Numerical values for the coefficients are given by (5–170), (5–171), (5–232), and (5–235).

The density model (5–240) is rigorous, simple, gives a concrete idea about possible density distributions, and is practically applicable. However, it is not very general because of the use of polynomial representations.

The general form of the density distributions discussed in the preceding section, by (5–239), (5–184), and (5–156) is

$$\rho(u,\theta) = \rho_0 + \rho_1 - (x^2+y^2)A(u) - z^2 B(u) + \frac{E^2 \sin^2\theta}{u^2 + E^2\cos^2\theta}\, h(u) \quad . \qquad (5\text{–}241)$$

The functions $\alpha_n(u)$ that correspond to this distribution according to (5–86), are computed from (5–186), (5–187), (5–188), and (5–189), as well as from the auxiliary formula for $\sin^2\theta$ given at the beginning of sec. 5.7:

$$\begin{aligned}
\frac{1}{4\pi}\,\alpha_0(u) &= \left(u^2 + \frac{1}{3}E^2\right)(\rho_0+\rho_1) + \frac{2}{3}E^2 h(u) + \\
&+ \left(-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4\right)A(u) + \left(-\frac{1}{3}u^4 - \frac{1}{5}E^2u^2\right)B(u), \\
\frac{1}{4\pi}\,\alpha_2(u) &= \frac{2}{3}E^2(\rho_0+\rho_1) - \frac{2}{3}E^2 h(u) + \\
&+ \left(\frac{2}{3}u^4 + \frac{4}{7}E^2u^2 - \frac{2}{21}E^4\right)A(u) + \left(-\frac{2}{3}u^4 - \frac{4}{7}E^2u^2\right)B(u), \\
\frac{1}{4\pi}\,\alpha_4(u) &= \frac{8}{35}\left[(E^2u^2+E^4)A(u) - E^2u^2B(u)\right] \quad ; \\
\alpha_n(u) &\equiv 0 \quad \text{if} \quad n>4 \quad .
\end{aligned} \qquad (5\text{–}242)$$

The functions $h(u)$, $A(u)$, and $B(u)$ are rather arbitrary; they must only satisfy the conditions (5–162) and (5–198), together with the constant ρ_1. The "Maclaurin density" ρ_0 follows from (5–164).

We clearly see that the present model is not of the simple form (5–121) but implies a nonzero $\alpha_4(u)$. We also remark that the function $h(u)$ introduced in sec. 5.3 by (5–113), is of very general significance and also enters in (5–242), whereas the other auxiliary function $g(u)$ introduced in (5–112) was of more limited applicability: it was used in sec. 5.3.1 and, as the constant Maclaurin density (5–133), has played a basic role in sec. 5.4; $g(u)$ was also still used in sec. 5.6 but later on it lost its significance together with (5–178).

5.10 Numerical Considerations and Problems

In this section we shall work with the density function (5–241). The unit of length will again be chosen equal to the semiminor axis b of the reference ellipsoid:

$$b = 1 \quad . \qquad (5\text{–}243)$$

Again we represent the functions $A(u)$ and $B(u)$ by polynomials

$$A(u) = b_0 + b_2 u^2 \quad , \tag{5-244}$$
$$B(u) = A(u)F(u) \quad , \tag{5-245}$$
$$F(u) = a_0 + a_2 u^2 + a_4 u^4 \quad . \tag{5-246}$$

This is in agreement with (5–240), except that we shall use a slightly different specification of the function $h(u)$ later on.

Flattening of the surfaces of constant density. We shall now try to estimate the flattening of the surfaces of constant density inside the earth. By a suitable selection of the functions $A(u)$ and $F(u) \doteq 1$ (as we did in sec. 5.8.1) we can achieve that these spheroidal (but not ellipsoidal!) surfaces are nearly spherical, so that the density distribution (5–241) may be approximated by the spherical distribution:

$$\begin{aligned} \sigma(r) &= \rho_0 + \rho_1 - r^2 A(r) \\ &= \rho_0 + \rho_1 - b_0 r^2 - b_2 r^4 \quad , \end{aligned} \tag{5-247}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ is the radius vector as usual.

