for $\varrho = 1.0$ e-02 are given in Table 3.13 and Table 3.14. The order of convergence of the error for the primal variables is similar as in the first case, see Table 3.13, where we considered the cost coefficient $\varrho = 1$. The difference here is the order of convergence for the adjoint velocity $\underline{w}$ and for the adjoint pressure r. We obtain second order of convergence for the adjoint velocity in the $[L_2(\Omega)]^n$–norm and 1.5 for the adjoint pressure in the $L_2(\Omega)$–norm. Moreover we present the results of the tracking functional $\|\underline{u}_h - \overline{\underline{u}}\|^2_{[L_2(\Omega)]^n}$ in Table 3.15 for the cost coefficients $\varrho = 1$ and $\varrho = 1.0$ e-02. For the latter case we obtain smaller values of the tracking functional, which is due to the cost functional. These values are very similar as in the Stokes case, see Table 3.5.

$H^{1/2}(\Gamma)$–control

Now we consider the control in $\left[H^{1/2}(\Gamma)\right]^n$, for different cost coefficients ϱ.

L	$\left\|\underline{u}_{h_9} - \underline{u}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\left\|\underline{z}_{h_9} - \underline{z}_h\right\|_{[L_2(\Gamma)]^n}$	eoc
0	3.52815 e-03	–	9.63493 e-02	–	9.64011 e-03	–
1	1.96260 e-03	0.85	4.67816 e-02	1.04	3.47652 e-03	1.47
2	9.23323 e-04	1.09	1.85836 e-02	1.33	1.72964 e-03	1.01
3	2.79243 e-04	1.73	7.52133 e-03	1.30	5.30675 e-04	1.70
4	8.56131 e-05	1.71	3.06950 e-03	1.29	1.77762 e-04	1.58
5	3.08205 e-05	1.47	1.25639 e-03	1.29	7.22122 e-05	1.30
6	1.27908 e-05	1.27	5.21080 e-04	1.27	3.21299 e-05	1.17
7	5.30486 e-06	1.27	2.13141 e-04	1.29	1.36679 e-05	1.23
8	1.75534 e-06	1.60	7.32841 e-05	1.54	4.55947 e-06	1.58
		1.3		1.3		1.3

Table 3.16: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

L	$\left\|\underline{w}_{h_9} - \underline{w}_h\right\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	7.16687 e-04	–	7.68693 e-03	–
1	4.59596 e-04	0.64	4.35538 e-03	0.82
2	1.44450 e-04	1.67	1.57428 e-03	1.47
3	3.94663 e-05	1.87	3.55084 e-04	2.15
4	1.01972 e-05	1.95	8.34337 e-05	2.09
5	2.56676 e-06	1.99	2.25641 e-05	1.89
6	6.36391 e-07	2.01	6.73316 e-06	1.74
7	1.52710 e-07	2.06	2.13871 e-06	1.65
8	3.18648 e-08	2.26	6.51752 e-07	1.71
		2		1.6

Table 3.17: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

In Table 3.16 and Table 3.17 we present the numerical results for the cost coefficient $\varrho = 1$. We give the errors of the primal velocity $\underline{u}$ in the $[L_2(\Omega)]^n$–norm, of the primal pressure p in the $L_2(\Omega)$–norm, and of the control $\underline{z}$ in the $[L_2(\Gamma)]^n$–norm. We obtain close to 1.3 as order of convergence for the primal velocity, for the primal pressure and, for the control. Moreover, we present the results for the adjoint velocity $\underline{w}$ in the $[L_2(\Omega)]^n$–norm and for the adjoint pressure r in the $L_2(\Omega)$–norm. We obtain second order convergence for the adjoint velocity and around 1.6 for the adjoint pressure.

