Chapter 5

The Equipotential Ellipsoid and Its Density Distributions

The *equipotential ellipsoid*, or *level ellipsoid*, is an ellipsoid of revolution which, by definition, is an equipotential surface of its gravity potential. It is being used as a standard reference surface for both the geometry and the external gravitational field of the earth: for the geometry from the very beginning (e.g., Bessel and Clarke around 1850) and for the gravity field since 1930 (International Gravity Formula). Both the Geodetic Reference System 1967 and the Geodetic Reference System 1980 (cf. IAG, 1980; Moritz, 1984) are based on it.

Although the gravity field outside the ellipsoid and at its surface can be fully determined without the knowledge of its internal mass distribution, Heiskanen and Moritz (1967, p. 64) were still obliged to write: "In fact, we do not know of any "reasonable" mass distribution for the level ellipsoid." However, physically possible continuous density distributions were described one year later in (Moritz, 1968a, b).

From sec. 3.2.4 we know that, except in the (for the earth) unrealistic case of a homogeneous mass distribution there cannot be a figure of equilibrium with ellipsoidal equisurfaces. However, heterogeneous non-equilibrium mass distributions for the level ellipsoid do exist and will be considered in this chapter.

5.1 Ellipsoidal Coordinates and Ellipsoidal Harmonics

Ellipsoidal coordinates u, $\bar{\theta}$, λ and ellipsoidal harmonics are natural generalizations of spherical coordinates r, θ, λ and spherical harmonics. They permit a treatment of the theory of the level ellipsoid by closed formulas.

Since ellipsoidal coordinates and harmonics are standard (Hobson, 1931, secs. 249 to 252; Heiskanen and Moritz, 1967, secs. 1-19 and 1-20; Sigl, 1985, sec. III.5), a brief review will be sufficient.

In a rectangular system a point P has the coordinates xyz (Fig. 5.1). Now we pass through P the surface of an ellipsoid of revolution whose center is the origin O,

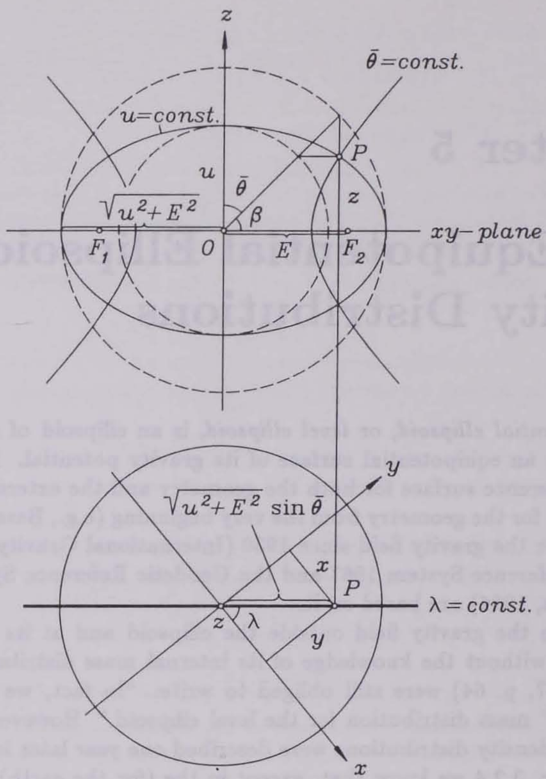

FIGURE 5.1: Ellipsoidal coordinates. Top: View from the front. Bottom: View from above.

whose axis coincides with the z-axis, and whose linear excentricity has the constant value E. The coordinate u is the semiminor axis of this "coordinate ellipsoid", $\bar{\theta}$ is the complement of the "reduced latitude" β of P with respect to this ellipsoid (for its definition cf. sec. 1.4), and λ is the geocentric longitude in the usual sense.

