sense (cf. the set F in Fig. 7.4): it cannot be "compressed" to a smaller set (the sphere can be compressed into a point without changing the external potential!).

The maximal analytical extension gives a uniquely defined solution. Still, it is an *improperly posed problem* in the sense of sec. 7.1 since the solution is not stable (cf. Lavrentiev, 1967, Chapter II). An arbitrarily small change in the external potential V can provoke a large change in the analytical continuation and may even completely alter the singularities: shift them, make them vanish or create new ones. This is an implication for the present problem of the Runge theorem (cf. Schulze and Wildenhain, 1977, sec. III.2.9; Moritz, 1980, sec. 8). We may also say that the analytical continuation into the interior of S is an initial-value (Cauchy) problem for Laplace's equation, which has for a long time been known to be improperly posed (Courant and Hilbert, 1962, pp. 227–229). In fact, S may be regarded as a "Cauchy surface", from which the analytical continuation into the interior starts.

A striking and simple example is given by the spherical harmonic series of the external potential, which always exists and converges outside any sphere that completely encloses S. Inside S, the analytical continuation may have very complicated singularities. Truncating the series at an arbitrarily high degree N (e.g., equal to 10^6) always provides a function that has a multipole singularity *at the origin only.* In fact, a spherical harmonic of degree n is equivalent to a multipole; this interpretation is due to Maxwell (cf. Courant and Hilbert, 1953, pp. 514–521; Hobson, 1931, secs. 79 to 84). Thus letting $N \to \infty$ makes the multipole singularity "explode" to form the arbitrarily complex original singularity structure!

In view of the instability of analytical continuation, the precise determination of the singularity structure of the earth (say) is practically impossible even if it were theoretically feasible. Thus the attractive idea of determining all possible density distributions by "blowing up" the singularity structure in various ways (much in the same way as a point mass singularity may be blown up to spheres of various sizes) is likely to remain science fiction.

7.6 Continuous Density Distributions for the Sphere

We have seen that the general gravitational inverse problem is very difficult and has not been solved generally so far.

However, restricting ourselves to continuous density distributions for the sphere, a rather general solution can be found in a simple and elementary way. A spherical earth is a good approximation for many geophysical purposes, especially for determining density anomalies from given potential anomalies. Furthermore, even "discontinuities" such as the core-mantle boundary may be regarded as continuous, though rather abrupt, transitions.

The approach is based on trying to find an approximate finite matrix equivalent to the Newtonian operator N, as we have already announced in sec. 7.1. The approach employs the usual spherical coordinates r (radius vector), θ (polar distance), and λ

(longitude). With respect to θ and λ, the base functions are the surface spherical harmonics, which are known to form a complete basis for functions defined on the sphere. For describing the dependence on the radius vector r, we take a polynomial representation. The powers r^n form a complete though not orthogonal basis in the space of continuous functions $f(r)$, in view of the famous *theorem of Weierstrass*: the polynomials are dense in the space of continuous functions, or in less abstract terms, any continuous function can be uniformly approximated by polynomials to any degree of accuracy. The coefficients of these polynomials form a finite-dimensional space. Our method thus is entirely elementary, avoiding Hilbert spaces considered in (Ballani and Stromeyer, 1983). As we have already mentioned, there is a certain similarity to the elegant approach of Dufour (1977) who uses orthogonalized (Jacobi) polynomials, but the present method is somewhat more general, admitting all even and odd powers of the radius vector r. We shall follow (Moritz, 1989).

7.6.1 Use of Spherical Harmonics

So let us use spherical coordinates r, θ and λ. For an internal sphere $r = \text{const.} = r_0$ the density ρ will be a function of θ and λ:

$$\rho = f_0(\theta, \lambda) = \rho(\theta, \lambda; r_0) \quad , \tag{7-22}$$

where r_0 enters as a parameter labeling the set of concentric spheres. This function may be expanded into a series of spherical surface harmonics (1–45):

$$f_0(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm} \cos\theta) \quad , \tag{7-23}$$

where P_{nm} are the standard Legendre functions and the coefficients a_{nm} and b_{nm} depend on r_0. Using the notation

$$\begin{aligned} Y_{nm}(\theta, \lambda) &= P_{nm}(\cos\theta) \cos m\lambda \ , \quad m = 0, 1, \ldots, n \ , \\ Y_{n,-m}(\theta, \lambda) &= P_{nm}(\cos\theta) \sin m\lambda \ , \quad m = 1, \ldots, n \ , \end{aligned} \tag{7-24}$$

we may write (7–23) in a more compact way:

$$f_0(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}(r_0) Y_{nm}(\theta, \lambda) = \sum_{n,m} f_{nm} Y_{nm} \quad . \tag{7-25}$$