Setting

$$\rho(u, \theta) = \sigma(r) + \Delta(u, \theta) \quad , \tag{5-248}$$

we have for the "density anomaly"

$$\Delta = (x^2 + y^2)[A(r) - A(u)] + z^2[A(r) - B(u)] + \frac{e'^2 \sin^2 \theta}{u^2 + e'^2 \cos^2 \theta} h(u) \quad ; \tag{5-249}$$

note that because of (5–243), E has been replaced by the second excentricity $e' = E/b$.

Consider now the deviation of the surfaces of constant density, $\rho =$ const., from the spherical surfaces $\sigma =$ const. Denoting the (variable) radius vector of a surface $\rho =$ const. by r, and the (constant) radius vector of the corresponding surface $\sigma =$ const. by r_0, and putting

$$r = r_0 + \zeta \quad , \tag{5-250}$$

then ζ represents the separation between these surfaces.

Then for the surface $\rho = C$ we have by (5–248)

$$\sigma(r) + \Delta = C \quad , \tag{5-251}$$

and for the surface $\sigma = C$ (with the same constant C),

$$\sigma(r_0) = C \quad . \tag{5-252}$$

On substituting (5–250) into (5–251) and expanding by Taylor's theorem we get

$$\sigma(r_0) + \sigma'(r_0)\zeta + \Delta = C \quad , \tag{5-253}$$

where

$$\sigma' = \frac{d\sigma}{dr} = -(2b_0 + 4b_2 r^2) r \quad . \tag{5-254}$$

In view of (5–252) this gives

$$\zeta = -\frac{\Delta}{\sigma'} \; . \tag{5–255}$$

The flattening of the surfaces of constant density may then be expressed by

$$f = \frac{(r_0 + \zeta_a) - (r_0 + \zeta_b)}{r_0 + \zeta_a} \doteq \frac{\zeta_a - \zeta_b}{r_0} \quad , \tag{5–256}$$

where ζ_a and ζ_b are the values of ζ at the equator and at the poles, respectively. By (5–255) this becomes (with r_0 replaced by r)

$$f = -\frac{\Delta_a - \Delta_b}{r\sigma'} = \frac{\Delta_a - \Delta_b}{r^2(2b_0 + 4b_2 r^2)} \quad . \tag{5–257}$$

The singularity in ellipsoidal coordinates. Before studying (5–257) further, we must consider a strange singularity of the ellipsoidal coordinate system. The equatorial plane, coinciding with the xy-plane in Fig. 5.4, is given by two equations: *outside*

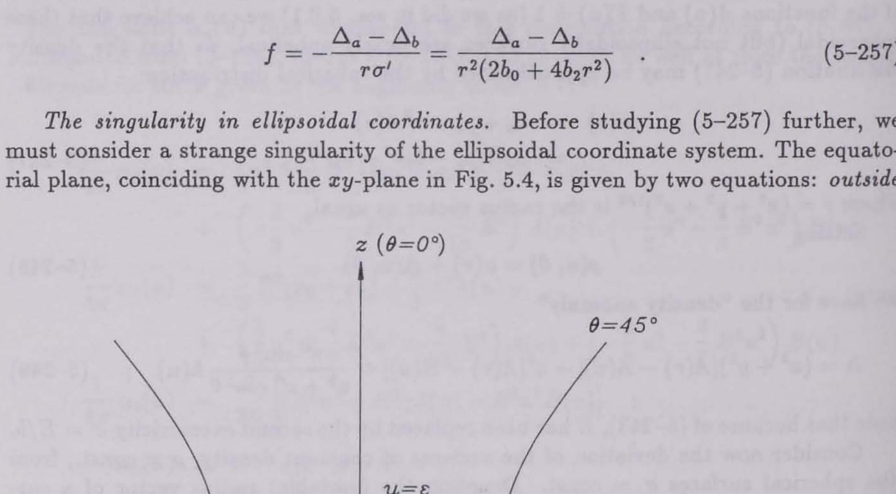

FIGURE 5.4: The "focal disc singularity" in the ellipsoidal coordinate system; ϵ is an arbitrary small number (dimensionless if $b = 1$)