The ellipsoidal coordinates u, $\bar{\theta}$, λ are related to x, y, z by the equations

$$\begin{aligned} x &= \sqrt{u^2+E^2}\sin\bar{\theta}\cos\lambda \quad , \\ y &= \sqrt{u^2+E^2}\sin\bar{\theta}\sin\lambda \quad , \\ z &= u\cos\bar{\theta} \quad , \end{aligned} \tag{5–1}$$

which can be read from the figure, considering that $\sqrt{u^2+E^2}$ is the semimajor axis of the ellipsoid whose surface passes through P.

If we take $u =$ const. we find

$$\frac{x^2+y^2}{u^2+E^2}+\frac{z^2}{u^2}=1 \quad , \tag{5–2}$$

which represents an ellipsoid of revolution. For $\bar{\theta} =$ const. we obtain

$$\frac{x^2+y^2}{E^2\sin^2\bar{\theta}}-\frac{z^2}{E^2\cos^2\bar{\theta}}=1 \quad , \tag{5–3}$$

which represents a hyperboloid of one sheet, and for $\lambda =$ const. we get the meridian plane

$$y = x\tan\lambda \quad . \tag{5–4}$$

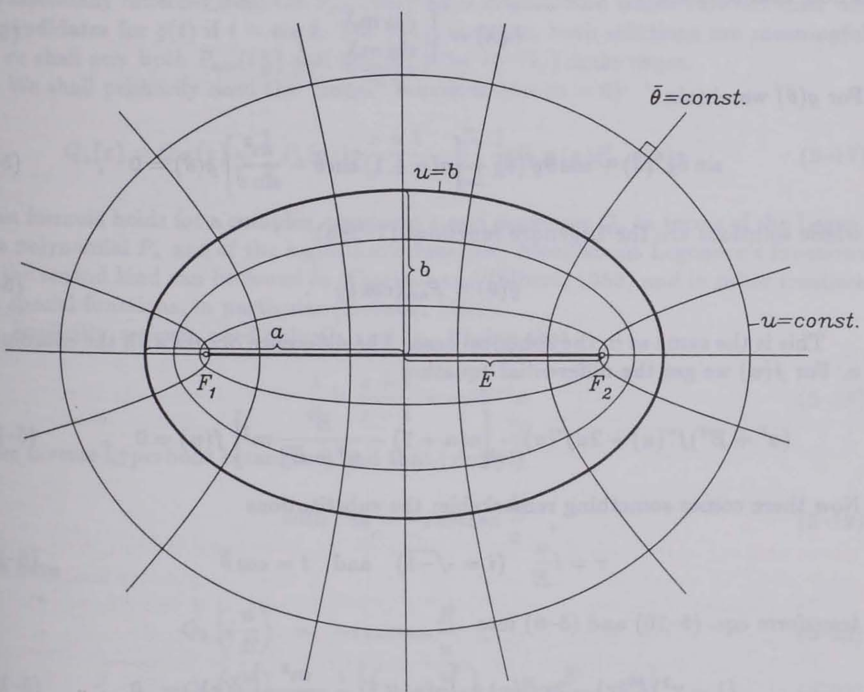

FIGURE 5.2: The confocal coordinate ellipsoids $u =$ const. and hyperboloids $\bar{\theta} =$ const., together with the reference ellipsoid $u = b$

The constant focal length (*linear excentricity*) $E = F_1O = OF_2$ which is the same for *all* ellipsoids $u =$ const., characterizes the coordinate system. For $E = 0$ we have the usual spherical coordinates r, θ, λ as a limiting case. Fig. 5.2 shows the set of coordinate surfaces u = const. and $\bar{\theta}$ = const., which intersect each other orthogonally. One of the set of coordinate ellipsoids u = const. is singled out as the *reference ellipsoid*, the constant u being its semiminor axis b.