Since f_{nm} ($= a_{nm}$ or b_{nm}) depends on r_0, we may generally write

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}(r) Y_{nm}(\theta, \lambda) \quad , \tag{7-26}$$

an equation which is valid and convergent under very weak conditions on the smoothness of the functions involved, which we shall always take for granted without explicitly mentioning it.

The functions $f_{nm}(r)$ are quite arbitrary; they only must decrease with increasing n in such a way that (7–26) converges and, above all, they must be compatible with the given external potential. Still, there remains an infinite set of possible $f_{nm}(r)$, which expresses the non-uniqueness of the problem.

Let us represent f_{nm} by a polynomial

$$f_{nm}(r) = \sum_{k=0}^{N} x_{nmk} r^k \tag{7–27}$$

or, more briefly,

$$f(r) = \sum_{k=0}^{N} x_k r^k \quad , \tag{7–28}$$

x_k denoting constant coefficients. According to Weierstrass' theorem, we get an arbitrarily good approximation by choosing N sufficiently large.

We write eq. (7–1) in the form

$$V(r, \theta, \lambda) = G \iiint \frac{\rho}{l} dv \quad , \tag{7–29}$$

where V denotes the gravitational potential, G the gravitational constant, the integral is extended over the volume of the body bounded by the sphere $r = R$ ($\doteq$ 6371 km for terrestrial applications), ρ is the density and dv the volume element as usual. The symbol l denotes the distance between the point $P(r, \theta, \lambda)$ to which V refers, and the point $Q(r', \theta', \lambda')$ to which ρ and dv refer. If P is outside the sphere, we have by (1–53)

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{7–30}$$

where P_n denotes Legendre's polynomials and ψ is the angle between r and r'.

We substitute (7–26) and (7–30) into (7–29), to get

$$V(r, \theta, \lambda) = G \sum_{n'=0}^{\infty} \frac{1}{r^{n'+1}} \sum_{n,m} \iint_{\sigma} Y_{nm}(\theta', \lambda') P_{n'}(\cos\psi) d\sigma \cdot \int_{r'=0}^{R} f_{nm}(r') r'^{n'+2} dr'. \tag{7–31}$$

Here we have replaced the index n in (7–30) by n', interchanged sum and integration without mathematical scruples, and put for the volume element

$$dv = r'^2 \sin\theta' dr' d\theta' d\lambda' = r'^2 dr' d\sigma \quad , \tag{7–32}$$

$d\sigma$ denoting the element of solid angle or the surface element of the unit sphere σ.

Now the integral over σ is zero unless $n' = n$ because of orthogonality, and there remains

$$V(r, \theta, \lambda) = \sum_{n,m} V_{nm} \frac{Y_{nm}(\theta, \lambda)}{r^{n+1}} \quad , \tag{7–33}$$

using (1–49) with $f(\theta', \lambda') = Y_{nm}(\theta', \lambda')$, so that

$$V_{nm} = \frac{4\pi G}{2n+1} \int_0^R f_{nm}(r') r'^{n+2} dr' \quad . \tag{7–34}$$

The substitution of the polynomial (7–27) finally yields on integration

$$V_{nm} = \frac{4\pi G}{2n+1} \sum_{k=0}^{N} \frac{R^{n+k+3}}{n+k+3} x_{nmk} = \sum_{k=0}^{N} a_{nmk} x_{nmk} \quad . \tag{7–35}$$

Now the coefficients V_{nm} are nothing else than the spherical-harmonic coefficients of the external gravitational potential, which are well-known on a smoothed global scale; cf. (Rapp, 1986).

Assuming them given, we thus have the system of equations

$$\sum_{k=0}^{N} a_{nmk} x_{nmk} = V_{nm} \tag{7–36}$$

for the unknown coefficients x_{nmk}.