L	$\|\underline{u}_{h_9} - \underline{u}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|p_{h_9} - p_h\|_{L_2(\Omega)}$	eoc	$\|\underline{z}_{h_9} - \underline{z}_h\|_{[L_2(\Gamma)]^n}$	eoc
0	7.79285 e-02	–	6.20632 e-01	–	2.41706 e-01	–
1	1.93613 e-02	2.01	3.36406 e-01	0.88	3.14637 e-02	2.94
2	7.12177 e-03	1.44	2.35124 e-01	0.52	2.04793 e-02	0.62
3	2.33467 e-03	1.61	1.43977 e-01	0.71	8.10727 e-03	1.34
4	7.21662 e-04	1.69	7.23078 e-02	0.99	2.42412 e-03	1.74
5	2.18373 e-04	1.72	3.34138 e-02	1.11	6.78719 e-04	1.84
6	6.80214 e-05	1.68	1.49044 e-02	1.16	1.97977 e-04	1.78
7	2.22623 e-05	1.61	6.36174 e-03	1.23	6.27923 e-05	1.66
8	6.52712 e-06	1.77	2.23991 e-03	1.51	1.83088 e-05	1.78
		1.6		1.2		1.7

Table 3.18: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1.0$ e-02.

L	$\|\underline{w}_{h_9} - \underline{w}_h\|_{[L_2(\Omega)]^n}$	eoc	$\|r_{h_9} - r_h\|_{L_2(\Omega)}$	eoc
0	5.62630 e-04	–	5.01039 e-03	–
1	9.04124 e-05	2.64	1.37691 e-03	1.86
2	2.79929 e-05	1.69	6.42450 e-04	1.10
3	7.23355 e-06	1.95	1.85970 e-04	1.79
4	1.87880 e-06	1.94	5.03063 e-05	1.89
5	4.77610 e-07	1.98	1.44939 e-05	1.80
6	1.19249 e-07	2.00	5.14806 e-06	1.49
7	2.89331 e-08	2.04	2.01234 e-06	1.36
8	6.22295 e-09	2.22	6.56161 e-07	1.62
		2		1.4

Table 3.19: Errors for $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1.0$ e-02.

Similar to the previous example, we present in Table 3.18 and Table 3.19 the numerical results for the cost coefficient $\varrho = 1.0$ e-02. We obtain close to 1.6 as order of convergence for the primal velocity in the $[L_2(\Omega)]^n$–norm, and around 1.2 for the primal pressure p in the $L_2(\Omega)$–norm. Moreover, we obtain close to 1.7 as order of convergence for the control $\underline{z}$ in $[L_2(\Gamma)]^n$–norm. For the adjoint velocity $\underline{w}$ we obtain second order of convergence in the $[L_2(\Omega)]^n$–norm, and around 1.4 for the adjoint pressure r in the $L_2(\Omega)$–norm.

	$\varrho = 1$	$\varrho = 1.0$ e-02
L	$\Vert\underline{u}_h - \overline{\underline{u}}\Vert^2_{[L_2(\Omega)]^n}$	$\Vert\underline{u}_h - \overline{\underline{u}}\Vert^2_{[L_2(\Omega)]^n}$
0	1.06438 e-02	1.03024 e-02
1	1.05282 e-02	1.38746 e-03
2	1.03925 e-02	8.22577 e-04
3	1.03025 e-02	7.24906 e-04
4	1.02749 e-02	6.93895 e-04
5	1.02675 e-02	6.84979 e-04
6	1.02656 e-02	6.82592 e-04
7	1.02651 e-02	6.81977 e-04
8	1.02650 e-02	6.81823 e-04

Table 3.20: Tracking functional for $\left[H^{1/2}(\Gamma)\right]^n$–control.

In Table 3.20 we present the numerical results for the tracking functional $\Vert\underline{u}_h - \overline{\underline{u}}\Vert^2_{[L_2(\Omega)]^n}$. Again, we consider the cost coefficients $\varrho = 1$ and $\varrho = 1.0$ e-02 and obtain similar results as in the Stokes case, see Table 3.10.