Ellipsoidal harmonics. We proceed in the same way as with spherical harmonics (sec. 1.3). We express Laplace's equation $\Delta V = 0$ in ellipsoidal coordinates and try to solve it by a product of three functions, each of which depends on one coordinate only:

$$V(u, \bar{\theta}, \lambda) = f(u) g(\bar{\theta}) h(\lambda) \quad . \tag{5-5}$$

For $h(\lambda)$ we get the ordinary differential equation

$$h''(\lambda) + m^2 h(\lambda) = 0 \quad , \tag{5-6}$$

whose solutions are

$$h(\lambda) = \begin{cases} \cos m\lambda & , \\ \sin m\lambda & . \end{cases} \tag{5-7}$$

For $g(\bar{\theta})$ we obtain

$$\sin\bar{\theta} g''(\bar{\theta}) + \cos\bar{\theta} g'(\bar{\theta}) + \left[n(n+1)\sin\bar{\theta} - \frac{m^2}{\sin\bar{\theta}} \right] g(\bar{\theta}) = 0 \quad , \tag{5-8}$$

whose solutions are the Legendre functions (1–28b):

$$g(\bar{\theta}) = P_{nm}(\cos\bar{\theta}) \quad . \tag{5-9}$$

This is the same as in the spherical case. The difference occurs with the coordinate u. For $f(u)$ we get the differential equation

$$(u^2 + E^2) f''(u) + 2u f'(u) - \left[n(n+1) - \frac{E^2}{u^2 + E^2} m^2 \right] f(u) = 0 \quad . \tag{5-10}$$

Now there comes something remarkable: the substitutions

$$\tau = i\frac{u}{E} \quad (i = \sqrt{-1}) \quad \text{and} \quad t = \cos\bar{\theta} \tag{5-11}$$

transform eqs. (5–10) and (5–8) into

$$(1 - \tau^2)\bar{f}''(\tau) - 2\tau\bar{f}'(\tau) + \left[n(n+1) - \frac{m^2}{1 - \tau^2} \right] \bar{f}(\tau) = 0 \quad , \tag{5-12}$$

$$(1 - t^2)\bar{g}''(t) - 2t\bar{g}'(t) + \left[n(n+1) - \frac{m^2}{1 - t^2} \right] \bar{g}(t) = 0 \quad , \tag{5-13}$$

where the overbar indicates that the functions f and g are expressed in terms of the new arguments τ and t. Both equations are essentially identical!

Thus $\bar{f}(\tau)$ satisfies formally the same differential equation as $\bar{g}(t)$, which is called Legendre's equation. A solution, by (5-9) and (5–11), is

$$\bar{g}(t) = P_{nm}(t) \quad . \tag{5–14}$$

Any second-order differential equation, however, admits two essentially different solutions, a simple example being $\cos m\lambda$ and $\sin m\lambda$ for (5–6). Thus we have two different solutions of (5–12):

$$\bar{f}(\tau) = P_{nm}(\tau) \tag{5–15}$$

and

$$\bar{f}(\tau) = Q_{nm}(\tau) \quad . \tag{5–16}$$

The first, *Legendre's functions*, $P_{nm}(\tau)$, corresponds to (5–14); the second solution, (5–16) is new. The $Q_{nm}(\tau)$ are called *Legendre's function of the second kind.* They are essentially different from the P_{nm}; they have singularities which rule out their use as candidates for $\bar{g}(t)$ if $t = \cos\bar{\theta}$. For $f(u)$, however, both solutions are meaningful as we shall see: both $P_{nm}(i\frac{u}{E})$ and $Q_{nm}(i\frac{u}{E})$ (by (5–11)) make sense.

We shall primarily need the "zonal" functions (for $m = 0$):

$$Q_n(z) = Q_{n0}(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(z) P_{n-k}(z) \quad . \tag{5–17}$$

This formula holds for a complex argument z and expresses Q_n in terms of the Legendre polynomial P_n and of the logarithmic function. More about Legendre's functions of the second kind can be found in (Courant and Hilbert, 1953) and in other treatises on special functions, in particular (Hobson, 1931).