7.6.2 A Very General Solution

The system (7–36) is much less formidable than it looks. First of all, all degrees n and orders m are separated! *This means that we can treat each term* (m, n) *individually.* (This seems to be an essential advantage as compared to the approach of Dufour to be treated in sec. 7.6.7.) We thus omit the symbols n, m as we already did in (7–28) to get, instead of (7–36), a linear equation of form

$$\sum_{k=0}^{N} a_k x_k = b \quad , \tag{7–37}$$

where, of course, b represents V_{nm}.

Given b and the coefficients a_k (by (7–35)), we can satisfy (7–37) by infinitely many $(N+1)$-tuples x_k. Geometrically speaking, (7–37) is the equation of a (hyper)plane in $(N+1)$-dimensional space, and the only condition that the vector $x = [x_0, x_1, \ldots, x_N]$ must satisfy is that it must lead to a point in the plane (7–37), cf. Fig. 7.5.

A very general solution of (7–37) is

$$x_k = \frac{\sum_{j=0}^{N} c_{kj} a_j}{\sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} a_i a_j} b \quad , \tag{7–38}$$

as one immediately sees on substituting into (7–37). The matrix $[c_{ij}]$ can be chosen symmetric and positive definite and is otherwise arbitrary. The set of all these matrices (for all n) characterizes the set of possible solutions!

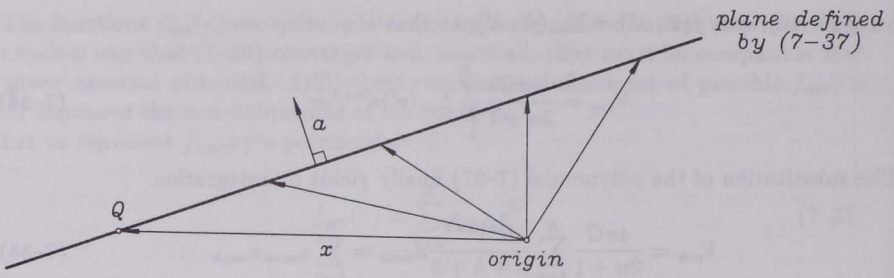

FIGURE 7.5: Possible choices of the vector x

The form (7–38) is motivated by the theory of generalized matrix inverses: if

$$Ax = b \tag{7–39}$$

is an underdetermined system of equations, the solution is formally given by

$$x = A^{-1}b \quad , \tag{7–40}$$

where the generalized inverse has the form (T denotes the transpose)

$$A^{-1} = CA^T(ACA^T)^{-1} \quad , \tag{7–41}$$

with any positive-definite symmetric square matrix C of appropriate dimension (cf. Bjerhammar, 1973, p. 110; Moritz, 1980, p. 164). Clearly (7–37) and (7–38) are special cases of (7–39) and (7–40) with (7–41).

The solution (7–38) satisfies the minimum condition

$$x^T Px = \text{minimum} \quad , \tag{7–42}$$

where $P = C^{-1}$. This means that x represents the "shortest" distance of the plane (7–37) from the origin, but of course in a non-orthogonal coordinate system whose metric tensor is P. That any point in the plane can be reached by a suitable choice of P can be seen in the following way (Krarup, 1972).

As we have mentioned, eq. (7–37) defines an N-dimensional hyperplane in our $(N+1)$-dimensional space (Fig. 7.5). Choose, for the first N base vectors, any set of N mutually orthogonal unit vectors (in the Euclidean sense) spanning the hyperplane. For the remaining $(N+1)$st base vector simply take the vector x leading from the origin to the desired point Q in the plane (Fig. 7.5). It is "orthogonal" to the hyperplane in the sense of the metric tensor P (though not in the Euclidean sense!) by the very condition (7–42), and its length is arbitrarily taken as unity.

Now we have found a set of $N+1$ linearly independent non-orthogonal vectors, and we must determine the metric tensor P for which they constitute an "orthonormal" set of base vectors. Let A now be the $(N+1)\times(N+1)$ matrix having as column vectors the

components of these $N+1$ base vectors in our original Cartesian coordinate system. Then A is not singular, and the condition that the given vectors be orthonormal with respect to P can be expressed as follows:

$$A^T P A = I \tag{7–43}$$

(I denotes the unit matrix), whence

$$P = (AA^T)^{-1} \tag{7–44}$$

is determined. Clearly, P and hence C are symmetric and positive definite matrices.

If one takes care of convergence, one may even let $N \to \infty$, but this is not really necessary because of Weierstrass' theorem mentioned above.