Comparison

In the following we consider the difference of a control in $[L_2(\Gamma)]^n$ and $\left[H^{1/2}(\Gamma)\right]^n$ in the case of an optimal control problem of the Navier–Stokes equations. First, let us consider the optimality condition for a control in $[L_2(\Gamma)]^n$, which is given by

$$-\nu(\nabla\underline{w})\underline{n} - (\underline{u}\cdot\underline{w})\underline{n} - (\underline{u}\cdot\underline{n})\underline{w} - r\underline{n} + \rho\underline{z} = \underline{0} \qquad \text{on } \Gamma.$$

Due to $\underline{w} = 0$ on Γ we conclude $(\nabla\underline{w})\underline{n} = \underline{0}$ in each corner point of the domain $\Omega = (0,1)^2$. Hence we conclude further

$$\varrho\underline{z} = r\underline{n}$$

in each corner point of the domain Ω. As in Section 3.1.3 we conclude

$$\underline{z} = \underline{0} \quad \text{and} \quad r = 0$$

in each corner point. Do to this reason we get an additional necessary condition on the control in $[L_2(\Gamma)]^n$. For an arbitrary domain see Remark 3.4.
Next, let us consider therefore Table 3.11 – Table 3.12 and Table 3.16 – Table 3.17, where the cost coefficient $\varrho = 1$ is considered. Similar to the Stokes case, we obtain better errors as well as a better order of convergence, when we consider the control in $\left[H^{1/2}(\Gamma)\right]^n$. This effect can also be seen for the cost coefficient $\varrho = 1.0$ e-02, see Table 3.13 – Table 3.14 and Table 3.18 – Table 3.19.
With respect to the tracking functional $\Vert\underline{u}_h - \overline{\underline{u}}\Vert^2_{[L_2(\Omega)]^n}$ we like to mention that the results

for the $\left[H^{1/2}(\Gamma)\right]^n$ case are better, but the difference decreases for smaller cost coefficients, as in the case of $\varrho = 1.0$ e-02.
In Figure 3.12 – Figure 3.17, we plot the absolute values of the primal velocity $\underline{u}_h$ for different cost coefficients. Again, we note that the velocity is tending to zero in the corner points for a control in $[L_2(\Gamma)]^n$, see Figure 3.12 and Figure 3.14. Indeed, for the $\left[H^{1/2}(\Gamma)\right]^n$ control we do not have this behavior. Considering the cost coefficient $\varrho = 1.0$ e-06, the results are similar, see Figure 3.16 and Figure 3.17.

Remark 3.7. *These results show, that the consideration of the control in* $\left[H^{1/2}(\Gamma)\right]^n$ *is more natural and leads to better numerical results. Actually we get a similar behavior for the control* $[L_2(\Gamma)]^n$*, as in the Stokes case, where the control is zero in the corner points.*

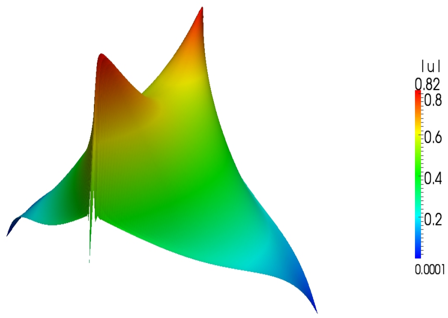

Figure 3.12: $[L_2(\Gamma)]^n$–control, $\varrho = 1$.

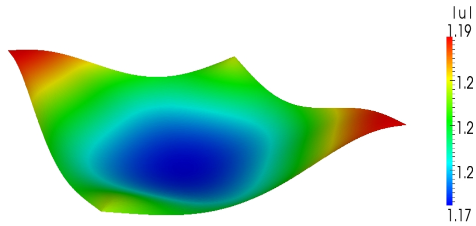

Figure 3.13: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 1$.