the "focal disc" obtained by rotating OF_2 around the z-axis and indicated by the segment F_1F_2 in Fig. 5.4, we have (e.g., for point A):

$$\theta = 90° \quad , \tag{5–258}$$

whereas *inside the focal disc* (e.g., for point B) the equatorial plane is characterized by

$$u = 0 \quad , \tag{5-259}$$

but $\theta \neq 90°$! In fact, the basic relation (5–51),

$$r^2 = u^2 + E^2 \sin^2\theta \tag{5-260}$$

between the ellipsoidal coordinates u, θ and the spherical coordinate r (radius vector), gives for $u = 0$:

$$r = E\sin\theta \quad , \tag{5-261}$$

$$\sin\theta = \frac{r}{E} \le 1 \tag{5-262}$$

and hence $0 \le \theta \le 90°$. The focal points F_1 and F_2 are limiting points for which simultaneously $u = 0$, $\theta = 90°$ holds; in space, this is the "focal circle" bounding the focal disc.

In spherical coordinates,

$$r = 0 \tag{5-263}$$

denotes a single point (the origin), whereas in ellipsoidal coordinates,

$$u = 0 \tag{5-264}$$

holds for the whole focal disc, the individual points of which must be distinguished by θ and λ.

This fact that $u = 0$ denotes a surface (the focal disc) rather than a point, is not so strange as such since in rectangular coordinates

$$z = 0$$

also defines a surface, namely the whole xy-plane. What is unpleasant with ellipsoidal coordinates is the fact that we have to make a rather "unnatural" distinction between $r < E$ and $r > E$, and that we are, so to speak, "spoiled" by the nice behavior of the spherical coordinates, which have as singularity only the origin $r = 0$, whereas in ellipsoidal coordinates the singularity $u = 0$ comprises the whole focal disc.

Flattening of the surfaces of constant density resumed. Now we are in a position to return to eq. (5–257).

For the poles ($\theta = 0°$ or $180°$, $u = r$) we find from (5–249):

$$\Delta_b = r^2 A(r)[1 - F(r)] \quad . \tag{5-265}$$

With $b = 1$, $E = e'$, eq. (5–260) becomes

$$r^2 = u^2 + e'^2 \sin^2\theta \quad . \tag{5-266}$$

For the equator we thus have to distinguish the two cases discussed above. First, if $r \geq e'$, the equator is represented by $\theta = 90°$, so that by (5–266)

$$u^2 = r^2 - e'^2 \quad ,$$

and hence by (5–249)

$$\Delta_a = r^2 \left[A(r) - A\left(\sqrt{r^2 - e'^2}\right)\right] + \frac{e'^2}{r^2 - e'^2} h\left(\sqrt{r^2 - e'^2}\right) \quad . \tag{5–267}$$

Secondly, if $r \leq E = e'$, the equator is represented by

$$u = 0 \,, \qquad \sin\theta = \frac{r}{e'} \quad , \tag{5–268}$$

so that, then,

$$\Delta_a = r^2[A(r) - A(0)] \quad , \tag{5–269}$$

provided that $h(u)$ goes to zero sufficiently strongly as $u \to 0$.

It will be convenient to split up the flattening f as given by (5–257) into two parts:

$$f = f_1 + f_2 \quad , \tag{5–270}$$

where f_1 represents the effect of the first part of (5–241) (the "Maclaurin part") and f_2 represents the effect of $h(u)$. By (5–244), (5–246), (5–265), (5–267), and (5–269) we get

$$\begin{aligned} \text{for} \quad r \geq e' : \quad f_1(r) &= \frac{(b_0 + b_2 r^2)(-1 + a_0 + a_2 r^2 + a_4 r^4) + b_2 e'^2}{2b_0 + 4b_2 r^2} \quad , \\ f_2(r) &= \frac{e'^2 h\left(\sqrt{r^2 - e'^2}\right)}{r^2(r^2 - e'^2)(2b_0 + 4b_2 r^2)} \quad ; \end{aligned} \tag{5–271}$$

$$\begin{aligned} \text{for} \quad r \leq e' : \quad f_1(r) &= \frac{(b_0 + b_2 r^2)(-1 + a_0 + a_2 r^2 + a_4 r^4) + b_2 r^2}{2b_0 + 4b_2 r^2} \quad , \\ f_2(r) &= 0 \quad . \end{aligned} \tag{5–272}$$

Obviously, both expressions give the same values at $r = e'$.