A minor point is that the degree $n = 1$ is usually missing: it can be made zero by a suitable choice of origin (Heiskanen and Moritz, 1967, p. 62). Also in order to have a well-defined density at the origin, it is necessary, except for $n = 0$, to start the summation in equations such as (7–27) or (7–37) with $k = 1$ rather than $k = 0$ (which reduces the dimension of our base space from $N+1$ to N).

7.6.3 Harmonic Densities

A possible solution of (7–36) is, of course, obtained by putting $a_{nmk} = 0$ except for $k = n$, which gives

$$x_{nm} = \frac{V_{nm}}{a_{nmn}} = \frac{(2n+1)(2n+3)}{4\pi G R^{2n+3}} V_{nm} \quad , \tag{7–45}$$

by (7–35); this solution is unique. Thus

$$f_{nm} = x_{nm} r^n = \text{const.} r^n \quad , \tag{7–46}$$

and (7–26) gives

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} x_{nm} r^n Y_{nm}(\theta, \lambda) \quad , \tag{7–47}$$

which is a series of (internal) spherical harmonics; cf. sec. 1.3. Thus (7–47) represents the *harmonic density* for the spherical case. It is uniquely defined as we have announced in sec. 7.3 (theorem of Lauricella).

Considering the behavior of the powers r^n (Fig. 7.6; we have put $R = 1$), we see that the higher the degree n, the more concentrated towards the earth's surface will be the corresponding contribution of the density. This about corresponds to the physical feeling that higher-frequency density anomalies should be situated in the earth's upper crust and mantle, but otherwise the harmonic densities do not have any meaningful physical interpretation. Their main usefulness is mathematical, as a uniquely defined continuous solution of the inverse problem; cf. sec. 7.3.

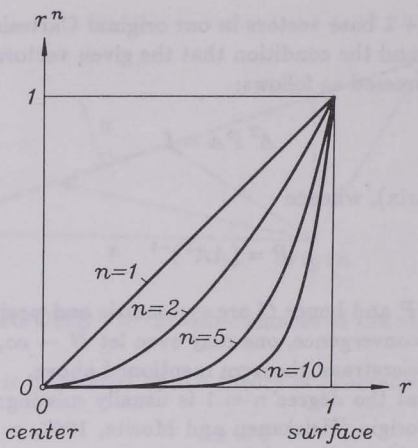

FIGURE 7.6: The powers r^n $(0 \leq r \leq 1)$

7.6.4 Zero-Potential Densities

The solution (7–38) gives $x_k = 0$ if the right-hand side of (7–37), $b = V_{nm}$, is zero. This is the case of the homogeneous equation corresponding to (7–36),

$$\sum_{k=0}^{N} a_{nmk} x_{nmk} = 0 \quad , \tag{7–48}$$

or briefly, corresponding to (7–37),

$$\sum_{k=0}^{N} a_k x_k = 0 \quad , \tag{7–49}$$

which represents the case of a mass distribution that produces zero external potential.

These are the "zero-potential densities" (sec. 7.2), forming the kernel of the Newtonian operator, for the present case. It is very easy to find non-zero solutions of (7–48) or (7-49): eq. (7–49) means simply that the vector x is normal to the given vector a (in the usual Euclidean metric)! Thus any vector x in the plane normal to a is admissible.

Finally we mention that the set of solutions of (7–49), forming *the vector* $x^{(2)}$ in (7–52) *is "orthogonal" to the vector* (7–38), denoted in (7–52) by $x^{(1)}$, if we again take P as metric tensor. This is geometrically evident and is also immediately verified by direct computation: using (7–38) in matrix notation, we have

$$x^{(1)T} P x^{(2)} = \frac{b}{a^T C a} a^T C P x^{(2)} = 0 \tag{7–50}$$

since $CP = I$ (unit matrix) and $a^T x^{(2)} = 0$ by (7–49).

7.6.5 Remarks on the General Solution

The proposed general set of solutions may be summarized as follows: the density is represented in the form (7–26) with (7–27):

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{k=0}^{N} x_{nmk} r^k Y_{nm}(\theta, \lambda) \quad , \tag{7–51}$$

where

$$x_{nmk} = x^{(1)}_{nmk} + x^{(2)}_{nmk} \quad , \tag{7–52}$$

$x^{(1)}$ corresponding to the solution (7–38) and $x^{(2)}$ to any solution of the homogeneous equation (7–48) as before. The coefficients a_{nmk} are given by (7–35):

$$a_{nmk} = \frac{4\pi G R^{n+k+3}}{(2n+1)(n+k+3)} \quad . \tag{7–53}$$

The set of solutions contains the following free parameters: an arbitrary positive definite symmetric matrix $[c_{ij}]$ in (7–38), different for each (m, n), and the "zero–potential–density vector" $x^{(2)}$ which is only subject to the condition that it satisfies (7–48). Evident restrictions such as the absence of the terms with $n = 1$ and of the terms $k = 0$ except for $n = 0$ have already been mentioned.