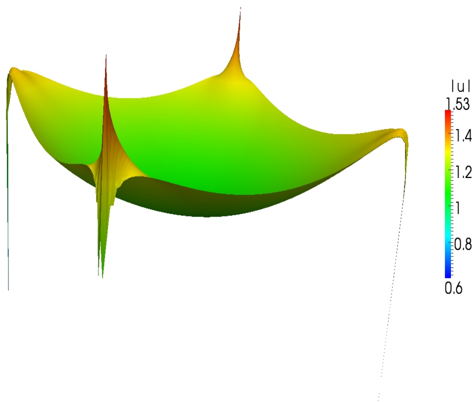

Figure 3.14: $[L_2(\Gamma)]^n$–control, $\varrho = 10^{-3}$.

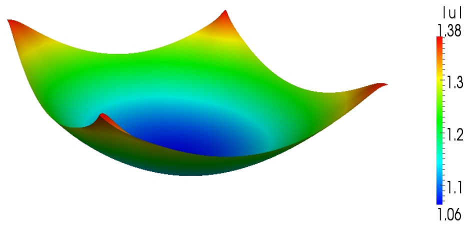

Figure 3.15: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 10^{-3}$.

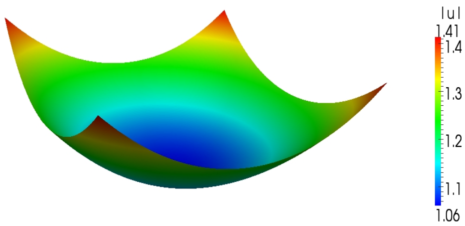

Figure 3.16: $[L_2(\Gamma)]^n$–control, $\varrho = 10^{-6}$.

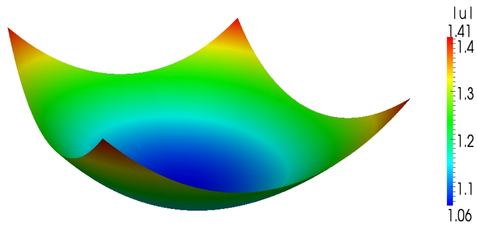

Figure 3.17: $\left[H^{1/2}(\Gamma)\right]^n$–control, $\varrho = 10^{-6}$.

Again, let us consider the previous examples with a cost coefficient $\varrho = 10^{-6}$, with an adaptive and a coarse mesh. The adaptive mesh is the same as considered in the Stokes case, see Section 3.1.3, with $N = 64642$ elements. For the coarse mesh we take the standard mesh with $N = 65536$ elements into account. Considering the control in $[L_2(\Gamma)]^n$, see Figure 3.18 and Figure 3.20, we observe non zero values in the corner points for the control for the coarse mesh, but for the adaptive mesh this effect is visible again. On the other hand, if we consider the control in $\left[H^{1/2}(\Gamma)\right]^n$ we do not have this behavior, see Figure 3.19 and Figure 3.21. This example shows, that adaptivity is an important aspect for optimal control problems.

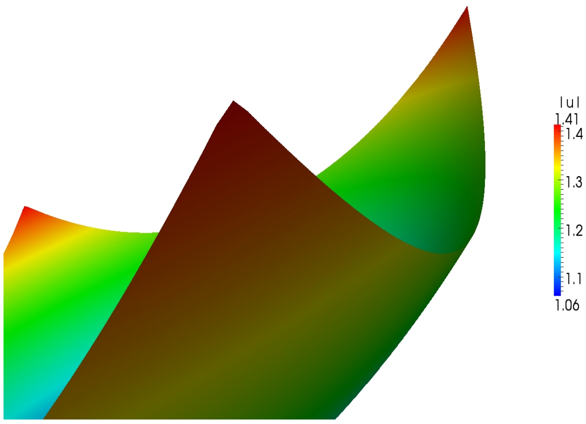

Figure 3.18: $[L_2(\Gamma)]^n$, coarse mesh.