Numerical results. In agreement with sec. 5.8.1 we shall use

$$\begin{aligned} b_0 &= 16.71 \quad , \\ b_2 &= -7.82 \quad , \\ e'^2 &= 0.00674 \quad , \\ e_0'^2 &= 0.00486 \quad , \\ a_0 &= 1.0049 \qquad (= 1 + e_0'^2) \quad , \\ a_2 &= 0.0259 \quad , \\ a_4 &= -0.0241 \quad ; \end{aligned} \tag{5–273}$$

as usual, e'^2 refers to the surface, and $e_0'^2$ to the center.

The values of f_1 obtained in this way are given in Table 5.1. We see that, if we go from the earth's surface ($r = 1$) to the center ($r = 0$), f_1 first increases markedly and only later decreases to the small central value.

This behavior of f_1 is, of course, unrealistic, but it is possible to compensate it by an f_2 that corresponds to a suitably chosen function $h(u)$.

Since f_1 has a maximum around $r = 0.9$, we must select $h(u)$ so that the corresponding values f_2 also have a maximum there. The function (5–165) will not fulfil this requirement; therefore we try instead

$$h(u) = \frac{D}{b^8} u^6(b^2 - u^2) \quad ; \tag{5–274}$$

temporarily we take $b \neq 1$. Then the condition (5–162) must be satisfied:

$$\int_0^b (u^2 + E^2)h(u)du = \frac{M}{4\pi}\left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) \quad . \tag{5–275}$$

The right-hand side has the numerical value (5–120):

$$-1.3646 \times 10^{23}\text{kg} \quad ,$$

and the left-hand side becomes on substituting (5–274) and integrating:

$$b^3 D\left(\frac{2}{99} + \frac{2}{63}e'^2\right) \quad . \tag{5–276}$$

We thus have the condition

$$\left(\frac{2}{99} + \frac{2}{63}e'^2\right) D = -0.5313\,\text{g/cm}^3 \quad , \tag{5–277}$$

from which we find

$$D = -26.02\,\text{g/cm}^3 \quad . \tag{5–278}$$

Then, (5–271) gives:

$$f_2 = e'^2 D\frac{(r^2 - e'^2)^2(1 - r^2 + e'^2)}{r^2(2b_0 + 4b_2r^2)} \quad . \tag{5–279}$$

Of course, r is again in terms of b ($\doteq R$) as a unit. The values f_2 corresponding to this function are also shown in Table 5.1.

The last column of Table 5.1 gives f, the flattening of the surfaces of constant density, as the sum of f_1 and f_2. We see that the effect of the maximum of f_1 at $r = 0.9$ has been quite sucessfully removed and f decreases monotonically from $r = 1.0$ to 0.1. (Because of the approximations involved in formulas such as (5–257), f for $r = 1$ turns out slightly too small, but the numbers are anyway to be considered illustrative rather than realistic.)

TABLE 5.1: Flattening of the surfaces of constant density according to Bullard's polynomial

r	f_1	f_2	f
0.0	0.0024	0.0000	0.0024
0.1	0.0010	0.0000	0.0010
0.2	0.0014	- 0.0002	0.0012
0.3	0.0020	- 0.0004	0.0016
0.4	0.0028	- 0.0008	0.0020
0.5	0.0036	- 0.0012	0.0024
0.6	0.0046	- 0.0018	0.0028
0.7	0.0055	- 0.0024	0.0031
0.8	0.0062	- 0.0030	0.0032
0.9	0.0064	- 0.0034	0.0030
1.0	0.0032	- 0.0005	0.0027

It would certainly be possible to obtain a strictly monotonic decrease by selecting a slightly different and more complicated function $h(u)$, but we shall not bother with this since our model shows a much stronger defect: the flattening increases again between $r = 0.1$ and $r = 0$.