Explicitly we even need only Q_0 and Q_2. Noting that

$$\frac{1}{2} \ln \frac{z+1}{z-1} = \coth^{-1} z \tag{5–18}$$

(the inverse hyperbolic cotangent) and that (verify!)

$$\coth^{-1} ix = -i \arctan \frac{1}{x} \quad , \tag{5–19}$$

we have

$$Q_0\left(i\frac{u}{E}\right) = -i \arctan \frac{E}{u} \quad , \tag{5–20}$$

$$Q_2\left(i\frac{u}{E}\right) = \frac{i}{2}\left[\left(1 + 3\frac{u^2}{E^2}\right) \arctan \frac{E}{u} - 3\frac{u}{E}\right] \tag{5–21}$$

(cf. Heiskanen and Moritz, 1967, p. 66).

The "tesseral" Q_{nm} for a complex argument are defined by

$$Q_{nm}(z) = (z^2 - 1)^{\frac{m}{2}} \frac{d^m Q_n(z)}{dz^m} \quad . \tag{5–22}$$

This is rather similar to the equation $P_{nm}(t) = (1-t^2)^{\frac{m}{2}} d^m P_n(t)/dt^m$ following from (1–30) and (1–32).

Thus we may summarize the possible solutions:

$$\begin{aligned} f(u) &= P_{nm}\left(i\frac{u}{E}\right) \quad \text{or} \quad Q_{nm}\left(i\frac{u}{E}\right) \quad ; \\ g(\bar\theta) &= P_{nm}(\cos\bar\theta) \quad ; \\ h(\lambda) &= \cos m\lambda \quad \text{or} \quad \sin m\lambda \quad . \end{aligned} \tag{5-23}$$

Here n and $m \le n$ are integers 0, 1, 2, ..., as before. (Note the analogy to (1–28)!) Hence the functions

$$\begin{aligned} V(u,\bar\theta,\lambda) &= P_{nm}\left(i\tfrac{u}{E}\right) P_{nm}(\cos\bar\theta) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\} \quad , \\ V(u,\bar\theta,\lambda) &= Q_{nm}\left(i\tfrac{u}{E}\right) P_{nm}(\cos\bar\theta) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\} \end{aligned} \tag{5-24}$$

are solutions of Laplace's equation $\Delta V = 0$, that is, harmonic functions.

From these functions we may by linear combination form the series

$$\begin{aligned} V_i(u,\bar\theta,\lambda) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{P_{nm}\left(i\frac{u}{E}\right)}{P_{nm}\left(i\frac{b}{E}\right)} \cdot \\ &\quad \cdot \left[a_{nm}P_{nm}(\cos\bar\theta)\cos m\lambda + b_{nm}P_{nm}(\cos\bar\theta)\sin m\lambda\right] \quad ; \end{aligned} \tag{5-25}$$

$$\begin{aligned} V_e(u,\bar\theta,\lambda) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{Q_{nm}\left(i\frac{u}{E}\right)}{Q_{nm}\left(i\frac{b}{E}\right)} \cdot \\ &\quad \cdot \left[a_{nm}P_{nm}(\cos\bar\theta)\cos m\lambda + b_{nm}P_{nm}(\cos\bar\theta)\sin m\lambda\right] \quad . \end{aligned} \tag{5-26}$$

Here b is the semiminor axis of our given reference ellipsoid; cf. Fig. 5.2. The division by $P_{nm}(ib/E)$ or $Q_{nm}(ib/E)$ is possible because they are constants; its purpose is to simplify the expressions and to make the coefficients a_{nm} and b_{nm} real. In fact, at the surface of the reference ellipsoid, both series (5–25) and (5–26) reduce to

$$\begin{aligned} V_i(b,\bar\theta,\lambda) &= V_e(b,\bar\theta,\lambda) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[a_{nm}P_{nm}(\cos\bar\theta)\cos m\lambda + b_{nm}P_{nm}(\cos\bar\theta)\sin m\lambda\right] \ . \end{aligned} \tag{5-27}$$

This is formally the same as for the sphere, with $\bar\theta = 90° - \beta$ instead of the polar distance θ; cf. (1–37) for $r = 1$ or (1–48).