Now there comes a surprise (Fig. 7.7). Unless $b = V_{nm}$ is zero, the end point of the

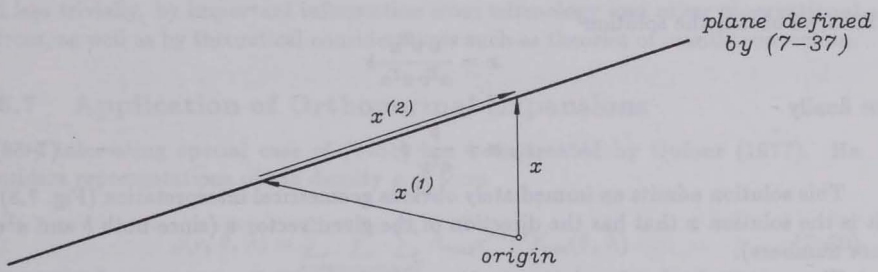

FIGURE 7.7: The sum $x = x^{(1)} + x^{(2)}$ again is of type $x^{(1)}$

vector x as given by (7–52) again lies in the hyperplane (7–37) and can therefore be represented in the form (7–38). Thus even the total solution (7–52), $x = x^{(1)} + x^{(2)}$, can be exclusively characterized by a certain matrix from our set of symmetric and positive definite matrices $[c_{ij}]$, so that we need only solutions of type $x^{(1)}$ as expressed by (7–38). Solutions of type $x^{(2)}$ are necessary only if $b = V_{nm} = 0$. Of course, on a closer look, this is not so surprising after all.

In statistical terms, $C = [c_{ij}]$ represents the covariance matrix of the vector x; in case it is given, (7–38) expresses a kind of least-squares (minimum norm) solution, by (7–42).

7.6.6 An Essential Simplification

The solution (7–38) may be written in the matrix form

$$x = Ca(a^T Ca)^{-1} b \quad , \tag{7–54}$$

where C is a square matrix and $a^T Ca$ is a number which must be $\neq 0$; b is also a number.

Every x satisfying (7–37) can be represented in the form (7–54), but, so to speak, it is "overrepresented": to each x there correspond infinitely many matrices C.

In fact (Rao and Mitra, 1971, p. 20), the matrix

$$CA^T(ACA^T)^{-1} \tag{7–55}$$

expresses the general form of a right inverse of A, according to the theory of generalized matrix inverses; the rank of ACA^T must be equal to the rank of A.

In our case, $A = a^T$ is a vector, supposed non-zero, that is, of rank 1. In this case *it is sufficient if* C *has rank* 1, that is, if it is of form

$$C = v\, v^T \quad , \tag{7–56}$$

where v is an arbitrary $(n+1)$ column vector which only satisfies

$$a^T v \neq 0 \quad . \tag{7–57}$$

Thus we obtain the solution

$$x = \frac{v\, v^T a}{a^T v\, v^T a} b$$

or finally

$$x = \frac{b}{a^T v} v \quad . \tag{7–58}$$

This solution admits an immediately obvious geometrical interpretation (Fig. 7.8): it is the solution x that has the direction of the given vector v (since both b and $a^T v$ are numbers).

This extremely simple solution is due to G. Zielke (Zielke and Moritz, 1989). It goes without saying that this is definitely preferable to (7–54) for practical applications, unless we exceptionally have some a-priori statistical or other information which we would like to incorporate into the matrix C. With (7–58) we get along with $N+1$ components of the vector v, instead of working with the elements (of order $N^2/2$) of a full-rank symmetric matrix C.

Obviously, $v = a$ gives the solution of minimum Euclidean norm (shortest length of x), i.e.,

$$x_{min} = \frac{b}{a^T a} a \quad , \tag{7–59}$$

which mathematically is the simplest solution but which does not seem to have a physical interpretation.