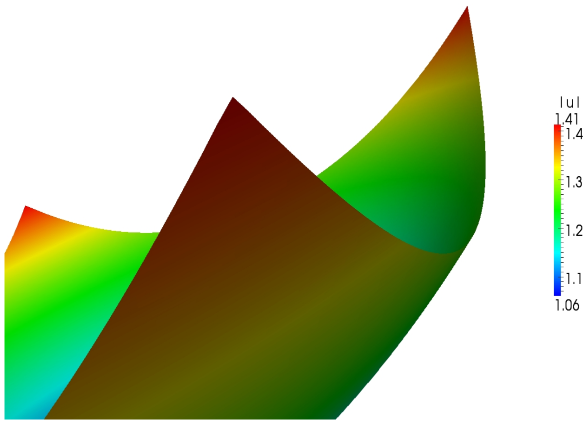

Figure 3.19: $\left[H^{1/2}(\Gamma)\right]^n$, coarse mesh.

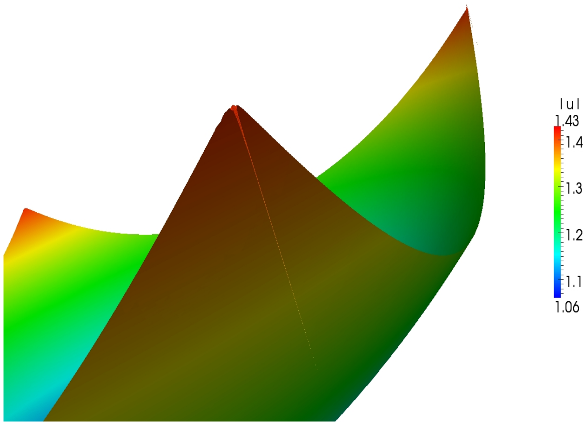

Figure 3.20: $[L_2(\Gamma)]^n$, adaptive mesh.

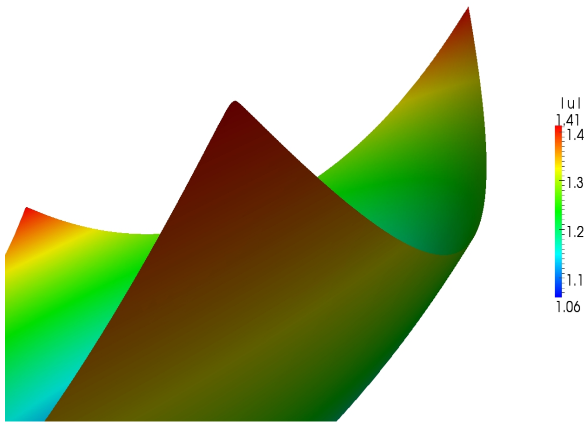

Figure 3.21: $\left[H^{1/2}(\Gamma)\right]^n$, adaptive mesh.

3.2.4 A more realistic example

In this section, we finally consider an optimal control problem of the Navier–Stokes equations, which is applied to an airfoil example. This example is similar as considered in [19]. The computational domain is given by a main airfoil, NACA 4412, at 8° angle of attack and a flap, NACA 4415, with 37° deflection, see Figure 3.22. The enclosed rectangle is considered as $(-1,6)\times(-1,1)$.