Thus we already understand the *surface* expansion (5–27). In order to better understand the *spatial* expansions (5–25) and (5–26), we consider the limit $E \to 0$. Then the ellipsoidal coordinates u, $\bar\theta$, λ become spherical coordinates r, θ, λ; the ellipsoid $u = b$ becomes the sphere $r = R$ (because then the semiaxes a and b are equal to the radius R); and we find

$$\lim_{E\to 0}\frac{P_{nm}\left(i\frac{u}{E}\right)}{P_{nm}\left(i\frac{b}{E}\right)} = \left(\frac{u}{b}\right)^n = \left(\frac{r}{R}\right)^n \quad , \qquad \lim_{E\to 0}\frac{Q_{nm}\left(i\frac{u}{E}\right)}{Q_{nm}\left(i\frac{b}{E}\right)} = \left(\frac{R}{r}\right)^{n+1} \quad . \tag{5-28}$$

Thus we see that the function $P_{nm}(iu/E)$ corresponds to r^n and $Q_{nm}(iu/E)$ corresponds to $r^{-(n+1)}$ in spherical harmonics.

Hence the series (5–25) is harmonic in the interior of the ellipsoid $u = b$, and the series (5–26) is harmonic in its exterior; this case is relevant to geodesy.

In fact, given $V = V(b, \bar{\theta}, \lambda)$ *at the surface* of the reference ellipsoid, the external potential $V_e(u, \bar{\theta}, \lambda)$ is fully determined by (5–26). In this way it is possible to determine the external potential *without knowledge of the internal mass distribution that produces it*, from the knowledge of its surface values as given by (5–27). This fact will be of basic importance in the next section. By the way, the (unique) determination of (5–25) or (5–26) from (5–27) is called the solution of *Dirichlet's problem* for the ellipsoid.

Reciprocal distance. We shall also need the expansion of the reciprocal distance, $1/l$, in ellipsoidal harmonics, corresponding to the spherical-harmonic expansion (1–53). We use the well-known addition theorem for spherical harmonics

$$\begin{aligned} P_n(\cos\psi) &= P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\cdot \\ &\cdot [R_{nm}(\theta,\lambda)R_{nm}(\theta',\lambda') + S_{nm}(\theta,\lambda)S_{nm}(\theta',\lambda')] \quad ; \end{aligned} \tag{5–29}$$

the notation follows sec. 1.3. Thus (1–53) becomes in spherical harmonics

$$\frac{1}{l} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} c_{nm}\frac{r'^n}{r^{n+1}}P_{nm}(\cos\theta)P_{nm}(\cos\theta')\cos m(\lambda'-\lambda) \tag{5–30}$$

with

$$c_{n0} = 1, \qquad c_{nm} = 2\frac{(n-m)!}{(n+m)!} \qquad (1 \le m \le n) \quad . \tag{5–31}$$

Using the correspondence (5–28) we expect that in ellipsoidal harmonics we shall have

$$\frac{1}{l} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{nm}Q_{nm}\left(i\frac{u}{E}\right)P_{nm}\left(i\frac{u'}{E}\right)P_{nm}(\cos\bar{\theta})P_{nm}(\cos\bar{\theta}')\cos m(\lambda'-\lambda) \quad . \tag{5–32}$$

The coefficients

$$C_{n0} = \frac{i}{E}(2n+1), \qquad C_{nm} = \frac{i}{E}2(-1)^m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 \tag{5–33}$$

can be found, e.g., in (Hobson, 1931, p. 430; note that Hobson has omitted the factor i which is necessary to make l real) or (Heine, 1961, vol. II, p. 106); cf. also (Hotine, 1969, p. 193). It is an interesting, though nontrivial, exercise to show that for $E \to 0$, (5–32) with (5–33) reduces to (5–30) with (5–31); hint: use eq. (22.49) of (Hotine, 1969) together with eq. (1–62) of (Heiskanen and Moritz, 1967), in both cases replacing $(1-t^2)^{m/2}$ by $(t^2-1)^{m/2}$ for complex t.