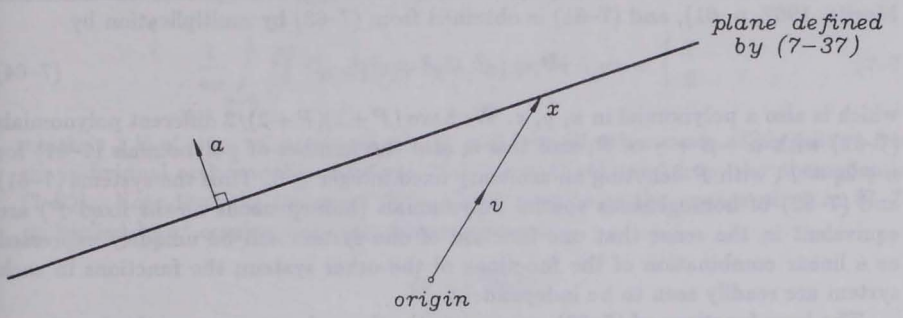

FIGURE 7.8: Representation of the solution x by an arbitrary vector v

Our set of density distributions comprises densities that are partly negative. As we have seen, this is not unphysical if V is regarded as a disturbing potential and ρ as a density anomaly, with respect to an underlying reference density model such as PREM. In fact, as mentioned before, this interpretation is of practical relevance if for V we take one of the global spherical harmonic expansions as discussed, e.g., in (Rapp, 1986).

The set of possible solutions can then be suitably restricted: by the obvious condition that the total density (reference density plus density anomaly) must be positive, and less trivially, by important information from seismology and other observational sources, as well as by theoretical considerations such as theories of mantle convection.

7.6.7 Application of Orthonormal Expansions

A very interesting special case of (7–51) has been treated by Dufour (1977). He considers representations of the density ρ of form

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{Q} \beta_{nmq} r^{n+2q} Y_{nm}(\theta, \lambda) \quad . \tag{7–60}$$

A first glance shows that (7–60) is less general than (7–51) because the powers r^k for $k < n$ are missing, as well as the powers r^{n+1}, r^{n+3}, r^{n+5}, ... However, the base functions

$$r^{n+2q} Y_{nm}(\theta, \lambda) = r^n Y_{nm}(\theta, \lambda) \cdot r^{2q} \tag{7–61}$$

are easily seen to be polynomials in the Cartesian coordinates x, y, z of form

$$x^\alpha y^\beta z^\gamma \quad , \tag{7–62}$$

α, β, γ being integers ≥ 0.

In fact, the solid harmonics (1–35a), or

$$r^n Y_n(\theta, \lambda) \quad , \tag{7–63}$$

are well known to be harmonic polynomials (cf. Kellogg, 1929, p. 141; Heiskanen and Moritz, 1967, p. 61), and (7–61) is obtained from (7–63) by multiplication by

$$r^{2q} = (x^2 + y^2 + z^2)^q \quad , \tag{7–64}$$

which is also a polynomial in x, y, z. We have $(P+1)(P+2)/2$ different polynomials (7–62) with $\alpha + \beta + \gamma = P$, and this is also the number of polynomials (7–61) for $n + 2q = P$, with P denoting an arbitrary fixed integer ≥ 0. Thus the systems (7–61) and (7–62) of homogeneous spatial polynomials (homogeneous means fixed P) are equivalent in the sense that one function of one system can be uniquely expressed as a linear combination of the functions of the other system; the functions in each system are readily seen to be independent.

The base functions of (7–60), as regards the dependence on r, are the functions

$$r^n[1, r^2, r^4, \ldots, r^{2Q}] \quad . \tag{7–65}$$

We may orthonormalize these functions, e.g., by the well–known Gram–Schmidt orthogonalization process (cf. Courant and Hilbert, 1953, pp. 4 and 50), obtaining the orthonormal system of polynomials

$$r^n P_{n,q}(r^2) \qquad (q = 0, 1, \ldots, Q) \tag{7–66}$$

equivalent to (7–65) in the sense just mentioned. They will satisfy the orthonormality relations

$$\int_0^1 r^{2n+2} P_{n,q}(r^2) P_{n,q'}(r^2) dr = \begin{cases} 1 & \text{if} \quad q' = q \, , \\ 0 & \text{if} \quad q' \neq q \, ; \end{cases} \tag{7–67}$$

we have put $R = 1$ without loss of generality. Substituting $r^2 = u$ we get

$$\frac{1}{2} \int_0^1 u^{n+\frac{1}{2}} P_{n,q}(u) P_{n,q'}(u) du = \begin{cases} 1 & \text{if} \quad q' = q \, , \\ 0 & \text{if} \quad q' \neq q \, . \end{cases} \tag{7–68}$$

Such polynomials (the factor $1/2$ is inessential) are special cases of Jacobi polynomials (cf. Courant and Hilbert, 1953, p. 88; Abramowitz and Stegun, 1965, sec. 22). We need not give them explicitly because we are mainly interested in the fundamental conceptual features of Dufour's theory.

