

Lichtenstein determines the geometry from the physics. Also, for Lichtenstein, the spherical configuration is the starting point, whereas for Wavre it is a singularity (0/0)!

Wavre's approach is also described in the books (Baeschlin, 1948) and (Ledersteger, 1969), whereas the basic book in English, (Jardetzky, 1958), does not present it, although it outlines an approximation method also due to Wavre ("procédé uniforme") which intends, by an ingenious but complicated trick, to circumvent the convergence problem of certain series of spherical harmonics. We shall not treat this here because the author believes that this problem can be tackled in a much simpler way as we shall see in sec. 4.1.5.

3.3 Stationary Potential Energy

In various domains of physics, equilibrium is associated with a stationary (maximum or minimum, depending on the sign) value of potential energy. Liapunov and Poincaré have treated *homogeneous* equilibrium figures from this point of view; a modern approach is found in the book (Macke, 1967, p. 543). Macke's method has been generalized to heterogeneous (terrestrial) equilibrium figures (Macke et al., 1964; Voss, 1965, 1966). This approach is interesting because it reflects the typical thinking and mathematical methods of modern theoretical physics.

3.3.1 Potential Energy

The gravitational energy of a material particle of mass m in a field of potential V is mV , and that of a system of particles thus

$$E = \sum m_i V_i \quad ; \quad (3-93)$$

the sign (+ or -) is conventional.

This holds for an *external* potential field V . If the field is produced by the mutual gravitational attraction of the particles themselves:

$$V_i = G \sum_j \frac{m_j}{l_{ij}} \quad (j \neq i) \quad , \quad (3-94)$$

then (3-93) gives

$$G \sum_{i,j} \frac{m_i m_j}{l_{ij}} .$$

Each term occurs twice, however (interchange i and j), so that we must divide by 2, obtaining

$$E_V = \frac{1}{2} G \sum_i \sum_j \frac{m_i m_j}{l_{ij}} \quad (j \neq i) \quad ; \quad (3-95)$$

cf. also (Kellogg, 1929, pp. 79-81) or (Poincaré, 1902, pp. 7-8).

The continuous analogue of (3-95) is

$$E_V = \frac{1}{2} G \iiint_{\mathfrak{v}} \iiint_{\mathfrak{v}'} \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{\|\mathbf{x} - \mathbf{x}'\|} dv dv' \quad (3-96)$$

with obvious notations: \mathbf{x} , \mathbf{x}' position vectors; v volume of the body; dv , dv' volume elements; and $l = \|\mathbf{x} - \mathbf{x}'\|$. Another form of (3-96) is

$$E_V = \frac{1}{2} \iiint_{\mathfrak{v}} V \rho dv \quad , \quad (3-97)$$

where V is the usual gravitational potential. Comparing with (3-93) note the factor 1/2, reflecting the fact that E_V is produced by an *internal field* created by the mass elements $dm = \rho dv$ themselves.

For the centrifugal part we have

$$E_{\Phi} = \sum m_i \Phi_i = \iiint_{\mathfrak{v}} \Phi \rho dv \quad , \quad (3-98)$$

in agreement with (3-93), since the centrifugal potential Φ acts as an *external field*.

The potential energy of the gravity potential $W = V + \Phi$ thus is the sum of (3-97) and (3-98):

$$E_W = \int \left(\frac{1}{2} V + \Phi \right) \rho dv \quad , \quad (3-99)$$

using only a simple integral sign for notational convenience.

3.3.2 Dirac's and Heaviside's Functions

We recall the basic definition of *Dirac's delta function* (cf. Moritz, 1980, pp. 28-30):

$$\begin{aligned} \delta(x) &= 0 && \text{except for } x = 0 \quad , \\ \delta(0) &= \infty && \text{in such a way that} \end{aligned} \quad (3-100)$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \quad . \quad (3-101)$$

It is a somewhat strange "function" but is extremely useful and popular in physics.

Its integral is Heaviside's step function:

$$\theta(x) = \int_{-\infty}^x \delta(x') dx' \quad . \quad (3-102)$$

From (3-100) and (3-101) it immediately follows that

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \quad , \\ 1 & \text{for } x > 0 \quad . \end{cases} \quad (3-103)$$

For $\theta(0)$ we may take the value 1/2.

From (3-102) there follows the basic relation

$$\delta(x) = \frac{d\theta(x)}{dx} = \theta'(x) \quad . \quad (3-104)$$

3.3.3 A Remarkable Expression for the Density

Assume the body to consist of n layers bounded by surfaces S_k and S_{k+1} (Fig. 3.3). The density within each layer is constant, denoted in our case by ρ_{k+1} .