Unfortunately, this defect has a deeper reason, which is seen as follows. For $r \leq e'$ we may neglect powers of r higher than r^2, so that (5–272) gives

$$\begin{aligned} f = f_1 &\doteq \frac{b_0 e_0'^2 + (b_2 a_0 + b_0 a_2) r^2}{2 b_0} \doteq \frac{1}{2} e_0'^2 + \frac{b_2}{2 b_0} r^2 \\ &= \frac{1}{2} e_0'^2 - 0.23\, r^2 < \frac{1}{2} e_0'^2 \doteq f_0 \quad . \end{aligned} \tag{5–280}$$

Thus, f in the neighborhood of the center is always smaller than the central value f_0. This is due to b_2 being negative; it cannot be helped by selecting different functions $F(u)$ or $h(u)$ (Moritz, 1973).

Difficulties with the focal disc singularity. If we look into the matter more closely, we see that the deeper reason of our problems is the nature of the ellipsoidal coordinate system, with its unpleasant disc singularity mentioned above. These problems seem to be very difficult to overcome if we are looking for a truly realistic density model such as some kind of ellipsoidal version of PREM (sec. 1.5). For the same reason, ellipsoidal coordinates have hardly been applied in the study of *heterogeneous* equilibrium figures, except that Poincaré (1886) used them for proving the impossibility of confocal ellipsoidal stratification for equilibrium figures.

For further studies of surfaces of constant density, their flattening, etc., it therefore seems best to give up ellipsoidal coordinates and change over to spherical coordinates. This will be done in Chapter 6.

The disadvantage of spherical coordinates is that series expansions must be used, as we have seen in the preceding chapters. The invaluable advantage of ellipsoidal coordinates is that they permit closed formulas. Therefore it is worthwhile to still use them to investigate problems in which closed formulas are important. This will be done in the last two sections of the present chapter.

5.11 Potential and Gravity Inside the Ellipsoid

Eq. (5–77) holds for the potential inside as well as outside the ellipsoid E, but the series for $1/l$, eq. (5–32), requires $u > u'$. If $u < u'$, then in this series we must interchange u and u'. This is completely analogous to the corresponding series for spherical harmonics, cf. (4-8) and (4–27). If the computation point $P(u, \theta, \lambda)$ lies inside the ellipsoid, we have to pass the coordinate ellipsoid S_P through it and use (5–32) directly for its interior I_P and, with u and u' interchanged, for the "shell" E_P between S_P and E; cf. Fig. 4.2 with the ellipsoid E instead of the spheroid S as boundary.

In agreement with eq. (4–6) we thus split up V as

$$V(u, \theta) = V_i(u, \theta) + V_e(u, \theta) \tag{5–281}$$

with

$$V_i(u, \theta) = G \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=0}^{u} \frac{1}{l} \rho(u', \theta') dv \quad , \tag{5–282}$$

$$V_e(u, \theta) = G \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=u}^{b} \frac{1}{l} \rho(u', \theta') dv \quad . \tag{5–283}$$

Now we proceed exactly as we did in sec. 5.3. For V_i we get the same expressions as (5–84), but with the upper limit of integration b replaced by u. Nonzonal terms are removed by orthogonality and there remains

$$V_i(u, \theta) = \sum_{n=0}^{\infty} A_n(u) Q_n \left(i\frac{u}{E}\right) P_n(\cos\theta) \tag{5–284}$$

with

$$A_n(u) = i\frac{G}{E}(2n+1) \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=0}^{u} \rho(u', \theta') P_n \left(i\frac{u'}{E}\right) P_n(\cos\theta') dv \quad , \tag{5–285}$$

in complete analogy to (5–74) and (5–85); of course, A_n is now a function of u.

Looking at (5–32), we immediately recognize that the interchange of u and u' is equivalent to the interchange of P_{nm} and Q_{nm} for u, with perfect symmetry. Applying these considerations to (5–284) and (5–285), we directly find

$$V_e(u, \theta) = \sum_{n=0}^{\infty} B_n(u) P_n \left(i\frac{u}{E}\right) P_n(\cos\theta) \tag{5–286}$$