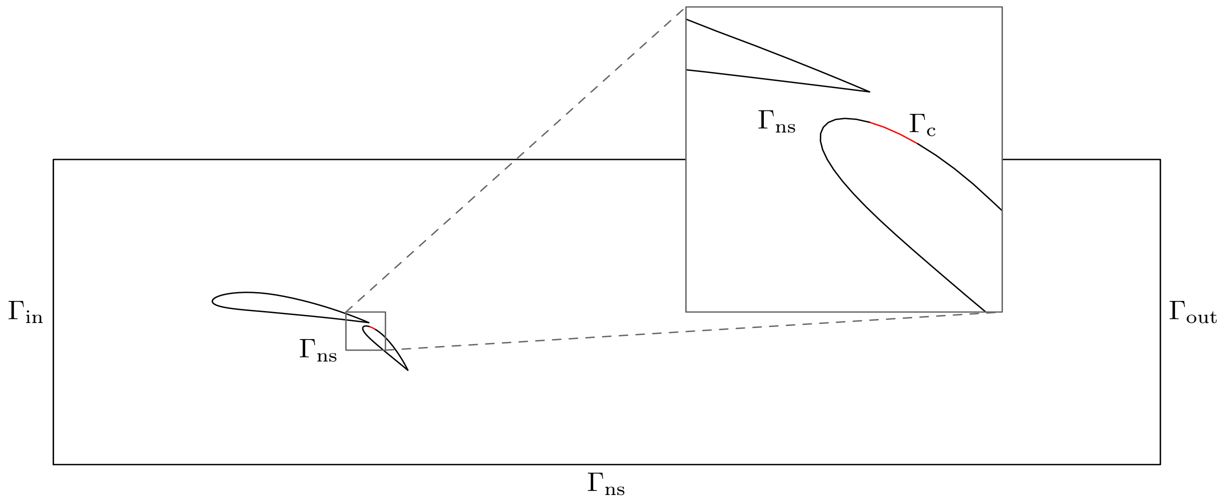

Figure 3.22: Airfoil.

We consider the boundary $\Gamma = \overline{\Gamma}_{\mathrm{ns}} \cup \overline{\Gamma}_{\mathrm{in}} \cup \overline{\Gamma}_{\mathrm{out}} \cup \overline{\Gamma}_{\mathrm{c}}$ with a noslip boundary Γ_{ns}, a inflow boundary Γ_{in}, a outflow boundary Γ_{out}, and a control boundary Γ_{c}. On a part of the boundary of the flap, the control boundary Γ_{c} is considered, where either some suction or blowing occurs, see Figure 3.22. The corresponding optimal control problem is given by (1.1)–(1.2), with the following data:

$$\overline{\underline{u}} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}, \quad \underline{f} = \underline{0}, \quad \underline{g} = \begin{pmatrix} 1-x_2^2 \\ 0 \end{pmatrix} \quad \text{on } \Gamma_{\mathrm{in}}.$$

Furthermore, we consider $\nu = 1/50$ as viscosity constant, which is about $Re \approx 100$, and the cost coefficient $\varrho = 1$. For the discretization we use stabilized equal order 1 elements. Again, we consider the control either in $[L_2(\Gamma_{\mathrm{c}})]^n$ or in $\left[H_{00}^{1/2}(\Gamma_{\mathrm{c}})\right]^n$. In particular, for the latter case, the control is considered in the subspace $\left[H_{00}^{1/2}(\Gamma_{\mathrm{c}})\right]^n$, since the control boundary Γ_{c} only intersects with a Dirichlet boundary Γ_{ns}, see Figure 3.22, i.e. $\Gamma_{\mathrm{ns}} \cap \Gamma_{\mathrm{c}} \neq 0$ and $\Gamma_{\mathrm{out}} \cap \Gamma_{\mathrm{c}} = 0$, see Section 1.2.2. In the following, we present numerical results for this example and focus especially on the difference in the choice of the control spaces. The calculations where done on a mesh with $N = 4321280$ elements.