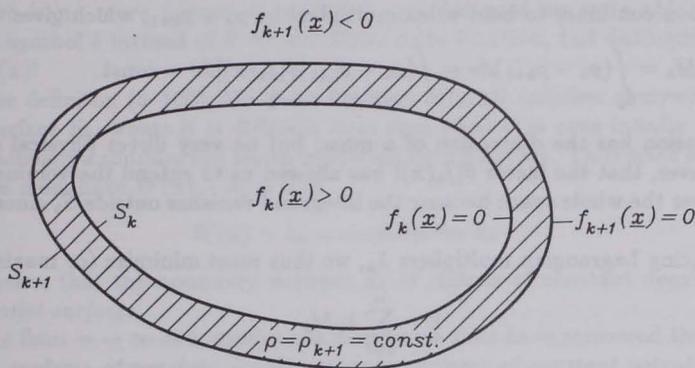


FIGURE 3.3: A layer of constant density (\underline{x} denotes \mathbf{x})

Let the surface S_k have the equation

$$f_k(\mathbf{x}) = 0 \quad , \quad (3-105)$$

and let f_k be monotonic with

$$f_k(\mathbf{x}) > 0 \quad \text{inside } S_k \quad (3-106)$$

(otherwise change the sign of f_k !).

Then the density everywhere within the stratified body can be described by the single expression

$$\rho(\mathbf{x}) = \sum_{k=1}^n (\rho_k - \rho_{k+1}) \theta[f_k(\mathbf{x})] \quad . \quad (3-107)$$

The reader is invited to verify this on the basis of (3-103) and (3-106). Eq. (3-107) holds with the understanding that $\rho_{n+1} = 0$ since the density is zero outside the boundary surface $S = S_n$.

3.3.4 Variation of the Potential Energy

Let us find the extremum of the potential energy $E = E_W$ as given by (3-99):

$$E = \int \left(\frac{1}{2} V + \Phi \right) \rho dv \quad , \quad (3-108)$$

where ρ is expressed by (3-107); since $\rho = 0$ outside S , we may extend the integral formally over the whole space. The *side condition* is that the volume enclosed by

any surface S_k (Fig. 3.3) remains unchanged during the variation δ to be performed below:

$$v_k = \int_{S_k} dv = \text{const.} \quad (3-109)$$

This equation continues to hold when multiplied by $\rho_k - \rho_{k+1}$, which gives

$$M_k = \int_{S_k} (\rho_k - \rho_{k+1}) dv = \int (\rho_k - \rho_{k+1}) \theta[f_k(\mathbf{x})] dv = \text{const.} \quad (3-110)$$

This expression has the dimension of a mass, but no very direct physical meaning. Note, however, that the factor $\theta[f_k(\mathbf{x})]$ has allowed us to extend the volume integral formally over the whole space because the integrand vanishes outside S_k since $f_k(\mathbf{x}) < 0$ there.

Introducing Lagrangian multipliers λ_k , we thus must minimize (or maximize)

$$E - \sum_{k=1}^n \lambda_k M_k .$$

This leads to the variational condition (δ is now the sign for variation and has nothing to do with the Delta function!):

$$\delta \left[E - \sum_{k=1}^n \lambda_k M_k \right] = 0 \quad (3-111)$$

or

$$\int (V + \Phi) \delta \rho dv - \sum_{k=1}^n \lambda_k \delta M_k = 0 . \quad (3-112)$$

Note that we are varying the density ρ by $\delta \rho$ and that, as compared to (3-108) the factor 1/2 seems to be missing. However, by (3-96), E_V is a *quadratic* functional of ρ . This introduces the usual factor of 2 on differentiation, which combines with 1/2 to 1. With the gravity potential $W = V + \Phi$ this reduces to

$$\int W \delta \rho dv - \sum_{k=1}^n \lambda_k \delta M_k = 0 . \quad (3-113)$$

Now we must express the density variations $\delta \rho$ by $\delta f_k(\mathbf{x})$ since $\delta \rho$ is caused by a change in the boundary surfaces only. Now our expression (3-107) comes in handy: we have

$$\delta \theta[f_k(\mathbf{x})] = \theta'[f_k(\mathbf{x})] \delta f_k(\mathbf{x}) , \quad (3-114)$$

where $\theta'(x) = \delta(x)$ is the delta function by (3-104); we prefer the notation θ' to avoid confusion with the variation δ .