The "conventional" spherical harmonics Y_{nm} (= R_{nm} or S_{nm}) are orthogonal but not normalized; cf. (1–41) and (1–42). By multiplying them by an obvious factor related to κ_{nm} in (1–42) it is possible to normalize them in the sense that

$$\frac{1}{4\pi} \iint_\sigma [\bar{Y}_{nm}(\theta, \lambda)]^2 d\sigma = 1 \quad , \tag{7–69}$$

$\bar{Y}_{nm}(\theta, \lambda)$ being "fully normalized" spherical harmonics (cf. Heiskanen and Moritz, 1967, sec. 1–14). We thus see that Dufour's spatial functions

$$r^n \bar{Y}_{nm}(\theta, \lambda) P_{n,q}(r^2) \equiv D_{nmq}(r, \theta, \lambda) \tag{7–70}$$

are orthonormal in the solid sphere $r \leq 1$:

$$\frac{1}{4\pi} \int_{r=0}^{1} \iint_{\sigma} D_{nmq}(r, \theta, \lambda) D_{n'm'q'}(r, \theta, \lambda) dv = \begin{cases} 1 \\ 0 \end{cases} , \tag{7-71}$$

which is 1 if $n' = n$, $m' = m$ and $q' = q$, and 0 in all other cases. This follows from the individual orthogonality relations (7–67) and (1–41) and from the normalization (7–69). Note that the factor r^{2n+2} in (7–67) results as the square of r^n in (7–70) multiplied by r^2 coming from the volume element

$$dv = r^2 dr d\sigma$$

in (7–71). Note also that the Jacobi polynomial $P_{n,q}$ in (7–70) has nothing to do with the Legendre function P_{nm} defined by (1–30)!

Let now finally the density ρ be expressed in terms of the orthonormal base functions (7–70):

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{Q} \alpha_{nmq} D_{nmq}(r, \theta, \lambda) \quad . \tag{7-72}$$

By what has been said, this expression is completely equivalent to (7–60), of course with different coefficients α_{nmq}.

Any function ρ that can be approximated by a linear combination of polynomials (7–62), can also be approximated by (7–72), and we may even let $Q \to \infty$ under certain circumstances. In fact, Weierstrass' approximation theorem has been extended to three– (and higher–) dimensional space, so that any continuous function in space can be uniformly approximated by a linear combination of polynomials (7–62) (cf. Davis, 1963, sec. 6.6).

Now it is clear that

$$P_{n,0}(r^2) \equiv \text{const.} \quad ; \tag{7-73}$$

this follows directly from the orthogonalization process leading from (7–65) to (7–66). Thus (7–67) gives immediately

$$\int_0^1 r^{2n+2} P_{n,q}(r^2) dr = 0 \quad \text{if} \quad q \neq 0 \quad . \tag{7-74}$$

Substituting

$$f_{nm}(r') = \sum_{q=0}^{Q} \alpha_{nmq} r'^n P_{n,q}(r'^2)$$

into (7–31), noting that $n' = n$ because of orthogonality, and considering (7–74), we see that *only the terms with* $q = 0$ *in* (7–72) *give a non–zero contribution to* V_{nm}.

Now $q = 0$ by (7–61) means the solid harmonic (7–63), so that the terms with $q = 0$ in (7–72), all other coefficients being taken to be zero, represent the *harmonic density* (7–47). The terms with $q \neq 0$ then give a zero–potential density, *the arbitrariness of*

the corresponding coefficients expressing the non-uniqueness of the density producing zero external potential.

Thus the main elegance of Dufour's method consists in a neat separation of the space D of possible density functions (as represented by polynomials) into two mutually orthogonal subspaces:

1. the set of harmonic densities D_1;
2. the set of zero-potential densities D_2.