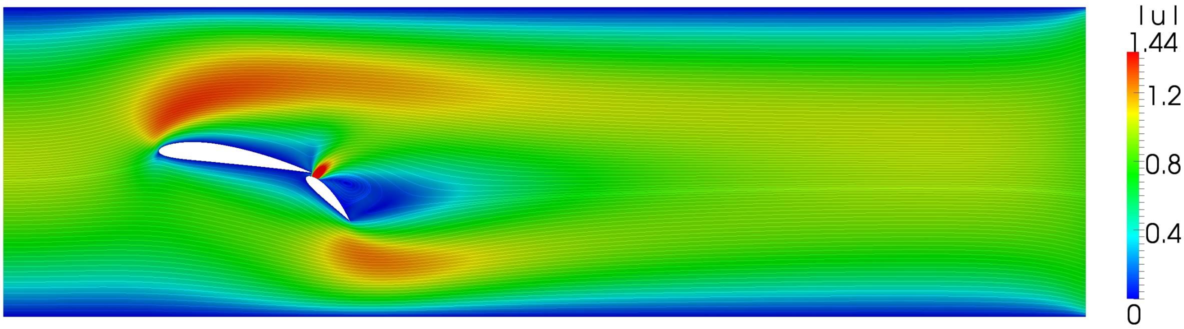

Figure 3.23: Absolute value of the state $\underline{u}_h$ for $[L_2(\Gamma_c)]^n$–control.

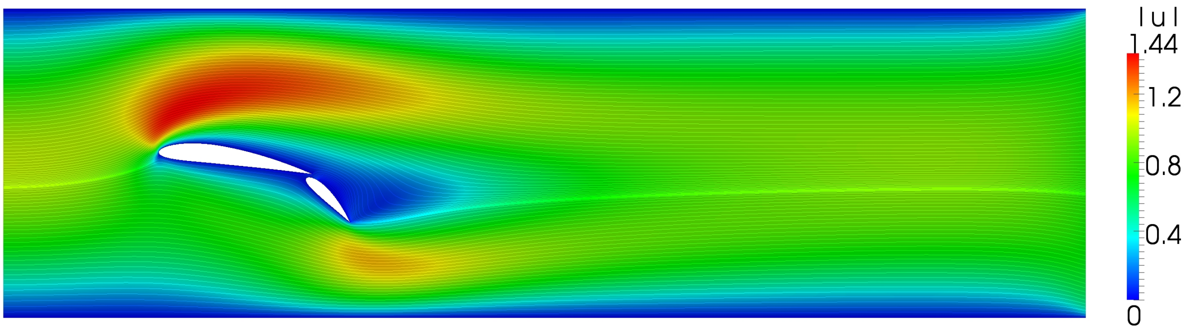

Figure 3.24: Absolute value of the state $\underline{u}_h$ for $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$–control.

In Figure 3.23 we give the numerical result for the control in $[L_2(\Gamma_c)]^n$ and the result for the control in $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$ is given in Figure 3.24. The former result, i.e. Figure 3.23, is scaled to make it comparable to Figure 3.24. We see that there is a huge difference between these two results. For the control in $[L_2(\Gamma_c)]^n$ a lot of blowing occurs at the control boundary. In the $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$ case, just a little occurs.
In Figure 3.25, we see a zoom on the control boundary Γ_c. We give the results without scaling. In the case where the control is considered in $[L_2(\Gamma_c)]^n$, we observe two singularities. If the cost coefficient is even smaller, i.e. blowing or suction is cheaper, the results would be even more different. However, if the control is considered in $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$, we get a parabolic outflow at the control boundary. Moreover, we recognize a huge difference in the absolute values at the control boundary.

Remark 3.8. *We see from this example, that the consideration of the control in the space* $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$ *makes more sense. However is the control considered in* $[L_2(\Gamma_c)]^n$, *we get singularities, this makes also no sense from a physical point of view.*