With (3-114) everything is straightforward: (3-107) gives $\delta \rho$, and (3-110) similarly gives δM_k . Thus (3-113) becomes

$$\int dv \left\{ \sum_{k=1}^n (\rho_k - \rho_{k+1}) (W(\mathbf{x}) - \lambda_k) \theta'[f_k(\mathbf{x})] \right\} \delta f_k(\mathbf{x}) = 0 . \quad (3-115)$$

The small deformations $\delta f_k(\mathbf{x})$ being arbitrary, the integrand between brackets $\{\}$ must vanish:

$$\sum_{k=1}^n (\rho_k - \rho_{k+1})(W(\mathbf{x}) - \lambda_k) \delta[f_k(\mathbf{x})] = 0 \quad (3-116)$$

Now there is no more danger of confusion, so that now we were able to use the standard symbol δ instead of θ' for the Dirac delta function, but distinguish $\delta[f_k(\mathbf{x})]$ from $\delta f_k(\mathbf{x})$!

By the definition (3-100), the delta function $\delta[f_k(\mathbf{x})]$ vanishes everywhere except on the surface S_k , where it is different from zero (that it is even infinite there gives mathematicians a shudder but leaves physicists entirely cold). Thus since $\delta[f_k(\mathbf{x})] \neq 0$ on S_k , we must have $W(\mathbf{x}) - \lambda_k = 0$ or

$$W(\mathbf{x}) = \lambda_k = \text{constant on } S_k \quad (3-117)$$

which means that the boundary surfaces S_k of regions of constant density *must be equipotential surfaces*.

In the limit $n \rightarrow \infty$ of a continuous density we thus have recovered the basic fact that *the surfaces of constant density must be surfaces of constant potential*. This is our well-known condition for equilibrium figures.

What is new? Formerly, in sec. 2.5, we have derived this condition from (2-98) by means of the *pressure* p , a quantity which we have not used afterwards any more. For some people's taste, it is not very elegant to introduce an auxiliary concept which plays the role of a *deus ex machina* and disappears again. Here we have derived our basic condition $\rho = \text{const.} \iff W = \text{const.}$ from the principle of stationary energy, which is logically more satisfactory for many people, especially in view of the fact that maximum or minimum principles play a fundamental role in physics.

Another beautiful fact: the Lagrange multiplier λ_k admits a natural physical interpretation; it is nothing else than the constant value of the potential W on S_k , cf. (3-117).

3.3.5 A General Integral Equation

Now we are also in a position to give an explicit representation for the functions $f_k(\mathbf{x})$ which characterize the equisurfaces S_k : we may simply put

$$f_k(\mathbf{x}) = W(\mathbf{x}) - \lambda_k \quad (3-118)$$

In fact, on S_k we have $f_k(\mathbf{x}) = 0$ by (3-117), and inside S_k there is $f_k(\mathbf{x}) > 0$ since W increases monotonically towards the center. Thus (3-105) and (3-106) are satisfied.

Now in

$$W(\mathbf{x}) = G \int \frac{\rho}{l} dv + \frac{1}{2} \omega^2 (x^2 + y^2) \quad (= V + \Phi) \quad (3-119)$$

we may substitute (3-107) together with (3-118), obtaining

$$W(\mathbf{x}) = G \int \frac{dv}{l} \sum_{k=1}^n (\rho_k - \rho_{k+1}) \theta[W(\mathbf{x}') - \lambda_k] + \frac{1}{2} \omega^2 (x^2 + y^2) \quad (3-120)$$

with $l = \|\mathbf{x} - \mathbf{x}'\|$. This is a nonlinear integral equation for $W(\mathbf{x})$; the Lagrangian parameters λ_k are determined by the condition of constant volume (3-109):

$$v_k = \int \theta[W(\mathbf{x}') - \lambda_k] dv, \quad k = 1, 2, \dots, n \quad (3-121)$$

It is straightforward to let $n \rightarrow \infty$ and perform the transition to a continuous (or piecewise continuous) mass distribution; then the index k becomes a continuous variable u :

$$W(\mathbf{x}) = -G \int \frac{dv}{l} \int_{u=0}^{\infty} d\rho(u) \theta[W(\mathbf{x}') - \lambda(u)] + \frac{1}{2} \omega^2 (x^2 + y^2) \quad , \quad (3-122)$$

$$v(u) = \int \theta[W(\mathbf{x}') - \lambda(u)] dv \quad . \quad (3-123)$$

Doubtlessly, this formulation for hydrostatic equilibrium figures is extremely elegant: it generally and completely expresses the fact that the surfaces of constant density are also surfaces of constant potential. Voss (1965, 1966) has tried to solve this equation by a series expression in terms of f , f^2 , etc. (f is the flattening as usual). The linear terms again give Clairaut's theory, essentially the linear integral equation (3-15). Voss has also computed second-order terms, but the details are very complicated and these terms can be calculated much more directly as we shall see in the next section. Another application of the potential energy approach is found in (Chandrasekhar and Roberts, 1963).

Thus the main importance of the method described in the present section is theoretical and conceptual, but it is great indeed.