The first subspace is represented by the terms with $q = 0$, the second subspace by the terms with $q \neq 0$. Taking a general expression (7–72) and putting all terms with $q \neq 0$ equal to zero thus corresponds to a projection of the function onto the subspace D_1, and putting the terms with $q = 0$ equal to zero amounts to a projection on the subspace D_2.

This beautiful result represents the principal theoretical value of Dufour's approach. From a practical point of view, the more general polynomial approximation considered in the previous subsections seems to be preferable because, for a given potential coefficient V_{nm}, one would certainly not like to miss the terms with $k < n$ in the polynomial (7–27). From a theoretical point of view, however, Dufour's approach represents the functional-analytic, "geometrical", aspect of the inverse Newtonian operator with unsurpassable clarity.

A simple example. Let us finally illustrate the situation by an extremely simple analogue. Let the space D be simplified to Euclidean three-dimensional space. Let the harmonic subspace D_1 correspond to the z-axis, and the orthogonal subspace D_2 to the xy-plane. A general 3-vector

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_0 \end{bmatrix}$$

will stand for the coefficients α_{nmq} in (7–72). The harmonic densities are then represented by the projection of this vector onto the z-axis,

$$\begin{bmatrix} 0 \\ 0 \\ \alpha_0 \end{bmatrix} ,$$

which also corresponds to the Newtonian operator N of sec. 7.1 (apart from a constant factor given by a_{nmn} in (7–35)). On the other hand, the zero-potential densities are represented by the projection of the vector onto the xy-plane:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} ;$$

such vectors form the kernel, or nullspace, of the Newtonian operator N (sec. 7.2).

A given external potential essentially amounts to prescribing α_0 only. The non-uniqueness of the inverse of the Newtonian operator then means nothing else than that the components α_1 and α_2 in the first vector can be arbitrarily chosen.

To be sure, this example is so simplified as to be almost trivial, but it illustrates the geometrical situation very clearly.

7.7 Lauricella's Use of Green's Function

Finally we shall treat a very general explicit solution of the gravimetric inverse problem due to Lauricella (1911, 1912), which forms part of important work done by Italian mathematicians such as T. Boggio, U. Crudeli, E. Laura, R. Marcolongo, C. Mineo, P. Pizzetti, and C. Somigliana between 1900 and 1930. This work is not so well known as it deserves; an excellent review is (Marussi, 1980), where also references to the original papers are found.

We shall here follow the book (Frank and Mises, 1961, pp. 845–862), translating that treatment from the two–dimensional to the three–dimensional case.

7.7.1 Application of Green's Identity

Green's second identity may be written:

$$\iiint_v (U\Delta F - F\Delta U)dv = \iint_S \left(U\frac{\partial F}{\partial n} - F\frac{\partial U}{\partial n}\right) dS \quad ; \tag{7–75}$$

this is eq. (1–28) of (Heiskanen and Moritz, 1967, p. 11) with F instead of V. It is valid for arbitrary functions U and F (which are, of course, "smooth", that is, sufficiently often differentiable, but this will be taken for granted in the sequel without mentioning). Here v denotes the volume enclosed by the surface S, with volume element dv and surface element dS as usual, Δ is Laplace's operator and $\partial/\partial n$ denotes the derivative along the normal pointing away from v. The formula (7–75) is standard in physical geodesy; derivations may be found in (Sigl, 1985, pp. 30–32) or (Kellogg, 1929, pp. 211–215).

We now put

$$F = \Delta V \quad , \tag{7–76}$$

the Laplacian of the gravitational potential V, obtaining

$$\iiint_v (U\Delta^2 V - \Delta V\Delta U)dv = \iint_S \left(U\frac{\partial \Delta V}{\partial n} - \Delta V\frac{\partial U}{\partial n}\right) dS \quad . \tag{7–77}$$

In this equation we interchange U and V and subtract the new equation from (7–77). The result is

$$\iiint_v (U\Delta^2 V - V\Delta^2 U)dv = \iint_S \left(-V\frac{\partial \Delta U}{\partial n} + \Delta U\frac{\partial V}{\partial n} - \Delta V\frac{\partial U}{\partial n} + U\frac{\partial \Delta V}{\partial n}\right) dS \quad . \tag{7–78}$$

Let us now daydream. Suppose we can select U such that

$$\Delta^2 U = 0 \tag{7–79}$$