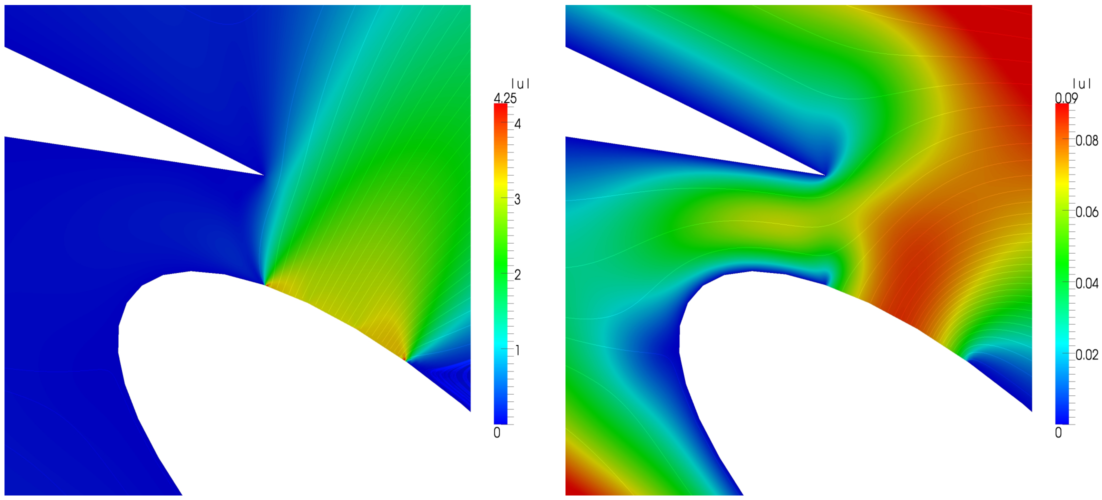

Figure 3.25: Absolute value of the state $\underline{u}_h$ at the control boundary for $[L_2(\Gamma_c)]^n$–control (left) and for $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$–control (right).

In this chapter, we have considered stabilized finite element methods for optimal control problems. In the first part, we considered the discretization of the optimal control problem of the Stokes equations. We presented the resulting linear system and some related numerical results. We obtain an additional necessary condition for a control in $[L_2(\Gamma)]^n$ and obtained better numerical results for a control in $\left[H^{1/2}(\Gamma_c)\right]^n$.
In the second part, we discretized the optimal control problem of the Navier–Stokes equations by using stabilized finite elements. Again, we observed better numerical results for a control in $\left[H^{1/2}(\Gamma_c)\right]^n$. Moreover, we considered the flow control around an airfoil. The results showed, that the consideration of a control in $\left[H_{00}^{1/2}(\Gamma_c)\right]^n$ is more natural.

Outlook

In this thesis, we have considered different optimal control problems in fluid mechanics. We treated optimal control problems for the Stokes equations and for the Navier–Stokes equations. In the first chapter, we considered the analysis of the optimal control problem for the Stokes equations and derived the optimality system as an equivalent problem. Moreover, we derived the optimality system for the Navier–Stokes equations.
In the second chapter, we focused on stabilized finite element methods. In the first part, these methods were considered for the Stokes equations, where the related numerical results confirmed the theory. In the second part, a stabilized finite element method for the Navier–Stokes equations is considered, and some numerical results are shown.
In the third chapter, stabilized finite element methods for optimal control problems of the Stokes equations and the Navier–Stokes equations are considered. Here we focused on the difference of a control in $[L_2(\Gamma)]^n$ and in the energy space $\left[H^{1/2}(\Gamma)\right]^n$, where we obtained better numerical results for the latter case.

In the second chapter stabilized finite element methods are considered. Here the choice of the stabilization parameter δ and the optimal stabilization parameter is still questionable. Especially for the stabilized finite element formulation for the Navier–Stokes equations it is not completely clear how to choose the constants λ_1, λ_2. In addition, it would be very interesting to apply these methods to time dependent problems.
For the optimal control problem of the Stokes equations and the Navier–Stokes equations, an error analysis would be interesting. Moreover, we already mentioned that we have to deal with large linear systems, when considering these problems. Up to now these are solved by a direct solver, but if we want to do 3D computations we have to think about appropriate solvers and preconditioners.
Another interesting topic, which is not considered in this thesis, are box constraints. Moreover, we want to mention an application to time dependent optimal control problems. Especially for the Navier–Stokes equations this would be interesting. Other interesting aspects are optimal control problems for hyperbolic equations and optimal control in fluid structure interaction.

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